Research Article

On 10-Centralizer Groups of Odd Order

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Let G be a group, and let $|\text{Cent}(G)|$ denote the number of distinct centralizers of its elements. A group G is called n -centralizer if $|\text{Cent}(G)| = n$. In this paper, we investigate the structure of finite groups of odd order with $|\text{Cent}(G)| = 10$ and prove that there is no finite nonabelian group of odd order with $|\text{Cent}(G)| = 10$.

1. Introduction

Throughout this paper all groups mentioned are assumed to be finite, and we will use usual notation; for example, C_n denotes the cyclic group of order n , and $C_n \rtimes C_p$ denotes the semidirect product of C_n and C_p with normal subgroup C_n , where n is a positive integer and p is a prime. For a group G , $Z(G)$ denotes the center of G , and $\text{Cent}(G) = \{C_G(x) \mid x \in G\}$, where $C_G(x)$ is the centralizer of the element x in G ; that is, $C_G(x) = \{y \in G \mid xy = yx\}$. A group G is a CA-group if $C_G(x)$ is abelian for every $x \in G \setminus Z(G)$. Many authors have studied the influence of $|\text{Cent}(G)|$ on finite group G (see [1–9]). It is clear that a group G is 1-centralizer if and only if it is abelian. In [6] Belcastro and Sherman proved that there is no n -centralizer group for $n = 2, 3$. They also proved that G is 4-centralizer if and only if $G/Z(G) \cong C_2 \times C_2$, and G is 5-centralizer if and only if $G/Z(G) \cong C_3 \times C_3$ or S_3 . In [2] Ashrafi proved that if G is 6-centralizer, then $G/Z(G) \cong D_8$, A_4 , $C_2 \times C_2 \times C_2$, or $C_2 \times C_2 \times C_2 \times C_2$. In [1] Abdollahi et al. proved that G is 7-centralizer if and only if $G/Z(G) \cong C_5 \times C_5$, D_{10} or $\langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$. They also proved that if G is 8-centralizer, then $G/Z(G) \cong C_2 \times C_2 \times C_2$, A_4 or D_{12} .

Our main result is as follows.

Theorem 1. *There is no finite nonabelian group G of odd order with $|\text{Cent}(G)| = 10$.*

2. Preliminary Results

By [1], a cover Γ for a group G is a collection of proper subgroups whose union is the whole G . We use the term n -cover for a cover with n members. A cover is called irredundant if no proper subcollection is also a cover. A cover is called a partition with kernel k if the intersection of pairwise members of the cover is k . Neumann in [10] obtained a uniform bound for the index of the intersection of an irredundant n -cover in terms of n , and Tomkinson [11] improved this bound. For a natural number n , let $f(n)$ denote the largest index $|G : D|$, where G is a group with an irredundant n -cover whose intersection of all of them is D . We know that $f(3) = 4$, $f(4) = 9$, $f(5) = 16$, and $f(6) = 36$ (see [12–15], resp.). Now we present some lemmas and propositions that will be used in the proof of Theorem 1.

Lemma 2 (Lemma 3.3 of [11]). *Let M be a proper subgroup of the finite group G , and let $H_1, H_2, \dots, H_k$ be subgroups of G with $|G : H_i| = \beta_i$ and $\beta_1 \leq \dots \leq \beta_k$. If $G = M \cup H_1 \cup \dots \cup H_k$, then $\beta_1 \leq k$. Furthermore, if $\beta_1 = k$, then $\beta_1 = \beta_2 = \dots = \beta_k = k$ and $H_i \cap H_j \leq M$ for any two distinct i and j .*

Definition 3 (Definition 2.1 of [1]). A nonempty subset $X = \{x_1, \dots, x_r\}$ of a finite group G is called a set of pairwise noncommuting elements if $x_i x_j \neq x_j x_i$ for all distinct $i, j \in \{1, \dots, r\}$. A set of pairwise noncommuting elements of G is

said to have maximal size if its cardinality is the largest one among all such sets.

Remark 4. Let G be a finite group, and let $\{x_1, \dots, x_r\}$ be a set of pairwise noncommuting elements of G having maximal size. Then

- (1) $\{C_G(x_i) \mid i = 1, \dots, r\}$ is an irredundant r -cover with the intersection $Z(G) = \cap_{i=1}^r C_G(x_i)$ (see Theorem 5.1 of [11]);
- (2) $|G/Z(G)| \leq f(r)$ (see Corollary 5.2 of [11]);
- (3) $f(3) = 4$, $f(4) = 9$, $f(5) = 16$, $f(6) = 36$, and $f(7) = 81$ (see [12–16], resp.);
- (4) let G be a group such that every proper centralizer in G is abelian. Then for all $a, b \in G \setminus Z(G)$ either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$.

