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Research Article

Nash Equilibria in Large Games

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This paper adds to the discussion, in a general setting, that given a Nash-Schmeidler (nonanonymous) game it is not always possible to define a Mas-Colell (anonymous) game. In the two games, the players have different strategic behaviours and the formulations of the two problems are different. Also, we offer a novel explanation for the lack of a Nash equilibrium in an infinite game. We consider this game as the limit of a sequence of approximate, finite games for which an equilibrium exists. However, the limiting pure strategy function is not measurable.

1. Introduction

This paper is in the context of "large games." First, we discuss the relationship of a particular Nash-Schmeidler, nonanonymous, formulation to a Mas-Colell anonymous game. We explain that, starting with the former, it is not always possible to define a Mas-Colell game. This is due to the different strategic behaviours of the players in the two formulations. The approach of Mas-Colell [1], although a general one, does not generalize that of Schmeidler [2]. The formulation of Schmeidler remains the more fundamental game theoretic approach.

The description "anonymous" refers to the fact that a player in Mas-Colell has no individual identity. What matters is the overall density of the choices, on which a set of possible utility functions depends. "Nonanonymous" refers to the fact that in Schmeidler [2] a player is specified in terms of an individual utility function which depends on the specific actions of the opponents.

The ideas above have an affinity to the issues discussed in the area of general equilibrium in markets with a continuum of traders. In the Schmeidler formulation, a game is defined as a measurable function from an atomless set of players to the set of players' characteristics. In the Mas-Colell alternative formulation, a game is defined anonymously, as a distribution on the players' characteristics.

The relationship and the link between the two different formulations have been studied in the literature, under various assumptions, a number of times. Rath [3] uses the idea of the representation of a game through a measurable function which induces a given distribution. He then shows that for a finite number of actions the two formulations are, in terms of equilibria, essentially the same. Ali Khan and Sun [4], in their synthetic treatment, untangle in detail the relationship between the two formulations and show that only in the case of finite action spaces the two formulations are essentially the same.

The two formulations have different implications when the set of actions is infinite. Rath et al. [5] discuss the issue in terms of symmetric equilibria; that is, players with the same characteristics choose the same action. They show that, for uncountably many actions, the two formulations are not equivalent. There is an anonymous game with no symmetric equilibrium distribution. Khan et al. [6] discuss also equilibria in nonatomic games and explain the implications of assuming that the set of actions has the cardinality of infinity. In this case, the two formulations are in general not equivalent.

This paper adds to the discussion in a different direction. It points out that a nonanonymous game cannot always be formulated in an anonymous, distributional form.

Second, we discuss a game for which it is known that there exists no equilibrium. It is due to Schmeidler, and it is also in the comprehensive survey by Ali Khan and Sun [7]. We consider this game as the limit of a sequence of finite approximate games for which an equilibrium exists. We explain that there is no equilibrium in the limit because the resulting pure strategy function is not measurable.

2. Nash and Mas-Colell Equilibria

In games with a continuum of players, we look at the connection between Nash (NE) and Mas-Colell's Cournot-Nash Equilibria (CNE), given a particular definition of the former which makes also a connection with the agents. We then point out that this definition is more restricted than the usual one and this implies that one cannot always relate the two types of games.

It is assumed that A is a nonempty, complete, and separable metric space of actions and $\mathscr{M}(A)$ the set of Borel probability measures on A. The set of possible utility functions $u : A\times\mathscr{M}(A) \to \mathbb{R}$ is given by $\mathscr{U}$, which is assumed to be a metric space, endowed with its Borel σ-algebra. Let T be the space of players and λ an atomless probability measure on Σ, the σ-algebra of Lebesgue sets. A game is given by a measurable function $U : T \to \mathscr{U}$.

A Borel measure τ on $\mathscr{U} \times A$ is a CNE if

(i) $\tau_{\mathscr{U}} = \lambda \circ U^{-1}$;

(ii) $\tau(\{(u, a) \in \mathscr{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1$.

