Research Article

Hypersurface Family with a Common Isoasymptotic Curve

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We handle the problem of finding a hypersurface family from a given asymptotic curve in $\mathbb{R}^4$. Using the Frenet frame of the given asymptotic curve, we express the hypersurface as a linear combination of this frame and analyze the necessary and sufficient conditions for that curve to be asymptotic. We illustrate this method by presenting some examples.

1. Introduction

Asymptotic curves are encountered in differential geometry frequently. A surface curve is called asymptotic if its tangent vectors always point in an asymptotic direction, that is, the direction in which the normal curvature is zero. In an asymptotic direction, the surface is not bending away from its tangent plane.

Asymptotic curves on a surface can be seen in many differential geometry books [1–5]. Rastogi [2] obtained the differential equation of hyperasymptotic curves by a new method and showed some properties of these curves. Aminov [6] established more general expressions for the curvature of asymptotic curves of submanifolds in the Riemannian space. Romero-Fuster et al. [3] studied asymptotic curves on generally immersed surfaces in $\mathbb{R}^5$. Both the general and rational developable surface pencils through an arbitrary parametric curve as its common asymptotic curve were analyzed by Liu and Wang [7]. Bayram et al. [8] tackled the problem of finding a surface pencil from a given asymptotic curve.

However, while differential geometry of a parametric surface in $\mathbb{R}^3$ can be found in textbooks such as those by Struik [5], Willmore [9], Stoker [4], and do Carmo [10], differential geometry of a parametric surface in $\mathbb{R}^n$ can be found in textbooks such as the contemporary literature on geometric modeling [11, 12]. Also, there is little literature on differential geometry of parametric surface family in $\mathbb{R}^3$ [8, 13–16] but not in $\mathbb{R}^4$. Besides, there is an ascending interest in fourth dimension [13, 14, 17].

Furthermore, various visualization techniques about objects in Euclidean n -space ($n \geq 4$) are presented [18–20]. The fundamental step to visualize a 4D object is projecting first into the 3-space and then into the plane. In many real world applications, the problem of visualizing three-dimensional data, commonly referred to as scalar fields, arises. The graph of a function $f(x, y, z) : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$, where U is open, is a special type of parametric hypersurface with the parametrization $(x, y, z, f(x, y, z))$ in 4-space. There exists a method for rendering such a 3-surface based on known methods for visualizing functions of two variables [21].

In this paper, we consider the four-dimensional analogue problem of constructing a parametric representation of a surface family from a given asymptotic as in Bayram et al. [8], who derived the necessary and sufficient conditions on the marching-scale functions for which the curve C is an asymptotic curve on a given surface. We express the hypersurface pencil parametrically with the help of the Frenet frame $\{T, N, B_1, B_2\}$ of the given curve. We find the necessary and sufficient constraints on the marching-scale functions, namely, coefficients of Frenet vectors, so that both the asymptotic and parametric requirements are met.

2. Preliminaries

Let us first introduce some notations and definitions. Bold letters such as $\mathbf{a}$, $\mathbf{R}$ will be used for vectors and vector functions. We assume that they are smooth enough so that

all the (partial) derivatives given in the paper are meaningful. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^4$ be an arc-length curve. If $\{T, N, B_1, B_2\}$ is the moving Frenet frame along α , then the Frenet formulas are given by

$$\begin{aligned} T' &= \kappa_1 N, \\ N' &= -\kappa_1 T + \kappa_2 B_1, \\ B_1' &= -\kappa_2 N + \kappa_3 B_2, \\ B_2' &= -\kappa_3 B_1, \end{aligned} \quad (1)$$

where T, N, B_1 , and B_2 denote the tangent, principal normal, first binormal, and second binormal vector fields, respectively, and κ_i ($i = 1, 2, 3$) denote the i th curvature functions of the curve α [20].