If $z \in (C_G(a) \cap C_G(b)) \setminus Z(G)$, then $C_G(z)$ contains both $C_G(a)$ and $C_G(b)$, since $C_G(a)$ and $C_G(b)$ are abelian. Since z is not in $Z(G)$, $C_G(z) \leq C_G(a)$ and $C_G(z) \leq C_G(b)$. Thus, $C_G(z) = C_G(a) = C_G(b)$. Hence, in such a group G , $\{C_G(x) \mid x \in G \setminus Z(G)\}$ forms a partition with kernel $Z(G)$. It follows that $\{C_G(x)/Z(G) \mid x \in G \setminus Z(G)\}$ forms a partition whose kernel is the trivial subgroup (see also Proposition 1.2 of [17]).

Lemma 5 (Lemma 2.4 of [1]). *Let G be a finite nonabelian group, and let $\{x_1, \dots, x_r\}$ be a set of pairwise noncommuting elements of G with maximal size. Then*

- (1) $r \geq 3$;
- (2) $r + 1 \leq |\text{Cent}(G)|$;
- (3) $r = 3$ if and only if $|\text{Cent}(G)| = 4$;
- (4) $r = 4$ if and only if $|\text{Cent}(G)| = 5$.

Proposition 6 (Proposition 2.5 of [1]). *Let G be a finite group, and let $X = \{x_1, \dots, x_r\}$ be a set of pairwise noncommuting elements of G having maximal size.*

- (a) If $|\text{Cent}(G)| < r + 4$, then
 - (1) for each element $x \in G$, $C_G(x)$ is abelian if and only if $C_G(x) = C_G(x_i)$ for some $i \in \{1, \dots, r\}$;
 - (2) if $C_G(x_i)$ is a maximal subgroup of G for some $i \in \{1, \dots, r\}$, then $Z(G) = C_G(x_i) \cap C_G(x_j)$ for all $j \in \{1, \dots, r\} \setminus \{i\}$. In particular, if $|G : C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$, then $|\text{Cent}(G)| = 4$, and if $|G : C_G(x_1)| \leq |G : C_G(x_2)| = 3$, then $|\text{Cent}(G)| = 5$.
- (b) If $|\text{Cent}(G)| = r + 2$, then there exists a proper nonabelian centralizer $C_G(x)$ which contains $C_G(x_{i_1})$, $C_G(x_{i_2})$, and $C_G(x_{i_3})$ for three distinct $i_1, i_2, i_3 \in \{1, \dots, r\}$.
- (c) If $|\text{Cent}(G)| = r + 3$, then there exists a proper nonabelian centralizer $C_G(x)$ which contains $C_G(x_{i_1})$ and $C_G(x_{i_2})$ for two distinct $i_1, i_2 \in \{1, \dots, r\}$.

Lemma 7 (Lemma 2.6 of [1]). *Let G be a finite nonabelian group. Then every proper centralizer of G is abelian if and only if $|\text{Cent}(G)| = r + 1$, where r is the maximal size of a set of pairwise noncommuting elements of G .*

Theorem 8 (Theorem 4.2 of [11]). *Suppose that G is covered by n abelian subgroups $A_1, A_2, \dots, A_n$; then*

- (i) if $G = \langle A_1, A_2 \rangle$, then $|G/Z(G)| \leq (n - 1)^2$;
- (ii) if $\langle A_1, A_2 \rangle < G$, then $|G/Z(G)| \leq 2(n - 2)^{\log_2(n-2)}$.

Definition 9 (see [18]). A group G is said to be an n -sum group if it can be written as the sum of n of its proper subgroups and of no smaller number. One then writes $G = \sum_{r=1}^n H_r$, where, for each r , H_r is a proper subgroup, which can be assumed to be maximal where convenient.

Theorem 10 (Theorem 1 of [18]). *If $G = \sum_{r=1}^n H_r$, then $|G| \leq \sum_{r=2}^n |H_r|$, with equality if and only if (a) $H_1 H_r = G$; $r \neq 1$ and (b) $H_r \cap H_s \subset H_1$; $r \neq s$.*

Remark 11. If A and B are subgroups of finite index in a group G , and $|G : A|$ and $|G : B|$ are relatively prime, then $G = AB$.

Proposition 12 (Proposition 2.2 of [19]). *Let p be the smallest prime that is dividing $|G|$. If $|G : Z(G)| = p^3$, then $|\text{Cent}(G)| = p^2 + p + 2$ or $p^2 + 2$.*

Proposition 13 (Proposition 2.9 of [19]). *Let $G/Z(G)$ be nonabelian, n an integer, and p a prime. If $G/Z(G) \cong C_n \rtimes C_p$, then $|\text{Cent}(G)| = n + 2$.*

Now we are ready to state the main result of this paper.

