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Research Article

Lattice Trace Operators

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A bounded linear operator T on a Hilbert space $\mathscr{H}$ is trace class if its singular values are summable. The trace class operators on $\mathscr{H}$ form an operator ideal and in the case that $\mathscr{H}$ is finite-dimensional, the trace $\operatorname{tr}(T)$ of T is given by $\sum_j a_{jj}$ for any matrix representation $\{a_{ij}\}$ of T. In applications of trace class operators to scattering theory and representation theory, the subject is complicated by the fact that if k is an integral kernel of the operator T on the Hilbert space $L^2(\mu)$ with μ a σ-finite measure, then $k(x,x)$ may not be defined, because the diagonal $\{(x,x)\}$ may be a set of $(\mu\otimes\mu)$-measure zero. The present note describes a class of linear operators acting on a Banach function space X which forms a *lattice* ideal of operators on X, rather than an *operator* ideal, but coincides with the collection of hermitian positive trace class operators in the case of $X = L^2(\mu)$.

1. Introduction

A *trace class operator* A on a separable Hilbert space $\mathscr{H}$ is a compact operator whose *singular values* $\lambda_j(A)$, $j = 1, 2, \ldots$, satisfy

$$\|A\|_1 = \sum_{j=1}^{\infty} \lambda_j(A) < \infty. \tag{1}$$

The decreasing sequence $\{\lambda_j(A)\}_{j=1}^{\infty}$ consists of eigenvalues of $(A^*A)^{1/2}$. Equivalently, A is trace class if and only if, for any orthonormal basis $\{h_j\}_{j=1}^{\infty}$ of $\mathscr{H}$, the sum $\sum_{j=1}^{\infty} |(Ah_j, h_j)|$ is finite. The number

$$\operatorname{tr}(A) = \sum_{j=1}^{\infty} \left(Ah_j, h_j\right) \tag{2}$$

is called the *trace* of A and is independent of the orthonormal basis $\{h_j\}_{j=1}^{\infty}$ of $\mathscr{H}$. *Lidskii's equality* asserts that $\operatorname{tr}(A)$ is actually the sum of the eigenvalues of the compact operator T [1, Theorem 3.7].

We refer to [1] for properties of trace class operators. The collection $\mathscr{C}_1(\mathscr{H})$ of trace class operators on $\mathscr{H}$ is an *operator ideal* and Banach space with the norm $\|\cdot\|_1$. The following facts are worth noting in the case of the Hilbert space $L^2([0,1])$ with respect to Lebesgue measure on the interval $[0,1]$.

(a) If $T : L^2([0,1]) \to L^2([0,1])$ is a trace class linear operator, then there exist $\phi_j, \psi_j \in L^2([0,1])$, $j = 1, 2, \ldots$, with $\sum_{j=1}^{\infty} \|\phi_j\|_2 \|\psi_j\|_2 < \infty$ and

$$(Tf)(x) = \int_0^1 k(x,y) f(y)\, dy, \quad \text{a.e. for } f \in L^2([0,1]), \tag{3}$$

where $k = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j$ a.e.. In particular, T is regular and $|T|$ has an integral kernel $|k| \le \sum_{j=1}^{\infty} |\phi_j| \otimes |\psi_j|$. Moreover,

$$\operatorname{tr}(T) = \sum_{j=1}^{\infty} \int_0^1 \phi_j(x) \psi_j(x)\, dx. \tag{4}$$

(b) Suppose that $T : L^2([0,1]) \to L^2([0,1])$ is a regular linear operator defined by formula (3) for a continuous function $k : [0,1] \times [0,1] \to \mathbb{C}$. If T is trace class, then $\int_0^1 |k(x,x)|\, dx < \infty$, and $\operatorname{tr}(T) = \int_0^1 k(x,x)\, dx$ [2, Theorem V.3.1.1].

(c) Suppose that the function $k : [0,1] \times [0,1] \to \mathbb{C}$ is continuous and positive definite; that is, $\sum_{j,\ell=1}^{n} z_j \overline{z_\ell} k(x_j, x_\ell) \ge 0$ for all $z_j \in \mathbb{C}$ and $x_j \in [0,1]$, $j = 1, \ldots, n$, and any $n = 1, 2, \ldots$. Then $k(x,x) \ge 0$ for

all $x \in [0,1]$. If $\int_0^1 k(x,x)\,dx < \infty$, then there exists a unique trace class operator defined by formula (3) [1, Theorem 2.12].