From elementary differential geometry we have

$$\begin{aligned} \alpha'(s) &= T(s), \\ \alpha''(s) &= \kappa_1(s)N(s), \quad \kappa_1(s) = \|\alpha''(s)\|. \end{aligned} \quad (2)$$

Using Frenet formulas one can obtain the following:

$$\begin{aligned} \alpha'''(s) &= -\kappa_1^2 T(s) + \kappa_1' N(s) + \kappa_1 \kappa_2 B_1(s), \\ \alpha^{(iv)}(s) &= -3\kappa_1 \kappa_1' T(s) + (-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) N(s) \\ &\quad + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') B_1(s) + \kappa_1 \kappa_2 \kappa_3 B_2(s). \end{aligned} \quad (3)$$

The unit vectors B_2 and B_1 are given by

$$\begin{aligned} B_2(s) &= \frac{\alpha'(s) \otimes \alpha''(s) \otimes \alpha'''(s)}{\|\alpha'(s) \otimes \alpha''(s) \otimes \alpha'''(s)\|}, \\ B_1(s) &= B_2(s) \otimes T(s) \otimes N(s), \end{aligned} \quad (4)$$

where $\otimes$ is the vector product of vectors in $\mathbb{R}^4$.

Since the vectors T, N, B_1 , and B_2 are orthonormal, the second curvature κ_2 and the third curvature κ_3 can be obtained from (3) as

$$\begin{aligned} \kappa_2(s) &= \frac{B_1(s) \cdot \alpha'''(s)}{\kappa_1(s)}, \\ \kappa_3(s) &= \frac{B_2(s) \cdot \alpha^{(iv)}(s)}{\kappa_1(s) \kappa_2(s)}, \end{aligned} \quad (5)$$

where “ $\cdot$ ” denotes the standard inner product.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for four-dimensional Euclidean space $\mathbb{R}^4$. The vector product of the vectors $u = \sum_{i=1}^4 u_i e_i$, $v = \sum_{i=1}^4 v_i e_i$, and $w = \sum_{i=1}^4 w_i e_i$ is defined by

$$u \otimes v \otimes w = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} \quad (6)$$

(see [22, 23]).

If u, v , and w are linearly independent, then $u \otimes v \otimes w$ is orthogonal to each of these vectors.

3. Hypersurface Family with a Common Isoasymptotic

A curve $r(s)$ on a hypersurface $P = P(s, t, q) \subset \mathbb{R}^4$ is called an isoparametric curve if it is a parameter curve; that is, there exists a pair of parameters t_0 and q_0 such that $r(s) = P(s, t_0, q_0)$. Given a parametric curve $r(s)$, it is called an *isoasymptotic* of a hypersurface P if it is both an asymptotic and an isoparametric curve on P .

Let $C : r = r(s)$, $L_1 \leq s \leq L_2$, be a C^3 curve, where s is the arc-length. To have a well-defined principal normal, assume that $r''(s) \neq 0$, $L_1 \leq s \leq L_2$.

Let $T(s), N(s), B_1(s)$, and $B_2(s)$ be the tangent, principal normal, first binormal, and second binormal, respectively; and let $\kappa_1(s), \kappa_2(s)$, and $\kappa_3(s)$ be the first, the second, and the third curvature, respectively. Since $\{T(s), N(s), B_1(s), B_2(s)\}$ is an orthogonal coordinate frame on $r(s)$, the parametric hypersurface $P(s, t, q) : [L_1, L_2] \times [T_1, T_2] \times [Q_1, Q_2] \rightarrow \mathbb{R}^4$ passing through $r(s)$ can be defined as follows:

$$\begin{aligned} P(s, t, q) &= r(s) + (u(s, t, q), v(s, t, q), w(s, t, q), x(s, t, q)) \\ &\quad \cdot \begin{pmatrix} T(s) \\ N(s) \\ B_1(s) \\ B_2(s) \end{pmatrix}, \end{aligned} \quad (7)$$

$$L_1 \leq s \leq L_2, \quad T_1 \leq t \leq T_2, \quad Q_1 \leq q \leq Q_2,$$

where $u(s, t, q), v(s, t, q), w(s, t, q)$, and $x(s, t, q)$ are all C^1 functions. These functions are called the *marching-scale functions*.

We try to find out the necessary and sufficient conditions for which a hypersurface $P = P(s, t, q)$ has the curve C as an isoasymptotic.

