Research Article

Normal Edge-Transitive Cayley Graphs of the Group U_{6n}

A. Assari and F. Sheikhmiri

Department of Basic Science, Jundi-Shapur University of Technology, Dezful 64616-18674, Iran

Correspondence should be addressed to A. Assari; amirassari@jsu.ac.ir

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A Cayley graph of a group G is called normal edge-transitive if the normalizer of the right representation of the group in the automorphism of the Cayley graph acts transitively on the set of edges of the graph. In this paper, we determine all connected normal edge-transitive Cayley graphs of the group U_{6n} .

1. Introduction

Let G be a group and let S be a subset of G . The Cayley graph of G with respect to S is a graph with the group G as the vertex set and (x, y) is an arc of the graph, if and only if for some $s \in S$ we have $y = sx$. We denote the Cayley graph by $\Gamma = \text{Cay}(G, S)$. Since G is a group, the Cayley graph cannot have any parallel edges. In addition we assume that the subset S does not contain the identity element of the group G to avoid having loops in the Cayley graph and also to be an inverse closed set; that is, $S = S^{-1}$, to have an undirected Cayley graph. Therefore, we focus on simple Cayley graphs.

Let $\Gamma = (V, E)$ be a simple graph with vertex set V and edge set E . In this paper, we denote the edge, joining the vertices x and y by $\{x, y\}$. An automorphism of the graph Γ is a permutation on the vertex set of Γ which preserves the edges. The set of all automorphisms of Γ forms a group with the composition of maps as the binary operation and is denoted by $\text{Aut}(\Gamma)$. Since $\text{Aut}(\Gamma)$ is a permutation group of V , it acts on V as well as E in the usual way. Γ is called vertex-transitive, edge-transitive, or arc-transitive, iff $\text{Aut}(\Gamma)$ acts transitively on the set of vertices, edges, or arcs of Γ , respectively. Γ is called half-arc-transitive iff it is vertex- and edge-transitive, but not arc-transitive.

Let G be a finite group and let S be an inverse closed subset of G that does not contain the identity. Set $\Gamma = \text{Cay}(G, S)$. Γ is connected iff S is a generating set of the group G . For $g \in G$, define the mapping $\rho_g : G \rightarrow G$ by $\rho_g(x) = xg$, for all $x \in G$. $\rho_g \in \text{Aut}(\Gamma)$ for every $g \in G$, and hence $R(G) = \{\rho_g \mid g \in G\}$

is a regular subgroup of $\text{Aut}(\Gamma)$ isomorphic to G , forcing Γ to be a vertex-transitive graph.

Let $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$. Normalizer of $R(G)$ in $\text{Aut}(\Gamma)$ is equal to

$$N_{\text{Aut}(\Gamma)}(R(G)) = R(G) \rtimes \text{Aut}(G, S), \quad (1)$$

where $\rtimes$ denotes the semidirect product of two groups [1].

In [2], the graph Γ is called normal if $R(G)$ is a normal subgroup of $\text{Aut}(\Gamma)$. And after that the normality of Cayley graphs has been extensively studied from different points of view. Among them, finding and classifying the normal Cayley graphs were an essential problem, since in normal Cayley graphs we know the exact automorphism group of the graph.

It was conjectured in [2] that most Cayley graphs are normal. For example, in [3] the authors determined all possible nonnormal (and, as a consequence, normal) Cayley graphs of groups of order $2p^2$, and lots of other authors have done similar works for groups of orders p , $2p$, $4p$, p^2 , and pq [4–8].

Another concept which was similar to the above one is introduced by Praeger in [9] in which a Cayley graph $\Gamma = \text{Cay}(G, S)$ of a group G with respect to S is called normal edge-transitive or arc-transitive if $N_{\text{Aut}(\Gamma)}(R(G))$ acts on the edges or arcs of Γ , respectively, and it is called normal half-arc-transitive Cayley graph if it is normal edge-transitive Cayley graph which is not normal arc-transitive. Obviously, any normal edge-transitive Cayley graph is edge-transitive.

Thus, this concept talks about the symmetric properties of a Cayley graph.

The latter concepts also were considered very much in the literature. For example, Alaeiyan in [10] found a class of normal edge-transitive Cayley graphs of abelian groups or, in [11], authors found all normal edge-transitive Cayley graphs of order $4p$ and as a consequence found a class of normal half-arc-transitive Cayley graphs which rarely happens. This motivated us to consider the Cayley graphs of groups U_{6n} and classify all normal edge-transitive Cayley graphs of groups U_{6n} .

2. Preliminary Results

One of the principle theorems that helps us to connect the group properties of a group G and normal edge-transitivity of $\Gamma = \text{Cay}(G, S)$ is the following theorem which is proved in [9].