Let $(\Sigma, \mathscr{B}, \mu)$ be a measure space. The *projective tensor product* $L^2(\mu)\widehat{\otimes}_\pi L^2(\mu)$ is the set of all sums:

$$k = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j \quad \text{a.e., with } \sum_{j=1}^{\infty} \|\phi_j\|_2 \|\psi_j\|_2 < \infty. \tag{5}$$

The norm of $k \in L^2(\mu)\widehat{\otimes}_\pi L^2(\mu)$ is given by $\|k\|_\pi = \inf\{\sum_{j=1}^{\infty} \|\phi_j\|_2 \|\psi_j\|_2\}$ where the infimum is taken over all sums for which the representation (5) holds. The Banach space $L^2(\mu)\widehat{\otimes}_\pi L^2(\mu)$ is actually the completion of the algebraic tensor product $L^2(\mu) \otimes L^2(\mu)$ with respect to the projective tensor product norm [3, Section 6.1].

There is a one-to-one correspondence between the space of trace class operators acting on $L^2(\mu)$ and $L^2(\mu)\widehat{\otimes}_\pi L^2(\mu)$, so that the trace class operator T_k has an integral kernel $k \in L^2(\mu)\widehat{\otimes}_\pi L^2(\mu)$. If the integral kernel k given by (5) has the property that

$$k(x,y) = \sum_{j=1}^{\infty} \phi_j(x)\,\psi_j(y) \tag{6}$$

for all $x, y \in \Sigma$ such that the sum $\sum_{j=1}^{\infty} |\phi_j(x)\psi_j(y)|$ is finite, then the equality

$$\operatorname{tr}(T_k) = \sum_{j=1}^{\infty} \int_\Sigma \phi_j(x)\,\psi_j(x)\,d\mu(x) = \int_\Sigma k(x,x)\,d\mu(x) \tag{7}$$

holds. Because the diagonal $\{(x,x) : x \in \Sigma\}$ may be a set of $(\mu\otimes\mu)$-measure zero in $\Sigma\times\Sigma$, it may be difficult to determine whether or not a given integral kernel $k : \Sigma \times \Sigma \to \mathbb{C}$ has such a *distinguished* representation.

The difficulty is addressed by Brislawn [4, 5], [1, Appendix D] who shows that, for a trace class operator $T_k : L^2(\mu) \to L^2(\mu)$ with integral kernel k, the equality

$$\operatorname{tr}(T_k) = \int_\Sigma \widetilde{k}(x,x)\,d\mu(x) \tag{8}$$

holds. The measure μ is supposed in [5] to be a σ-finite Borel measure on a second countable topological space Σ and the regularised kernel $\widetilde{k}$ is defined from k by averaging with respect to the product measure $\mu\otimes\mu$. Extending the result (c) of M. Duflo given above, Brislawn [5, Theorem 4.3] shows that a hermitian positive Hilbert-Schmidt operator T_k is a trace class operator if and only if $\int_\Sigma \widetilde{k}(x,x)\,d\mu(x) < \infty$.

The present paper examines the space $\mathfrak{C}_1(X)$ of absolute integral operators $T_k : X \to X$ defined on a Banach function space for which $\int_\Sigma |\widetilde{k}(x,x)|\,d\mu(x) < \infty$. Elements of $\mathfrak{C}_1(X)$ are called *lattice trace operators* because $\mathfrak{C}_1(X)$ is a *lattice ideal* in the Banach lattice of regular operators on X, whereas the collection $\mathscr{C}_1(\mathscr{H})$ of trace class operators on a Hilbert space $\mathscr{H}$ is an *operator ideal* in the Banach algebra $\mathscr{L}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$. The intersections of $\mathfrak{C}_1(X)$ and $\mathscr{C}_1(L^2(\mu))$ with the hermitian positive operators on $L^2(\mu)$ are equal for locally square integrable kernels; see Proposition 4.

The regularised kernel $\widetilde{k} : \Sigma \times \Sigma \to \mathbb{C}$ of an absolute integral operator T_k is defined by adapting the method of Brislawn [5] to positive operators with an integral kernel. The generalised trace $\int_\Sigma \widetilde{k}(x,x)\,d\mu(x)$ may be viewed alternatively as a bilinear integral $\int_\Sigma \langle T_k, dm\rangle$ with respect to the measure $m : E \mapsto \chi_E$, $E \in \mathscr{B}$. Lattice trace operators are employed in the proof of the Cwikel-Lieb-Rosenblum inequality for dominated semigroups [6].