First, to satisfy the isoparametricity condition there should exist $t_0 \in [T_1, T_2]$ and $q_0 \in [Q_1, Q_2]$ such that $P(s, t_0, q_0) = r(s)$, $L_1 \leq s \leq L_2$; that is,

$$\begin{aligned} u(s, t_0, q_0) &= v(s, t_0, q_0) = w(s, t_0, q_0) = x(s, t_0, q_0) \equiv 0, \\ t_0 &\in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2. \end{aligned} \quad (8)$$

Secondly, the curve C is an asymptotic curve on the hypersurface $P(s, t, q)$ if and only if the normal curvature $\kappa_n = S(T) \cdot T = 0$ along the curve, where S is the shape operator and T is the tangent vector to the curve. The normal $\hat{n}(s, t_0, q_0)$ of the hypersurface can be obtained by calculating the vector product of the partial derivatives and using the Frenet formula as follows:

$$\begin{aligned} \frac{\partial P(s, t, q)}{\partial s} &= \left(1 + \frac{\partial u(s, t, q)}{\partial s} - v(s, t, q) \kappa_1(s) \right) T(s) \\ &\quad + \left(u(s, t, q) \kappa_1(s) + \frac{\partial v(s, t, q)}{\partial s} \right. \\ &\quad \left. - w(s, t, q) \kappa_2(s) \right) N(s) \end{aligned}$$

$$\begin{aligned}
& + \left(\mathbf{v}(s, t, q) \kappa_2(s) + \frac{\partial \mathbf{w}(s, t, q)}{\partial s} \right. \\
& \quad \left. - \mathbf{x}(s, t, q) \kappa_3(s) \right) \mathbf{B}_1(s) \\
& + \left(\mathbf{w}(s, t, q) \kappa_3(s) + \frac{\partial \mathbf{x}(s, t, q)}{\partial s} \right) \mathbf{B}_2(s), \\
\frac{\partial \mathbf{P}(s, t, q)}{\partial t} &= \frac{\partial \mathbf{u}(s, t, q)}{\partial t} \mathbf{T}(s) + \frac{\partial \mathbf{v}(s, t, q)}{\partial t} \mathbf{N}(s) \\
& + \frac{\partial \mathbf{w}(s, t, q)}{\partial t} \mathbf{B}_1(s) + \frac{\partial \mathbf{x}(s, t, q)}{\partial t} \mathbf{B}_2(s), \\
\frac{\partial \mathbf{P}(s, t, q)}{\partial q} &= \frac{\partial \mathbf{u}(s, t, q)}{\partial q} \mathbf{T}(s) + \frac{\partial \mathbf{v}(s, t, q)}{\partial q} \mathbf{N}(s) \\
& + \frac{\partial \mathbf{w}(s, t, q)}{\partial q} \mathbf{B}_1(s) + \frac{\partial \mathbf{x}(s, t, q)}{\partial q} \mathbf{B}_2(s).
\end{aligned} \tag{9}$$

Remark 1. Because

$$\begin{aligned}
\mathbf{u}(s, t_0, q_0) &= \mathbf{v}(s, t_0, q_0) = \mathbf{w}(s, t_0, q_0) = \mathbf{x}(s, t_0, q_0) \equiv 0, \\
t_0 &\in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2,
\end{aligned} \tag{10}$$

along the curve C , by the definition of partial differentiation, we have

$$\begin{aligned}
\frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} &= \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} = \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} \\
&= \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \equiv 0,
\end{aligned} \tag{11}$$

$$t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2.$$

Using (8) we have

$$\begin{aligned}
\hat{\mathbf{n}}(s, t_0, q_0) &= \frac{\partial \mathbf{P}(s, t_0, q_0)}{\partial s} \otimes \frac{\partial \mathbf{P}(s, t_0, q_0)}{\partial t} \\
&\otimes \frac{\partial \mathbf{P}(s, t_0, q_0)}{\partial q} \\
&= \phi_1(s, t_0, q_0) \mathbf{T}(s) - \phi_2(s, t_0, q_0) \mathbf{N}(s) \\
&\quad + \phi_3(s, t_0, q_0) \mathbf{B}_1(s) - \phi_4(s, t_0, q_0) \mathbf{B}_2(s),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
&\phi_1(s, t_0, q_0) \\
&= \begin{vmatrix} \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{vmatrix} = 0,
\end{aligned}$$

$$\begin{aligned}
&\phi_2(s, t_0, q_0) \\
&= \begin{vmatrix} 1 + \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{vmatrix} \\
&= \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} - \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t},
\end{aligned}$$

$$\begin{aligned}
&\phi_3(s, t_0, q_0) \\
&= \begin{vmatrix} 1 + \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{vmatrix} \\
&= \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} - \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t},
\end{aligned}$$

$$\begin{aligned}
&\phi_4(s, t_0, q_0) \\
&= \begin{vmatrix} 1 + \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} \end{vmatrix} \\
&= \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} - \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t}.
\end{aligned} \tag{13}$$