3. The Proof of Theorem 1

In this section, we give the proof of the main theorem.

Suppose, on the contrary, that G is a finite nonabelian group of odd order with $|\text{Cent}(G)| = 10$. Let $\{x_1, \dots, x_r\}$ be a set of pairwise noncommuting elements of finite nonabelian group G having maximal size. Then $X_i = C_G(x_i)$, $1 \leq i \leq r$, is an irredundant r -cover with intersection $Z(G)$. Assume that $|G : X_i| = \alpha_i$, where $\alpha_1 \leq \dots \leq \alpha_r$. Since G is a 10-centralizer group, therefore, by Lemma 5, we have $5 \leq r \leq 9$.

Suppose $r = 5$. By Remark 4, $|G/Z(G)| \leq 16$ and $G = X_1 \cup X_2 \cup \dots \cup X_5$; therefore by Lemma 2, $|G : X_2| \leq 4$. Since $|G|$ is odd, then $|G : X_2| = 3$. On the other hand, 3 is a divisor of $|G/Z(G)|$; therefore, we have $|G/Z(G)| \in \{9, 15\}$. If $|G/Z(G)| = 9$, then $|\text{Cent}(G)| = 5$ by [6, Theorem 6], which is a contradiction. Now if $|G/Z(G)| = 15$, then G is abelian which is not possible. It implies that $r \neq 5$.

Suppose $r = 6$. By Remark 4, $|G/Z(G)| \leq 36$ and $G = X_1 \cup X_2 \cup \dots \cup X_6$; therefore by Lemma 2, $|G : X_2| \leq 5$. If $|G : X_2| = 5$, since 5 is a divisor of $|G/Z(G)|$, then $|G/Z(G)| \in \{15, 25, 35\}$. If $|G/Z(G)| \in \{15, 35\}$, then G is abelian, which is not possible. If $|G/Z(G)| = 25$, then by [6, Theorem 6], $|\text{Cent}(G)| = 7$, a contradiction. Now suppose $|G : X_2| = 3$; since 3 is a divisor of $|G/Z(G)|$, therefore $|G/Z(G)| \in \{9, 15, 21, 27, 33\}$. If $|G/Z(G)| \in \{15, 33\}$, then G is abelian which is not possible. If $|G/Z(G)| = 21$, then $G/Z(G) \cong C_7 \rtimes C_3$, and by Proposition 13, $|\text{Cent}(G)| = 9$. If $|G/Z(G)| = 9$, then $|\text{Cent}(G)| = 5$. If $|G/Z(G)| = 27$, then by Proposition 12, $|\text{Cent}(G)| = 11$ or 14. Hence, $r \neq 6$ and so $|\text{Cent}(G)| < r + 4$.

Now suppose $r = 7$. By Remark 4, $G = X_1 \cup X_2 \cup \dots \cup X_7$, and by Lemma 2, $|G : X_2| \leq 6$. Now if $|G : X_2| = 5$, then $|G : X_1| = 3$ or 5. If $|G : X_1| = 3$, then by Remark 11, $G = X_1 X_2$, and by Proposition 6, $X_1 \cap X_2 = Z(G)$. Hence, $|G/Z(G)| = 15$ which is not possible, since G is nonabelian. If $|G : X_1| = 5$, then $|G/Z(G)| \leq 25$, and since 5 is a divisor of $|G/Z(G)|$, then $|G/Z(G)| \in \{15, 25\}$, which is again a contradiction. Hence, $r \neq 7$.

Now suppose $r = 8$. By Remark 4, $G = X_1 \cup X_2 \cup \dots \cup X_8$, and by Lemma 2, $|G : X_2| \leq 7$. Now if $|G : X_2| = 3$, by Proposition 6, $|\text{Cent}(G)| = 5$, a contradiction. If $|G : X_2| = 5$, then $|G : X_1| = 3$ or 5. If $|G : X_1| = 3$, then by Remark 11, $G = X_1 X_2$, and by Proposition 6, $X_1 \cap X_2 = Z(G)$. Hence, $|G/Z(G)| = 15$ which is not possible, since G is nonabelian. If $|G : X_1| = 5$, then $|G/Z(G)| \leq 25$. Now since 5 is a divisor of $|G/Z(G)|$, $|G/Z(G)| \in \{15, 25\}$, which is again a contradiction. Finally, suppose that $|G : X_2| = 7$. By Lemma 2, $|G : X_2| = |G : X_3| = \dots = |G : X_8| = 7$. Therefore, $|G| = \sum_{i=2}^8 |X_i|$, and by Theorem 10, $G = X_1 X_2$. Again by Proposition 6, $X_1 \cap X_2 = Z(G)$. Since $|G|$ is odd, $|G : X_1| = 3, 5$, or 7. If $|G : X_1| = 3$, then $|G/Z(G)| = 21$. So $G/Z(G) \cong C_7 \rtimes C_3$, and by Proposition 13, $|\text{Cent}(G)| = 9$, which is a contradiction. If $|G : X_1| = 5$, then $|G/Z(G)| = 35$. Therefore, G is abelian, which is not possible. If $|G : X_1| = 7$, then $|G/Z(G)| = 49$. Now by [6, Theorem 6], $|\text{Cent}(G)| = 9$, which is a contradiction. Hence, $r \neq 8$.