The basic definitions of Banach function spaces and operators with an integral kernel which act upon them are set out in Section 2. The martingale regularisation of the integral kernel of an operator between Banach function spaces is set out in Section 3 and the connection with trace class operators on $L^2(\mu)$ is set out in Section 4.

2. Banach Function Spaces and Regular Operators

Let Σ be a second countable topological space with Borel σ-algebra $\mathscr{B}$. The diagonal $\operatorname{diag}(\Sigma \times \Sigma) = \{(x,x) : x \in \Sigma\}$ is a closed subset of the Cartesian product $\Sigma\times\Sigma$. Because the Borel σ-algebra of $\Sigma\times\Sigma$ is equal to $\mathscr{B}\otimes\mathscr{B}$, the diagonal $\operatorname{diag}(\Sigma\times\Sigma)$ belongs to the σ-algebra $\mathscr{B}\otimes\mathscr{B}$.

We suppose that $(\Sigma, \mathscr{B}, \mu)$ is a σ-finite measure space. The space of all μ-equivalence classes of Borel measurable scalar functions is denoted by $L^0(\mu)$. It is equipped with the topology of convergence in μ-measure over sets of finite measure and vector operations pointwise μ-almost everywhere. Any Banach space X that is a subspace of $L^0(\mu)$ with the properties that

(i) X is an order ideal of $L^0(\mu)$, that is, if $g \in X$, $f \in L^0(\mu)$, and $|f| \le |g|$ μ-a.e., then $f \in X$ and

(ii) if $f, g \in X$ and $|f| \le |g|$ μ-a.e., then $\|f\|_X \le \|g\|_X$,

is called a *Banach function space* (based on $(\Sigma, \mathscr{B}, \mu)$). The Banach function space X is necessarily Dedekind complete; that is, every order bounded set has a sup and an inf [7, page 116]. The set of $f \in X$ with $f \ge 0$ μ-a.e. is written as X_+.

We suppose that X contains the characteristic functions of sets of finite measure and $m : S \mapsto \chi_S$, $S \in \mathscr{S}$, is σ-additive in X on sets of finite measure; for example, X is σ-order continuous; see [8, Corollary 3.6]. If X is reflexive and μ is finite and nonatomic, then it follows from [8, Corollary 3.23] that the values of the variation $V(m)$ of m are either zero or infinity. In particular, this is the case for $X = L^p([0,1])$ with $1 < p < \infty$.

Following the account of Brislawn [5], we extend the mapping $T \mapsto \int_\Sigma \langle T, dm\rangle$ from the space $\mathscr{C}_1(L^2(\mu))$ of trace class linear operators to a larger class of regular operators by representing T by a "regularised" kernel, so that the collection of regular operators T for which $\int_\Sigma \langle |T|, dm\rangle < \infty$ is a vector sublattice of the Riesz space of regular operators—a property not necessarily enjoyed by the trace class operators.

Let X be a Banach function space based on the σ-finite measure space $(\Sigma, \mathscr{B}, \mu)$ as above. A continuous linear operator $T : X \to X$ is called *positive* if $T : X_+ \to X_+$. The collection of all positive continuous linear operators on X is written as $\mathscr{L}_+(X)$. If the real and imaginary parts of a continuous linear operator $T : X \to X$ can be written as the difference of two positive operators, it is said to be *regular*. The *modulus* $|T|$ of a regular operator T is defined by

$$|T| f = \sup_{|g| \le f} |Tg|, \quad f \in X_+. \tag{9}$$

The collection of all regular operators is written as $\mathscr{L}_r(X)$ and it is given the norm $T \mapsto \||T|\|$, $T \in \mathscr{L}_r(X)$ under which it becomes a Banach lattice [7, Proposition 1.3.6].

A continuous linear operator $T : X \to X$ has an integral kernel k if $k : \Sigma \times \Sigma \to \mathbb{C}$ is a Borel measurable function such that $T = T_k$ for the operator given by

$$(T_k f)(x) = \int_\Sigma k(x, y) f(y) \, d\mu(y), \quad \mu\text{-almost all } x \in \Sigma, \tag{10}$$

for each $f \in X$, in the sense that $\int_\Sigma |k(x, y) f(y)| d\mu(y) < \infty$ for μ-almost all $x \in \Sigma$ and $x \mapsto \int_\Sigma k(x, y) f(y) d\mu(y)$ is an element of X. Then k is $(\mu \otimes \mu)$-integrable on any product set $A \times B$ with finite measure. If $T_k \ge 0$, then $k \ge 0 (\mu \otimes \mu)$-a.e. on $\Sigma \times \Sigma$ [7, Theorem 3.3.5].