So

$$\kappa_n = S(T) \cdot T = 0 \iff \hat{\mathbf{n}} \cdot \mathbf{N} = 0 \iff$$

$$\phi_2(s, t_0, q_0) \equiv 0,$$

$$\begin{aligned} \phi_3^2(s, t_0, q_0) + \phi_4^2(s, t_0, q_0) &\neq 0, \\ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2. \end{aligned} \quad (14)$$

Thus, any hypersurface defined by (7) has the curve C as an isoasymptotic if and only if

$$\begin{aligned} \mathbf{u}(s, t_0, q_0) &= \mathbf{v}(s, t_0, q_0) = \mathbf{w}(s, t_0, q_0) = \mathbf{x}(s, t_0, q_0) \equiv 0, \\ \phi_2(s, t_0, q_0) &\equiv 0, \\ \phi_3^2(s, t_0, q_0) + \phi_4^2(s, t_0, q_0) &\neq 0, \\ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2 \end{aligned} \quad (15)$$

is satisfied. We call the set of hypersurfaces defined by (7) and satisfying (15) an *isoasymptotic hypersurface family*.

4. Examples

Example 1. Let $\mathbf{r}(s) = ((1/2)\cos(s), (1/2)\sin(s), (1/2)s, (\sqrt{2}/2)s)$, $0 \leq s \leq 2\pi$, be a curve parametrized by arc-length. For this curve,

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{r}'(s) = \left(-\frac{1}{2}\sin(s), \frac{1}{2}\cos(s), \frac{1}{2}, \frac{\sqrt{2}}{2} \right), \\ \mathbf{N}(s) &= (-\cos(s), -\sin(s), 0, 0), \\ \mathbf{B}_2(s) &= \frac{\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)}{\|\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)\|} = \left(0, 0, \frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3} \right), \\ \mathbf{B}_1(s) &= \mathbf{B}_2 \otimes \mathbf{T} \otimes \mathbf{N} \\ &= \left(-\frac{\sqrt{3}}{2}\sin(s), \frac{\sqrt{3}}{2}\cos(s), -\frac{\sqrt{3}}{6}, -\frac{\sqrt{6}}{6} \right). \end{aligned} \quad (16)$$

Let us choose the marching-scale functions as

$$\begin{aligned} \mathbf{u}(s, t, q) &= (t - t_0)(q - q_0), \\ \mathbf{v}(s, t, q) &= t - t_0, \\ \mathbf{w}(s, t, q) &\equiv 0, \\ \mathbf{x}(s, t, q) &= q - q_0, \\ t_0 \in [0, 1], \quad q_0 \in [0, 1], \quad 0 \leq s \leq 2\pi. \end{aligned} \quad (17)$$

So, we have the hypersurface

$$\begin{aligned} \mathbf{P}(s, t, q) &= \mathbf{r}(s) + \mathbf{u}(s, t, q)\mathbf{T}(s) + \mathbf{v}(s, t, q)\mathbf{N}(s) \\ &\quad + \mathbf{w}(s, t, q)\mathbf{B}_1(s) + \mathbf{x}(s, t, q)\mathbf{B}_2(s) \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1}{2}\cos(s) - \frac{1}{2}(t - t_0)(q - q_0)\sin(s) - (t - t_0)\cos(s), \right. \\ &\quad \frac{1}{2}\sin(s) + \frac{1}{2}(t - t_0)(q - q_0)\cos(s) - (t - t_0)\sin(s), \\ &\quad \frac{1}{2}s + \frac{1}{2}(t - t_0)(q - q_0) + \frac{\sqrt{6}}{3}(q - q_0), \\ &\quad \left. \frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}(t - t_0)(q - q_0) - \frac{\sqrt{3}}{3}(q - q_0) \right), \end{aligned} \quad (18)$$

where $0 \leq s \leq 2\pi$, $0 \leq t \leq 1$, $0 \leq q \leq 1$, $t_0 \in [0, 1]$, and $q_0 \in [0, 1]$, which is a member of the isoasymptotic hypersurface family, since it satisfies (15).