Thus, $r = 9$. By Lemma 7, G is a CA-group, and by Remark 4, $X_i \cap X_j = Z(G)$ for all distinct $i, j \in \{1, 2, \dots, 9\}$. Now $G = X_1 \cup X_2 \cup \dots \cup X_9$, and by Lemma 2, $|G : X_2| \leq 8$. If $|G : X_2| = 3$, then $|\text{Cent}(G)| = 5$, a contradiction. If $|G : X_2| = 5$, then $|G : X_1| = 3$ or 5. If $|G : X_1| = 3$, then by Remark 11, $G = X_1 X_2$. Since $X_1 \cap X_2 = Z(G)$, $|G/Z(G)| = 15$, which is a contradiction. If $|G : X_1| = 5$, then $|G/Z(G)| \leq 25$, and since 5 is a divisor of $|G/Z(G)|$, $|G/Z(G)| \in \{15, 25\}$, which is again a contradiction. If $|G : X_2| = 7$, then $|G : X_1| = 3, 5$, or 7. If $|G : X_1| = 3$, then by Remark 11, $G = X_1 X_2$, and since $X_1 \cap X_2 = Z(G)$, therefore $|G/Z(G)| = 21$ and so $G/Z(G) \cong C_7 \rtimes C_3$, and by Proposition 13, $|\text{Cent}(G)| = 9$, which is a contradiction. Similarly, if $|G : X_1| = 5$, then $|G/Z(G)| = 35$, a contradiction. Finally, if $|G : X_1| = 7$, then $G = \langle X_1, X_2 \rangle$. Therefore by Theorem 8, $|G/Z(G)| \leq 64$. Since $|G|$ is odd and G is nonabelian, $|G/Z(G)| \in \{21, 35, 49, 63\}$. If $|G/Z(G)| = 21$, then $G/Z(G) \cong C_7 \rtimes C_3$, and by Proposition 13, $|\text{Cent}(G)| = 9$, which is a contradiction. If $|G/Z(G)| = 35$, then G is abelian which is not possible. If $|G/Z(G)| = 49$, then by [6, Theorem 6], $|\text{Cent}(G)| = 9$, which is a contradiction. Now suppose that $|G/Z(G)| = 63$. If X_1 or X_2 is normal in G , then $G = X_1 X_2$. Since $X_1 \cap X_2 = Z(G)$, $|G/Z(G)| = 49$, and by [6, Theorem 6], $|\text{Cent}(G)| = 9$, a contradiction. It is easy to see that $C_G(g)/Z(G) \leq C_{G/Z(G)}(gZ(G))$ for any $g \in G \setminus Z(G)$. Therefore, $|G : X_i| = 7, 9$, or 21, for $i \in \{1, 2, \dots, 9\}$. Let $xZ(G)$ be an element of order 7 in $G/Z(G)$. Then $C_{G/Z(G)}(xZ(G))$ is the normal Sylow 7-subgroup of $G/Z(G)$. Since $C_G(x)/Z(G) \leq C_{G/Z(G)}(xZ(G))$, $C_G(x)/Z(G) = C_{G/Z(G)}(xZ(G))$. Hence by [19, Lemma 3.1], G has exactly one centralizer of index 9. Suppose $|G : X_3| = 9$. Then $|G : X_4| = |G : X_5| = |G : X_6| = |G : X_7| = |G : X_8| = |G : X_9| = 21$ and $|G| > \sum_{i=2}^9 |X_i|$, which is a contradiction

by Theorem 10. Therefore, $|G : X_3| = 7$. Similarly, we can show that $|G : X_4| = |G : X_5| = |G : X_6| = |G : X_7| = 7$. So we have $|G : X_1| = \dots = |G : X_7| = 7$, $|G : X_8| = 9$, and $|G : X_9| = 21$. By $G = X_1 \cup X_2 \cup \dots \cup X_9$ and the property $X_i \cap X_j = Z(G)$ for all distinct $i, j \in \{1, 2, \dots, 9\}$, it is easy to see that $|G| = |X_1| + \dots + |X_9| - 8|Z(G)|$. So we have $|G/Z(G)| < 63$, a contradiction.

Now the proof of Theorem 1 is complete.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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