A continuous linear operator T is an *absolute integral operator* if it has an integral kernel k for which $T_{|k|}$ is a bounded linear operator on $L^2(\mu)$. Then $|T_k| = T_{|k|}$ [7, Theorem 3.3.5]. The collection of all absolute integral operators is a lattice ideal in $\mathscr{L}_r(X)$ [7, Theorem 3.3.6].

Suppose that $T \in \mathscr{L}(X)$ has an integral kernel $k = \sum_{j=1}^n f_j \chi_{A_j}$, that is, an X-valued simple function with $\mu(A_j) < \infty$. Then it is natural to view

$$\int_\Sigma \langle T, dm \rangle = \sum_{j=1}^n \int_{A_j} f_j \, d\mu = \int_\Sigma k(x, x) \, d\mu(x) \tag{11}$$

as a bilinear integral. Our aim is to extend the integral to a wider class of absolute integral operators.

3. Martingale Regularisation

Let $\mathscr{U} = \{U_1, U_2, \ldots\}$ be a countable base for the topology of Σ. An increasing family of countable partitions $\mathscr{P}_n$, $n = 1, 2, \ldots$, is defined recursively by setting $\mathscr{P}_1$ equal to a partition of Σ into Borel sets of finite μ-measure and

$$\mathscr{P}_{j+1} = \left\{ P \cap U_j, P \setminus U_j : P \in \mathscr{P}_j \right\} \tag{12}$$

for $j = 1, 2, \ldots$. For each $n = 1, 2, \ldots$, let $\mathscr{E}_n$ be the σ-algebra for all countable unions of elements of $\mathscr{P}_n$.

Suppose that $k \ge 0$ is a Borel measurable function defined on $\Sigma \times \Sigma$ that is integrable on every set of finite $(\mu \otimes \mu)$-measure.

For each $x \in \Sigma$, the set $U_n(x)$ is the unique element of the partition $\mathscr{P}_n$ containing x. For each $n = 1, 2, \ldots$, the conditional expectation $k_n = \mathbb{E}(k \mid \mathscr{E}_n \otimes \mathscr{E}_n)$ can be represented for μ-almost all $x, y \in \Sigma$ as

$$\begin{aligned} &\mathbb{E}(k \mid \mathscr{E}_n \otimes \mathscr{E}_n)(x, y) \\ &\quad = \frac{1}{\mu(U_n(x)) \mu(U_n(y))} \int_{U_n(x)} \int_{U_n(y)} k(s, t) \, d\mu(s) \, d\mu(t) \\ &\quad = \sum_{U, V \in \mathscr{P}_n} \frac{\int_{U \times V} k \, d(\mu \otimes \mu)}{\mu(U) \mu(V)} \chi_{U \times V}(x, y). \end{aligned} \tag{13}$$

Let $\mathscr{N}$ be the set of all $x \in \Sigma$ for which there exists $n = 1, 2, \ldots$ such that $\mu(U_n(x)) = 0$. Then $\mu(U_m(x)) = 0$ for all $m > n$ because $\mathscr{P}_m$ is a refinement of $\mathscr{P}_n$ if $m > n$. Moreover $\mathscr{N}$ is μ-null because $\mathscr{N} \subset \bigcup_{n=1}^\infty \bigcup \{U \in \mathscr{P}_n : \mu(U) = 0\}$. If $0 \le k_1 \le k_2 (\mu \otimes \mu)$-a.e., then

$$\begin{aligned} &\mathbb{E}(k_1 \mid \mathscr{E}_n \otimes \mathscr{E}_n)(x, y) \\ &\qquad \le \mathbb{E}(k_2 \mid \mathscr{E}_n \otimes \mathscr{E}_n)(x, y), \quad n = 1, 2, \ldots, \end{aligned} \tag{14}$$

for all $(x, y) \in \mathscr{N}^c \times \mathscr{N}^c$. In particular,

$$\begin{aligned} &\mathbb{E}(k_1 \mid \mathscr{E}_n \otimes \mathscr{E}_n)(x, x) \\ &\qquad \le \mathbb{E}(k_2 \mid \mathscr{E}_n \otimes \mathscr{E}_n)(x, x), n = 1, 2, \ldots, \end{aligned} \tag{15}$$

for all $x \in \mathscr{N}^c$. Although diag $(\Sigma \times \Sigma)$ may be a set of $(\mu \otimes \mu)$-measure zero, the application of the conditional expectation operators $k \mapsto \mathbb{E}(k \mid \mathscr{E}_n \otimes \mathscr{E}_n)$, $n = 1, 2, \ldots$, has the effect of regularising k. By an appeal to the martingale convergence theorem, k_n converges $(\mu \otimes \mu)$-a.e. to k as $n \to \infty$.