Theorem 1. *Let $\Gamma = \text{Cay}(G, S)$ be a connected Cayley graph on S . Then, Γ is normal edge-transitive if and only if $\text{Aut}(G, S)$ is either transitive on S or has two orbits in S in the form of T and T^{-1} , where T is a nonempty subset of S such that $S = T \cup T^{-1}$.*

The following corollary comes from Theorem 1 which is also mentioned in [11].

Corollary 2. *Let $\Gamma = \text{Cay}(G, S)$ and let H be the subset of all involutions of the group G . If H does not generate the group G and Γ is connected normal edge-transitive, then the valency of Γ is even.*

Theorem 1 with a result in [12] characterizes arc-transitive and half-arc-transitive Cayley graphs as described in the following.

Theorem 3. *Let $\Gamma = \text{Cay}(G, S)$ be a connected normal edge-transitive Cayley graph. Γ is normal arc-transitive if $\text{Aut}(G, S)$ acts transitively on S and Γ is normal half-arc-transitive if $S = T \cup T^{-1}$, where T is an orbit of the action of $\text{Aut}(G, S)$ on S .*

3. Group U_{6n}

Group U_{6n} has the presentation

$$U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle, \quad (2)$$

which we can write its elements as follows:

$$U_{6n} = \{a^i b^j \mid 0 \leq i \leq 2n-1, 0 \leq j \leq 2\}. \quad (3)$$

One can see that the group U_{6n} has order $6n$.

In Theorem 1, there are some relations between the normal edge-transitive Cayley graphs and the automorphism group of the relying group. Therefore, first we will find the automorphism group of the group U_{6n} .

Theorem 4. *Every automorphism of the group U_{6n} is in the form of $\sigma_{i,j} : U_{6n} \rightarrow U_{6n}$ which sends a to a^i and b to b^j , or*

$\delta_{i,j,s} : U_{6n} \rightarrow U_{6n}$ sends a to $a^i b^j$ and b to b^s , where $(i, 2n) = 1$ for $0 \leq i \leq 2n-1$ and $1 \leq j, s \leq 2$.

Proof. First of all we find some relations between elements of the group U_{6n} .

Since $a^{-1}ba = b^{-1}$, equivalently

$$ba = ab^{-1}. \quad (4)$$

Hence, we have

$$b^2 a = bba = bab^2 = ab^2 b^2 = ab. \quad (5)$$

One can check by induction that, for $0 \leq i \leq 2n-1$ and $1 \leq j \leq 2$, we will have

$$b^j a^i = \begin{cases} a^i b^j & \text{if } i \text{ is even;} \\ a^i b^{-j} & \text{if } i \text{ is odd.} \end{cases} \quad (6)$$

If we denote the order of element of x by $o(x)$, then, for $0 \leq i \leq 2n-1$, we have

$$o(a^i) = \frac{2n}{(i, 2n)}. \quad (7)$$

For finding the order of elements of the form $a^i b^j$ for $j = 1, 2$, we have to observe that, for odd i and arbitrary k , we have $(a^i b)^{2k} = a^{2ik}$ and $(a^i b)^{2k+1} = a^{i(2k+1)} b$ and, for even i and arbitrary k , we have $(a^i b)^{3k} = a^{3ik}$, $(a^i b)^{3k+1} = a^{i(3k+1)} b$, and $(a^i b)^{3k+2} = a^{i(3k+2)} b^2$ which can be obtained from (6).

Therefore, if the order of $x = a^i b$ is L , for odd i , then $L = 2k$ for some integer k . Since order of a is $2n$, we should have

$$2n \mid 2ik \implies n \mid ik \implies \frac{n}{(n, i)} \mid k. \quad (8)$$

But k should be the least integer satisfying the condition and implies $k = n/(n, i)$; that is,

$$o(a^i b) = \frac{2n}{(n, i)}, \quad \text{if } i \text{ is odd,} \quad (9)$$

where $(\cdot, \cdot)$ is the greatest common divisor.