A pure strategy is measurable function $f : T \to A$. A pure strategy will be a NE if

$$\lambda\left(\left\{t \in T : U(t)\left(f(t), \lambda \circ f^{-1}\right) \geq U(t)\left(A, \lambda \circ f^{-1}\right)\right\}\right) = 1. \tag{1}$$

Mas-Colell looks at equilibria in pure strategies.

We now state the following useful result. The proof is straightforward and hence omitted.

Theorem 1. *If f is a NE, then $\tau = \lambda \circ (U, f)^{-1}$ is a CNE.*

However, in general, the two types of large games are clearly different and not compatible. In Mas-Colell's anonymous games, only the average distribution of opponents' actions matters. In Schmeidler's nonanonymous games, the strategic behaviour of players can depend on the specific choice of each of the opponents. As we see below, $\lambda \circ f^{-1}$ is not always necessarily present in order to define a Nash-Schmeidler game and to obtain a NE.

We now look at two Schmeidler type examples, one without a NE and one with a NE, which cannot be cast in the Mas-Colell form.

Example 2. Consider $T = [0, 1]$, $A = \{-1, 1\}$. The utility of player $t \in T$ is

$$u_t(\alpha, f) = \left|\alpha - \int_0^t f(x)\, d\lambda(x)\right|, \tag{2}$$

where $\alpha \in A$ is a choice for player t, $f : T \to A$ is a measurable assignment of choices for all players, and λ is Lebesgue measure on T. The above example is in Remark 3 in Schmeidler [2]. It is also on page 1785 in Ali Khan and Sun [7] (Example 4), where any $f \in L_1(\lambda, \{-1, 1\})$ partitions T into two sets:

$$T_- = \{t \in T : f(t) = -1\}, \qquad T_+ = \{t \in T : f(t) = 1\}. \tag{3}$$

Then, the integral of f over any measurable set S of T is

$$\int_S f(x)\, d\lambda = \lambda(S \cap T_+) - \lambda(S \cap T_-). \tag{4}$$

Further, f induces a measure on $A = \{-1, 1\}$, that is, an element $(1 - q, q)$ of $\mathscr{M}(A)$, given by $q = \lambda(T_+)$. Mas-Colell considers utility functions defined on $A \times \mathscr{M}(A)$—hence having the form $u(a, q)$.

However, Example 2 would not allow a consistent definition of a Mas-Colell utility function. Consider the two following functions:

$$\begin{gathered} f : T \longrightarrow A \text{ such that } T_- = \left[0, \frac{1}{2}\right], \qquad T_+ = \left(\frac{1}{2}, 1\right], \\ g : T \longrightarrow A \text{ such that } T_- = \left[0, \frac{1}{4}\right] \cup \left(\frac{1}{2}, \frac{3}{4}\right], \\ T_+ = \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{3}{4}, 1\right]. \end{gathered} \tag{5}$$

Although f, g induce the same element of $\mathscr{M}(A)$, namely, $(1/2, 1/2)$,

$$\int_0^{1/2} f(x)\, d\lambda = \lambda(\emptyset) - \lambda\left(\left[0, \frac{1}{2}\right]\right) = -\frac{1}{2}, \tag{6}$$

whereas

$$\int_0^{1/2} g(x)\, d\lambda = \lambda\left(\left(\frac{1}{4}, \frac{1}{2}\right]\right] - \lambda\left(\left[0, \frac{1}{4}\right]\right) = 0. \tag{7}$$

It follows from (6) and (7) that $u_{1/2}(a, f)$ and $u_{1/2}(a, g)$ cannot be represented consistently by the same Mas-Colell utility $u(a, 1/2)$.

The following question arises: which part of the mathematical discussion above breaks down? So we cannot define a Mas-Colell game. The example above shows that, given a measurable function $f : T \to A$, $\lambda \circ f^{-1}$ does not have to be present in order to define a Nash-Schmeidler game. In the context of Theorem 1 of Ali Khan and Sun [7], u_t is allowed to depend only on the actions of the agents before t. On the other hand, $\lambda \circ f^{-1}$ requires dependence on the distribution of the actions of all the agents. Here, $\mathscr{M}(A)$ is just $q \in [0, 1]$.