Let $\widetilde{k}(x, y) = \limsup_{n \to \infty} \mathbb{E}(k \mid \mathscr{E}_n \otimes \mathscr{E}_n)(x, y)$ for all $x, y \in \Sigma$ and we set

$$\int_\Sigma \langle T, dm \rangle = \int_\Sigma \widetilde{k}(x, x) \, d\mu(x) \in [0, \infty]. \tag{16}$$

If $\int_\Sigma \langle T, dm \rangle < \infty$, then $A \mapsto \int_A \langle T, dm \rangle = \int_A \widetilde{k}(x, x) \, d\mu(x)$, $A \in \mathscr{B}$, is a finite measure. For a regular operator $T = T_+ - T_-$ with positive and negative parts $T_\pm$, we set

$$\int_\Sigma \langle T, dm \rangle = \int_\Sigma \langle T_+, dm \rangle - \int_\Sigma \langle T_-, dm \rangle \tag{17}$$

if one of the integrals on the right-hand side of the equation is finite. The integral $\int_\Sigma \langle T, dm \rangle$ is defined by linearity for each regular operator $T : X \to X$. It is clear from the construction that the collection of absolute integral operators T such that $\int_\Sigma \langle |T|, dm \rangle < \infty$ is a vector sublattice $\mathfrak{C}_1(X)$ of the space of regular operators on $L^2(\mu)$. We call elements of $\mathfrak{C}_1(X)$ *lattice trace operators*.

Theorem 1. *The space $\mathfrak{C}_1(X)$ is a lattice ideal in $\mathscr{L}_r(X)$; that is, if $S, T \in \mathscr{L}_r(X)$, $|S| \le |T|$ and $T \in \mathfrak{C}_1(X)$, then $S \in \mathfrak{C}_1(X)$. Moreover, $\mathfrak{C}_1(X)$ is a Dedekind complete Banach lattice with the norm*

$$T \longmapsto \||T|\| + \int_\Sigma \langle |T|, dm \rangle, \quad T \in \mathfrak{C}_1(X). \tag{18}$$

The map $T \mapsto \int_\Sigma \langle T, dm \rangle$ is a positive continuous linear function on $\mathfrak{C}_1(X)$.

Proof. If $S, T \in \mathscr{L}_r(X)$ and $|S| \leq |T|$, then S is an absolute integral operator by [7, Theorem 3.3.6]. If k_1 is the integral kernel of S and k_2 is the integral kernel of T, then by [7, Theorem 3.3.5], the inequality $|k_1| \leq |k_2|$ holds $(\mu \otimes \mu)$-a.e.. Then $|\widetilde{k}_1(x,x)| \leq |\widetilde{k}_2(x,x)|$ for μ-almost all $x \in \Sigma$, so that

$$\int_\Sigma \langle |S|, dm\rangle \leq \int_\Sigma \langle |T|, dm\rangle < \infty. \tag{19}$$

Hence, $S \in \mathfrak{C}_1(X)$.

To show that $\mathfrak{C}_1(X)$ is complete in its norm, suppose that

$$\sum_{j=1}^{\infty}\left(\left\|\left|T_j\right|\right\| + \int_\Sigma \left\langle \left|T_j\right|, dm\right\rangle\right) < \infty \tag{20}$$

for $T_j \in \mathfrak{C}_1(X)$. Then $T = \sum_{j=1}^{\infty} T_j$ in the space of regular operators on X. The inequality $|T| \leq \sum_{j=1}^{\infty} |T_j|$ ensures that T is an absolute integral operator with kernel k by [7, Theorem 3.3.6] and $|k| \leq \sum_{j=1}^{\infty} |k_j|$ $(\mu \otimes \mu)$-a.e..