Similar argument can apply for $a^i b^2$ in the case i is odd to obtain

$$o(a^i b^2) = \frac{2n}{(n, i)}, \quad \text{if } i \text{ is odd.} \quad (10)$$

Now for even i , assume $i = 2j$ for some integer j and $o(a^i b) = L = 3k$ for some integer k . Thus, we have

$$(a^i b)^{3k} = (a^{2j} b)^{3k} = a^{6jk} = 1. \quad (11)$$

Compare it with the order of a ; we get

$$2n \mid 6jk \implies n \mid 3k \implies \frac{n}{(n, j)} \mid 3k. \quad (12)$$

Since $3 \mid 3k$, one can conclude that $[n/(n, j), 3] \mid 3k$, where $[\cdot, \cdot]$ is the least common multiple, and finally we obtain

$$o(a^i b) = \left[\frac{n}{(n, i/2)}, 3 \right], \quad \text{if } i \text{ is even.} \quad (13)$$

Similar argument can be discussed to prove

$$o(a^i b^2) = \left[\frac{n}{(n, i/2)}, 3 \right], \quad \text{if } i \text{ is even.} \quad (14)$$

Suppose σ is an automorphism of the group U_{6n} ; thus, σ preserves the order of elements; hence, $o(\sigma(a)) = o(a) = 2n$ and $o(\sigma(b)) = o(b) = 3$.

By the order of elements of U_{6n} , if we define the following sets

$$\begin{aligned} A &= \{a^i \mid (i, 2n) = 1\}, \\ B &= \{a^i b^j \mid (i, 2n) = 1\} \\ &\text{(observe that in this case } i \text{ is odd)} \\ C &= \{a^{2n/3}, a^{4n/3}\}, \\ D &= \left\{ a^{2r} b^s \mid r = \frac{n}{3}, \frac{2n}{3}, s = 1, 2 \right\}, \\ E &= \{b, b^2\}, \end{aligned} \quad (15)$$

by the order of elements of the group U_{6n} , we can say that $\sigma(a) \in A \cup B$ and $\sigma(b) \in C \cup D \cup E$. Let $x = \sigma(a)$ and $y = \sigma(b)$. But all of the following cases which may happen for x and y yield a contradiction.

- (1) $x \in A$ and $y \in C$ contradicts with the fact that $\{\sigma(a), \sigma(b)\}$ is a generating set of U_{6n} .
- (2) $x \in A$ and $y \in D$. The equation $a^{-1}ba = b^{-1}$ implies $x^{-1}yx = y^{-1}$ which also implies $r = 0$ or $r = n/2$. But none of them can occur, since $r \in \{n/3, 2n/3\}$.
- (3) $x \in B$ and $y \in C$. Thus, $x = a^i b^j$ for some odd i and $y = a^r$ for some $r \in \{2n/3, 4n/3\}$. Hence, r is even and $r + i$ is odd and we have

$$x^{-1}yx = a^{-i}b^j a^r a^i b^j = a^r. \quad (16)$$

But $x^{-1}yx = y^{-1} = a^{-r}$ because σ is an automorphism and hence $a^r = a^{-4}$ yields $r = n$, which is a contradiction.

- (4) $x \in B$ and $y \in D$. We can write $x = a^i b^j$, for odd i , and $y = a^{2r} b^s$, for $r \in \{n/3, 2n/3\}$ and $1 \leq s \leq 2$. Thus, i and, hence, $2r + i$ are odd and we obtain

$$\begin{aligned} x^{-1}yx &= a^{-i}b^j a^{2r} b^s a^i b^j = a^{-i}b^j a^{2r+i} b^{-s+j} \\ &= a^{-i+2r+i} b^{-j-s+j} = a^{2r} b^{-s}. \end{aligned} \quad (17)$$

But we have $x^{-1}yx = y^{-1} = a^{-2r} b^{-s}$. Thus, we should have $a^{2r} = a^{-2r}$; that is, $r = n/2$, which is a contradiction.

Therefore, the only cases that may happen are the cases $x \in A$, $y \in E$ and $x \in B$, $y \in E$ and the theorem is proved. $\square$

And from Theorem 4, we can obtain that the automorphism group of the group U_{6n} is as follows:

$$\begin{aligned} \text{Aut}(U_{6n}) &= \{ \sigma_{i,j}, \delta_{i,j,s} : U_{6n} \longrightarrow U_{6n} \mid (i, 2n) = 1, j, s \in \{1, 2\}, \\ \sigma_{i,j}(a) &= a^i, \sigma_{i,j}(b) = b^j, \delta_{i,j,s}(a) = a^i b^j, \\ \delta_{i,j,s}(b) &= b^s \}. \end{aligned} \quad (18)$$

Now, we are ready to find the orbits of the group U_{6n} under the action of $\text{Aut}(U_{6n})$.

Lemma 5. *In the action of group $\text{Aut}(U_{6n})$ on U_{6n} , each element falls into one of the following orbits, depending on k :*

- (I) $\{a^{ki} \mid k \text{ is even}, (i, 2n) = 1\}$;
- (II) $\{a^{ki} b^j \mid k \text{ is even}, 1 \leq j \leq 2\}$;
- (III) $\{a^{ki} b^j \mid k \text{ is odd}, 0 \leq j \leq 2\}$;
- (IV) $\{b, b^2\}$.