By changing the parameters t_0 and q_0 we can adjust the position of the curve $\mathbf{r}(s)$ on the hypersurface. Let us choose $t_0 = 1/2$ and $q_0 = 0$. Now the curve $\mathbf{r}(s)$ is again isoasymptotic on the hypersurface $\mathbf{P}(s, t, q)$ and the equation of the hypersurface is

$$\begin{aligned} \mathbf{P}(s, t, q) &= \left(\cos s - t \cos s + \frac{1}{4}q \sin s - \frac{1}{2}qt \sin s, \right. \\ &\quad \sin s - \frac{1}{4}q \cos s - t \sin s + \frac{1}{2}qt \cos s, \\ &\quad \frac{1}{2}s - \frac{1}{4}q + \frac{1}{3}\sqrt{6}q + \frac{1}{2}qt, \\ &\quad \left. \frac{1}{2}\sqrt{2}s - \frac{1}{3}\sqrt{3}q - \frac{1}{4}\sqrt{2}q + \frac{1}{2}\sqrt{2}qt \right). \end{aligned} \quad (19)$$

The projection of a hypersurface into 3-space generally yields a three-dimensional volume. If we fix each of the three parameters, one at a time, we obtain three distinct families of 2-spaces in 4-space. The projections of these 2-surfaces into 3-space are surfaces in 3-space. Thus, they can be displayed by 3D rendering methods.

So, if we (parallel) project the hypersurface $\mathbf{P}(s, t, q)$ into the $\mathbf{w} = 0$ subspace and fix $q = 0$, we obtain the surface

$$\mathbf{P}_w(s, t, 0) = \left(\cos s - t \cos s, \sin s - t \sin s, \frac{1}{2}s \right), \quad (20)$$

where $0 \leq s \leq 2\pi$, $0 \leq t \leq 1$ in 3-space illustrated in Figure 1.

Example 2. Given the curve $\mathbf{r}(s) = ((1/2)\sin(s), (1/2)\cos(s), 0, (\sqrt{3}/2)s)$, $0 \leq s \leq 3$, it is easy to show that

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{r}'(s) = \left(\frac{1}{2}\cos(s), -\frac{1}{2}\sin(s), 0, \frac{\sqrt{3}}{2} \right), \\ \mathbf{N}(s) &= (-\sin(s), -\cos(s), 0, 0), \\ \mathbf{B}_2(s) &= \frac{\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)}{\|\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)\|} = (0, 0, -1, 0), \\ \mathbf{B}_1(s) &= \mathbf{B}_2 \otimes \mathbf{T} \otimes \mathbf{N} \\ &= \left(\frac{\sqrt{3}}{2}\cos(s), -\frac{\sqrt{3}}{2}\sin(s), 0, -\frac{1}{2} \right). \end{aligned} \quad (21)$$

FIGURE 1: Projection of a member of the hypersurface family and its isoasymptotic.

FIGURE 2: Projection of a member of the hypersurface family and its isoasymptotic.

Let us choose the marching-scale functions as

$$\begin{aligned} \mathbf{u}(s, t, q) &= (t - t_0), \\ \mathbf{v}(s, t, q) &= (s + t + 1)(q - q_0), \\ \mathbf{w}(s, t, q) &\equiv 0, \\ \mathbf{x}(s, t, q) &= (s + 1)(t - t_0). \end{aligned} \quad (22)$$

From (15), the hypersurface

$$\begin{aligned} \mathbf{P}(s, t, q) &= \mathbf{r}(s) + \mathbf{u}(s, t, q)\mathbf{T}(s) + \mathbf{v}(s, t, q)\mathbf{N}(s) \\ &\quad + \mathbf{w}(s, t, q)\mathbf{B}_1(s) + \mathbf{x}(s, t, q)\mathbf{B}_2(s) \\ &= \left(\frac{1}{2} \sin(s) - (s + t + 1)(q - q_0) \sin(s) \right. \\ &\quad \left. + \frac{1}{2}(t - t_0) \cos(s), \right. \\ &\quad \frac{1}{2} \cos(s) - (s + t + 1)(q - q_0) \cos(s) \\ &\quad \left. - \frac{1}{2}(t - t_0) \sin(s), \right. \\ &\quad \left. - (s + 1)(t - t_0), \frac{\sqrt{3}}{2}s + \frac{\sqrt{3}}{2}(t - t_0) \right), \end{aligned} \quad (23)$$

where $0 \leq s \leq 3$, $0 \leq t \leq 1$, and $0 \leq q \leq 1$, is a member of the hypersurface family having the curve $\mathbf{r}(s)$ as an isoasymptotic.