It is not always possible to go from a Nash-Schmeidler game to a Mas-Colell one, as their strategic structures are different. A utility function appropriate for Nash-Schmeidler might not be defined on $\mathscr{M}(A)$.

Example 3. We also point out that we can define the utility function such that

$$u_t = \left|\int_0^t f(x)\, d\lambda\right|. \tag{8}$$

Now, any admissible function f is a NE, but we could not define a Mas-Colell problem.

3. Existence of NE

We now consider the question of existence of a NE in Example 2. There is no measurable function f which could serve as a NE. The lack of existence of a NE in Example 2 is discussed by Schmeidler [2] on page 299.

Then, in an attempt to give a further explanation, we consider a sequence of approximate games. We show that a NE exists for the sequence and explain why it fails to do so for the limiting game.

3.1. Lack of Existence of NE in Example 2. We quote Schmeidler's result without proof.

Lemma 4. *There is no measurable function which can serve as a NE in Example 2.*

The proof there is rather dense and relies on a result concerning the nonexistence of a certain Lebesgue-measurable set. An alternative, more direct, and simpler proof was produced, but it became unnecessarily long.

An intuitive interpretation of the result is as follows. Let $\int_0^t f(x)d\lambda > 0$, while player t chooses $\alpha = 1$. We go down to the next one who plays 1 and the integral remains positive. Each such agent can change his/her action and become better off. Analogously, if $\int_0^t f(x)d\lambda < 0$, a number of players benefit by changing their actions. Hence, for equilibrium, we require $\int_0^t f(x)d\lambda = 0$ for almost all agents, which contradicts $A = \{-1, 1\}$.

3.2. Finite Approximations of Example 2. We now consider Example 2 as the limiting game of a sequence of finite games each of which has a NE.

Lemma 5. *There is a NE in finite approximations of Example 2.*

Proof of Lemma 5. In the first type of approximation, the interval is divided to equal segments and in the second one to arbitrary segments.

(1) We partition $[0, 1]$ into equal segments of length z with end points $[t_0 = 0, t_1, \ldots, t_n = 1]$. We place an agent on each end point. The expression for u_t is replaced by

$$u_k = \left|\alpha_k - \sum_{r=1}^{k} \alpha_r z\right|. \tag{9}$$

We prove by induction that $a_k = 1$ for odd and $\alpha_k = -1$ for even t are a NE. We do two steps. First, we note that we can go from u_1 to u_2 to u_3 alternating the signs as prescribed, choosing the first and third one arbitrarily equal to 1. This is seen as follows. From the expression above, we obtain $u_1 = |(1 - z)\alpha_1|$, $u_2 = |(1 - z)\alpha_2 - z|$, and $u_3 = |(1 - z)\alpha_3|$, which justifies the signs. Now, suppose that $k-1$ is odd. Then, $u_k = |(1 - z)\alpha_k - z|$, as all previous pairs of 1, –1 up to $k - 2$ cancel, and this is maximized for $\alpha_k = 1$. In the next step, we get $u_{k+1} = |(1-z)a_{k+1}|$ which can be maximized for $\alpha_{k+1} = 1$. In conclusion, for every odd number, we choose 1 arbitrarily, and for the even end points we put –1.

Dividing $[0, 1]$ further, we can always get a NE with alternating values of 1 and –1. In the limit though there is no measurable function f which is a NE as we showed in the discussion of the continuous Example 2.

(2) We now consider a generalization of the above. Let there be n players. Partition the interval $[0, 1]$ by the set of points $\{t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1\}$. We will consider the λ-measure of the kth player to be $t_k - t_{k-1}$. So we will replace the continuous utility

$$u_t = \left|\alpha_t - \int_0^t f(x)\, d\lambda(x)\right| \tag{10}$$

by the finite approximation

$$u_k = \left|\alpha_k - \sum_{r=1}^{k} \alpha_r (t_r - t_{r-1})\right|. \tag{11}$$

Consider the first few cases to see the pattern. Assume that α is NE.