Proof. By Relation (6), one can show that, for odd i , if k is even, then $(a^i b^j)^k = a^{ki}$, and if k is odd, then $(a^i b^j)^k = a^{ki} b^j$ for $1 \leq j \leq 2$. Therefore, images of elements of U_{6n} under $\sigma_{i,j}$ and $\delta_{i,j,s}$ in $\text{Aut}(U_{6n})$ are as follows (recall that $(i, 2n) = 1$, $1 \leq j, s \leq 2$).

If k is even, then

$$\begin{aligned} \sigma_{i,j}(a^k) &= (a^i)^k = a^{ki}, \\ \sigma_{i,j}(a^k b^l) &= (a^i)^k (b^j)^l = a^{ki} b^{lj}, \\ 1 \leq l, j \leq 2, \text{ i.e. } 1 \leq lj \leq 2, \\ \delta_{i,j,s}(a^k) &= (a^i b^j)^k = a^{ki}, \\ \delta_{i,j,s}(a^k b^l) &= (a^i b^j)^k (b^l)^s = a^{ki} b^{j+ls}. \end{aligned} \quad (19)$$

If k is even or odd, then $\sigma_{i,j}(b^l) = b^{lj}$ and $\sigma_{i,j,s}(b^l) = b^{ls}$. Therefore, the orbit of a^k for an even k is $\{a^{ki} \mid k \text{ is even}, (i, 2n) = 1\}$ and for odd k is $\{a^{ki} b^j \mid k \text{ is odd}, 0 \leq j \leq 2\}$. The orbit of $a^k b^l$ for an even k is $\{a^{ki} b^j \mid k \text{ is even}, 1 \leq j \leq 2\}$ and for odd k is $\{a^{ki} b^j \mid k \text{ is odd}, 0 \leq j \leq 2\}$. And finally the orbit of b is $\{b, b^2\}$. $\square$

Now we bring a sufficient condition under which a Cayley graph of group U_{6n} can be normal edge-transitive.

Theorem 6. *If $\Gamma = \text{Cay}(U_{6n}, S)$ is connected normal edge-transitive Cayley graph, then $|S|$ is even, greater than 2, and S is contained in $\{a^{ki} b^j \mid k \text{ is odd}, 0 \leq j \leq 2\}$.*

Proof. From Corollary 2, $|S|$ is even.

Since Γ is normal edge-transitive, by Theorem 1, $\text{Aut}(U_{6n}, S)$ acts transitively on S or $S = T \cup T^{-1}$ where T is

an orbit of the action of $\text{Aut}(U_{6n}, S)$. Since $\text{Aut}(U_{6n}, S)$ is a subgroup of $\text{Aut}(U_{6n})$, thus S or, in the latter case, T should be contained in one of orbits of $\text{Aut}(U_{6n})$ on U_{6n} . But, from Lemma 5, we have 4 kinds of orbits in this action. But from (6), if i and j are even, $a^i b^s a^j b^{s'} = a^{i+j} b^{s+s'}$. Thus, $i + j$ is even and a is not generated by such elements. Therefore, if S is contained in an orbit of kind I, II, or IV, S will not generate U_{6n} . Therefore, Γ will not be connected. Thus, S is contained in III. $\square$

Theorem 6 has a critical corollary.

Corollary 7. $\Gamma = \text{Cay}(U_{6n}, S)$ is never connected normal half arc-transitive Cayley graph.

Proof. For odd i and $0 \leq j \leq 2$, we have $(a^i b^j)^{-1} = b^{-j} a^{-i} = a^{-i} b^j$. From Theorem 6, S contains such elements. Since S is closed under inversion, S is union of $A_{ij} = \{a^i b^j, a^{-i} b^j\}$ for some odd i and some $0 \leq j \leq 2$. $\sigma_{-1,1}$ is an automorphism of G which preserves A_{ij} , for all i and j , and hence preserves S ; that is, $\sigma_{-1,1} \in \text{Aut}(G, S)$. Accordingly, if we set $S = T \cup T^{-1}$, then each element of T is sent to an element of T^{-1} by $\sigma_{-1,1}$. T is not an orbit of the action of $\text{Aut}(G, S)$ on S . $\square$

By Theorems 1 and 6 and Corollary 7 we can obtain the following theorem.

Theorem 8. $\Gamma = \text{Cay}(U_{6n}, S)$ is connected normal edge-transitive Cayley graph if $|S|$ is even, greater than 2, S is contained in $\{a^{ki} b^j \mid k \text{ is odd}, 0 \leq j \leq 2\}$, and $\text{Aut}(U_{6n}, S)$ acts transitively on S .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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