Suppose first that X is a real Banach function space. Each positive part T_j^+ of T_j, $j = 1, 2, \ldots$, has an integral kernel k_j^+ such that

$$\int_\Sigma \left\langle T_j^+, dm\right\rangle = \int_\Sigma \widetilde{k_j^+}(x,x)\, d\mu(x). \tag{21}$$

By monotone convergence, there exists a set of full μ-measures on which

$$\mathbb{E}\left(k^+ \mid \mathscr{E}_n \otimes \mathscr{E}_n\right)(x,x) \leq \sum_{j=1}^{\infty} \mathbb{E}\left(k_j^+ \mid \mathscr{E}_n \otimes \mathscr{E}_n\right)(x,x) \tag{22}$$

for each $n = 1, 2, \ldots$. Taking the limsup and applying the monotone convergence theorem pointwise and under the sum show that

$$\widetilde{k^+}(x,x) \leq \sum_{j=1}^{\infty} \widetilde{k_j^+}(x,x) \tag{23}$$

for μ-almost all $x \in \Sigma$ and $\int_\Sigma \widetilde{k^+}(x,x) < \infty$. Applying the same argument to T^- and then the real and imaginary parts of T ensures that $T \in \mathfrak{C}_1(X)$ and

$$\||T|\| + \int_\Sigma \langle |T|, dm\rangle \leq \sum_{j=1}^{\infty}\left(\left\|\left|T_j\right|\right\| + \int_\Sigma \left\langle \left|T_j\right|, dm\right\rangle\right). \tag{24}$$

Dedekind completeness is inherited from $\mathscr{L}_r(X)$ [7, Theorem 1.3.2] and $L^1(\mu)$ [7, Example v, page 9]. The bound

$$\left|\int_\Sigma \langle T, dm\rangle\right| \leq \||T|\| + \int_\Sigma \langle |T|, dm\rangle \tag{25}$$

defines a positive continuous linear function on $\mathfrak{C}_1(X)$. □

Example 2 (see [4, Example 3.2]). There exist lattice-positive, compact linear operators $T : L^2([0,1]) \to L^2([0,1])$ such that $\int_0^1 \langle T, dm\rangle$ is finite but T is not a trace class linear operator on the Hilbert space $L^2([0,1])$.

In particular, the Volterra operator T is defined by

$$(Tf)(x) = \int_0^x f(y)\, dy, \quad x \in [0,1], \text{ for } f \in L^2([0,1]). \tag{26}$$

The (lattice) positive linear map $T : L^2([0,1]) \to L^2([0,1])$ is a Hilbert-Schmidt operator but not trace class. Nevertheless, $\int_0^1 \langle T, dm\rangle = 1/2$.

4. Trace Class Operators

Proposition 3 (see [5, Theorem 3.1]). *If $T : L^2(\mu) \to L^2(\mu)$ is a trace class linear operator, then, for any function $k : \Sigma \times \Sigma \to \mathbb{C}$ such that $T = T_k$, where*

$$k = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j, \qquad (\mu \otimes \mu)\text{-a.e.}, \tag{27}$$

with $\sum_{j=1}^{\infty} \|\phi_j\|_2 \|\psi_j\|_2 < \infty$, the equalities

$$\operatorname{tr}(T) = \int_\Sigma \langle T, dm\rangle = \int_\Sigma \widetilde{k}(x,x)\, d\mu(x) \tag{28}$$

hold.

If k is continuous almost everywhere along the diagonal $\operatorname{diag}(\Sigma \times \Sigma)$, then $\widetilde{k}(x,x) = k(x,x)$ for μ-almost all $x \in \Sigma$ [5, Theorem 2.4].

For positive operators in the Hilbert space sense, we have the following sufficient condition for traceability. The operator $u \mapsto \chi_B u$, $u \in L^2(\mu)$, for a Borel set B, is denoted by $Q(B)$.

Proposition 4. *Let $T : L^2(\mu) \to L^2(\mu)$ be an absolute integral operator whose integral kernel is square integrable on any set of finite $(\mu \otimes \mu)$-measure. If $(Tu, u) \geq 0$ for all $u \in L^2(\mu)$, then T is trace class if and only if $\int_\Sigma \langle T, dm\rangle$ is finite, and in this case*

$$\operatorname{tr}(T) = \int_\Sigma \langle T, dm\rangle. \tag{29}$$

Proof. The case where T is assumed to be trace class is covered by Proposition 3 above. Suppose that $T : L^2(\mu) \to L^2(\mu)$ is an absolute integral operator such that $(Tu, u) \geq 0$ for all $u \in L^2(\mu)$ and $\int_\Sigma \langle T, dm\rangle$ is finite.