Setting $t_0 = 1/2$ and $q_0 = 0$ yields the hypersurface

$$\begin{aligned} \mathbf{P}(s, t, q) &= \left(\frac{1}{2} \sin(s) - (s + t + 1)q \sin(s) + \frac{1}{2}\left(t - \frac{1}{2}\right) \cos(s), \right. \\ &\quad \frac{1}{2} \cos(s) - (s + t + 1)q \cos(s) - \frac{1}{2}\left(t - \frac{1}{2}\right) \sin(s), \\ &\quad \left. - (s + 1)\left(t - \frac{1}{2}\right), \frac{\sqrt{3}}{2}s + \frac{\sqrt{3}}{2}\left(t - \frac{1}{2}\right) \right). \end{aligned} \quad (24)$$

By (parallel) projecting the hypersurface $\mathbf{P}(s, t, q)$ into the subspace $\mathbf{w} = \mathbf{0}$ and fixing $q = 0$, we get the surface

$$\begin{aligned} \mathbf{P}_{\mathbf{w}}(s, t, 0) &= \left(\frac{1}{2} \sin(s) + \frac{1}{2}\left(t - \frac{1}{2}\right) \cos(s), \right. \\ &\quad \frac{1}{2} \cos(s) - \frac{1}{2}\left(t - \frac{1}{2}\right) \sin(s), \\ &\quad \left. -(s + 1)\left(t - \frac{1}{2}\right) \right), \end{aligned} \quad (25)$$

where $0 \leq s \leq 3$, $0 \leq t \leq 1$ in 3-space, illustrated in Figure 2.

For the same curve in question let us choose marching-scale functions as

$$\begin{aligned} \mathbf{u}(s, t, q) &\equiv 0, \\ \mathbf{v}(s, t, q) &= \sin(s(q - q_0)), \\ \mathbf{w}(s, t, q) &\equiv 0, \\ \mathbf{x}(s, t, q) &= sq^2(t - t_0). \end{aligned} \quad (26)$$

Thus, from (15) the curve $\mathbf{r}(s)$ is isoasymptotic on the hypersurface

$$\begin{aligned} \mathbf{P}(s, t, q) &= \mathbf{r}(s) + \mathbf{u}(s, t, q)\mathbf{T}(s) + \mathbf{v}(s, t, q)\mathbf{N}(s) \\ &\quad + \mathbf{w}(s, t, q)\mathbf{B}_1(s) + \mathbf{x}(s, t, q)\mathbf{B}_2(s) \\ &= \left(\frac{1}{2} \sin(s) - \sin(s) \sin(s(q - q_0)), \right. \\ &\quad \frac{1}{2} \cos(s) - \cos(s) \sin(s(q - q_0)), \\ &\quad \left. - sq^2(t - t_0), \frac{\sqrt{3}}{2}s \right), \end{aligned} \quad (27)$$

where $0 < s \leq \pi/2$, $0 \leq t \leq 1$, and $0 < q < 1$.

FIGURE 3: Projection of a member of the hypersurface family and its isoasymptotic.

By taking $t_0 = 1$ and $q_0 = 1/2$ we have the following hypersurface:

$$\begin{aligned} \mathbf{P}(s, t, q) = & \left(\frac{1}{2} \sin(s) - \sin(s) \sin\left(s\left(q - \frac{1}{2}\right)\right), \right. \\ & \frac{1}{2} \cos(s) - \cos(s) \sin\left(s\left(q - \frac{1}{2}\right)\right), \\ & \left. -sq^2(t-1), \frac{\sqrt{3}}{2}s \right). \end{aligned} \quad (28)$$

Hence, if we (parallel) project the hypersurface $\mathbf{P}(s, t, q)$ into the $\mathbf{z} = 0$ subspace, we get the surface

$$\begin{aligned} \mathbf{P}_z(s, q) = & \left(\frac{1}{2} \sin(s) - \sin(s) \sin\left(s\left(q - \frac{1}{2}\right)\right), \right. \\ & \left. \frac{1}{2} \cos(s) - \cos(s) \sin\left(s\left(q - \frac{1}{2}\right)\right), \frac{\sqrt{3}}{2}s \right), \end{aligned} \quad (29)$$

where $0 < s \leq \pi/2$, $0 < q < 1$, in 3-space shown in Figure 3.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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