Consider that $u_1 = |(1-t_1)\alpha_1|$ which maximizes for either choice $\alpha_1 = \pm 1$. Suppose $\alpha_1 = 1$ (the alternative case can be obtained from what follows by changing signs of all α's).

We have

$$u_2 = |\alpha_2 - t_1 - \alpha_2 (t_2 - t_1)| = |(1 + t_1 - t_2)\alpha_2 - t_1|. \tag{12}$$

The expression in (12) is maximized by $\alpha_2 = -1$. Consider

$$\begin{aligned} u_3 &= |\alpha_3 - t_1 + (t_2 - t_1) - \alpha_3 (t_3 - t_2)| \\ &= |(1 + t_2 - t_3)\alpha_3 + t_2 - 2t_1|. \end{aligned} \tag{13}$$

Then,

$$\alpha_3 = \begin{cases} +1 & \text{if } t_2 - 2t_1 > 0, \\ -1 & \text{if } t_2 - 2t_1 < 0, \\ \text{arb} & \text{if } t_2 - 2t_1 = 0. \end{cases} \tag{14}$$

We will ignore the last possibility as being nongeneric; we are approximating the integral in Schmeidler's utility by arbitrary simple functions and intending to take the limit, so generic cases are sufficient to give us all the information we need. Using these values in the next step, we have

$$u_4 = \begin{cases} |(1 - t_4 + t_3)\alpha_4 + 2(t_2 - t_1) - t_3| & \text{if } t_2 > 2t_1, \\ |(1 - t_4 + t_3)\alpha_4 + t_3 - 2t_1| & \text{if } t_2 < 2t_1. \end{cases} \tag{15}$$

Let us note from these examples that at any stage, say the kth, the expression we are looking at to determine α_k takes the form

$$u_k = |(1 - t_k + t_{k-1})\alpha_k + L_{k-1}|, \tag{16}$$

where

$$\begin{aligned} L_{k-1} = &-t_1 + (t_2 - t_1) - \alpha_3 (t_3 - t_2) \\ &- \alpha_4 (t_4 - t_3) - \cdots - \alpha_{k-1} (t_{k-1} - t_{k-2}). \end{aligned} \tag{17}$$

Note. If we had decided to omit the kth term in the approximating simple function, we would have $u_k = |\alpha_k + L_{k-1}|$, but the following argument would be unaltered.

To maximize u_k, α_k will be chosen with the same sign as L_{k-1}. The issue we face is what happens if the sign of L_{k-1} does not change for a number of steps. Suppose therefore (w.l.o.g.) that u_k has been maximized by $\alpha_k = +1$, which means that this is the sign of L_{k-1} also, but that subsequent values of L_i, up to say L_j, remain of the same sign—the same then being true for corresponding α's. Then, we have

$$L_j = L_{k-1} - \sum_{i=k}^{j} \alpha_i \left(t_i - t_{i-1}\right) = L_{k-1} - \left(t_j - t_{k-1}\right). \tag{18}$$

The sign of α_{j+1} will now change to negative if

$$t_j > L_{k-1} + t_{k-1}, \tag{19}$$

that is, once t_j is sufficiently large. So if the intervals of the partition are small, a number of them must be aggregated together to make a sufficiently large step. But this will always occur unless we come to the extreme right hand end of $[0, 1]$ and merely amount to aggregating the given intervals together into a suitable size.

For any arbitrary partition of $[0, 1]$, we have a finite game for which a NE exists. In the limit as n tends to infinity, we will have an infinite number of sign changes for the α's, so the limiting function is not measurable. □

Hence, by considering finite approximations (recent contributions by Khan and coworkers on approximations and on exact equilibria in games with hyperfinite spaces and with a biosocial topology are discussed in the works of JME and JET. See, for example, [8]) of the game in Example 2, we have provided an explanation for the lack of existence of a NE.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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