If the integral kernel k of T is square integrable on any set of finite $(\mu \otimes \mu)$-measure, then for any Borel set B with $\mu(B) < \infty$, the operator $Q(B)TQ(B)$ is a positive Hilbert-Schmidt operator. If $B_j \uparrow \Sigma$ as $j \to \infty$, then

$$\begin{aligned}\sup_j \sup_{n \geq m} \mathbb{E}\left(\left(\chi_{B_j} \otimes \chi_{B_j}\right) k \mid \mathscr{E}_n \otimes \mathscr{E}_n\right)(x,x) \\ = \sup_{n \geq m} \mathbb{E}\left(k \mid \mathscr{E}_n \otimes \mathscr{E}_n\right)(x,x)\end{aligned} \tag{30}$$

by monotone convergence, so

$$\sup_j \int_\Sigma \left\langle Q(B_j) T Q(B_j), dm\right\rangle \leq \int_\Sigma \langle T, dm\rangle. \tag{31}$$

By choosing $B_j = \cup_{m=1}^{j} \Sigma_m$ for $\Sigma_m \in \mathscr{P}_1$ for $m, j = 1, 2 \ldots$, we have

$$\begin{aligned} &\mathbb{E}\left(\left(\chi_{B_j} \otimes \chi_{B_j}\right) k \mid \mathscr{E}_n \otimes \mathscr{E}_n\right)(x, x) \\ &\quad = \mathbb{E}\left(k \mid \mathscr{E}_n \otimes \mathscr{E}_n\right)(x, x)\, \chi_{B_j}(x) \end{aligned} \tag{32}$$

for all $n, j = 1, 2 \ldots$, so

$$\int_\Sigma \left\langle Q\left(B_j\right) T Q\left(B_j\right), dm\right\rangle = \int_{B_j} \langle T, dm\rangle \longrightarrow \int_\Sigma \langle T, dm\rangle \tag{33}$$

as $j \to \infty$. According to [5, Theorem 4.3], $Q(B_j)TQ(B_j)$ is trace class and

$$\operatorname{tr}\left(Q\left(B_j\right) T Q\left(B_j\right)\right) = \int_{B_j} \langle T, dm\rangle . \tag{34}$$

For every $u \in L^2(\mu)$, the inequality

$$\begin{aligned} \left(Q\left(B_j\right) T Q\left(B_j\right) u, u\right) &\le \|u\|^2 \operatorname{tr}\left(Q\left(B_j\right) T Q\left(B_j\right)\right) \\ &\le \|u\|^2 \int_\Sigma \langle T, dm\rangle . \end{aligned} \tag{35}$$

By polarisation, $Q(B_j)TQ(B_j) \to T$ in the weak operator topology as $j \to \infty$, so

$$\begin{aligned} |\operatorname{tr}(TC)| &\le \|C\| \lim_{j\to\infty} \left|\operatorname{tr}\left(Q\left(B_j\right) T Q\left(B_j\right)\right)\right| \\ &\le \|C\| \int_\Sigma \langle T, dm\rangle \end{aligned} \tag{36}$$

for very finite rank operator C. By [1, Theorem 2.14], T is a trace class operator and an appeal to Proposition 3 gives (28). □

Proposition 5. *If $(\Sigma, \mathscr{B}, \mu)$ is an atomic measure space with countably many atoms, then $\mathfrak{C}_1(L^2(\mu)) = \mathscr{C}_1(L^2(\mu))$ and*

$$\operatorname{tr}(T) = \int_\Sigma \langle T, dm\rangle , \quad T \in \mathscr{C}_1\left(L^2(\mu)\right). \tag{37}$$

5. Lattice Properties

Let $J : \Sigma \to \operatorname{diag}(\Sigma \times \Sigma)$ be the diagonal embedding $J(x) = (x, x)$, $x \in \Sigma$. Let $\nu = (\mu \otimes \mu) + \mu \circ J^{-1}$. If $\limsup_{n\to\infty} \mathbb{E}(k \mid \mathscr{E}_n \otimes \mathscr{E}_n)$ converges pointwise ν-a.e. and in $L^1(\nu)$, then there exist scalars c_j and Borel sets C_j, D_j such that

$$\sum_{j=1}^{\infty} \left|c_j\right| \nu\left(C_j \times D_j\right) < \infty, \tag{38}$$

and we can write

$$k(x, y) = \sum_{j=1}^{\infty} c_j\, \chi_{C_j \times D_j}(x, y) \tag{39}$$

for every $x, y \in \Sigma$ such that $\sum_{j=1}^{\infty} |c_j|\, \chi_{C_j\times D_j}(x, y) < \infty$ and $k(x, x) = \tilde{k}(x, x)$ for μ-almost all $x \in \Sigma$; see [9].

Proposition 6. *Let $T : X \to X$ be a positive kernel operator. For any nonnegative μ-measurable functions V_1, V_2, the equalities*

$$\begin{aligned} \int_\Sigma \langle Q(V_2) T Q(V_1), dm\rangle &= \int_\Sigma \langle Q(V_1 V_2) T, dm\rangle \\ &= \int_\Sigma \langle T Q(V_1 V_2), dm\rangle \end{aligned} \tag{40}$$

of extended real numbers hold.

For any essentially bounded μ-measurable function V,

$$\left|\int_\Sigma \langle Q(V) T, dm\rangle\right| \le \|V\|_\infty \int_\Sigma \langle T, dm\rangle \in [0, \infty] . \tag{41}$$

Proof. If the kernel k of T has the representation (39), then for any sets $W_1, W_2 \in \mathscr{B}$, we have

$$\begin{aligned} (Q(W_1) k Q(W_2))^{\sim}(x, x) &= (Q(W_1) Q(W_2) k)^{\sim}(x, x) \\ &= (k Q(W_1) Q(W_2))^{\sim}(x, x) \end{aligned} \tag{42}$$

is equal to

$$\sum_{j=1}^{\infty} c_j\, \chi_{C_j \cap D_j \cap W_1 \cap W_2}(x) \tag{43}$$

for μ-almost all $x \in \Sigma$. The result follows by linearity and approximating V_1 and V_2 by simple functions. □

It is well known that if T is a trace class operator on a Hilbert space $\mathscr{H}$ and B is any bounded linear operator on $\mathscr{H}$ then BT and TB are also trace class operators (i.e., $\mathscr{C}_1(\mathscr{H})$ is an *operator ideal*) and [1, Corollary 3.8]

$$\operatorname{tr}(BT) = \operatorname{tr}(TB) . \tag{44}$$

By contrast, the space $\mathfrak{C}_1(L^2(\mu))$ is a *lattice ideal* in $\mathscr{L}_r(L^2(\mu))$. For $T \in \mathfrak{C}_1(L^2(\mu)))$ and $B \in \mathscr{L}(L^2(\mu))$, the operator BT may not even be a kernel operator, but we have the following trace property.

Proposition 7. *Let $T_j : X \to X$, $j = 1, 2$, be positive kernel operators. Then the equalities*

$$\int_\Sigma \langle T_1 T_2, dm\rangle = \int_\Sigma \langle T_2 T_1, dm\rangle \tag{45}$$

of extended real numbers hold.

Proof. Suppose that the kernels k_j of T_j, $j = 1, 2$, have the representation (39).

If $\mathscr{E}_n$, $n = 1, 2 \ldots$, is an increasing sequence of sub-σ-algebras of $\mathscr{B}$ such that the σ-algebra $\sigma(k_j)$ generated by k_j is contained in $\vee_n \mathscr{E}_n \otimes \mathscr{E}_n$ for $j = 1, 2$, then

$$\int_\Sigma \langle T_1 T_2, dm\rangle = \int_\Sigma \left(\int_\Sigma \tilde{k}_1(x, y) \tilde{k}_2(y, x)\, d\mu(y)\right) d\mu(x) . \tag{46}$$

By the Fubini-Tonelli Theorem this is equal to

$$\int_\Sigma \left(\int_\Sigma \tilde{k}_2(y,x)\tilde{k}_1(x,y)\,d\mu(x)\right) d\mu(y) = \int_\Sigma \langle T_2T_1, dm\rangle . \tag{47}$$

□

We also note that a bilinear version of the Fubini-Tonelli Theorem holds.

Let $(\Xi, \mathscr{E}, \nu)$ be a σ-finite measure space. For any function $f : \Xi \to \mathscr{L}_+(X)$ such that $\int_\Xi \int_\Sigma \langle f(\xi), dm\rangle\, d\nu(\xi) < \infty$, we say that f is $(m \otimes \nu)$-integrable if for each $u \in X$, $v \in X'$, the scalar function $\langle fu, v\rangle : \xi \mapsto \langle f(\xi)u, v\rangle$ is ν-integrable and there exists $T \in \mathfrak{C}_1(X)$ such that

$$\int_\Xi \int_\Sigma \langle f(\xi), dm\rangle\, d\nu(\xi) = \int_\Sigma \langle T, dm\rangle , \tag{48}$$

$$\int_\Xi \langle f(\xi)u, v\rangle\, d\nu = \langle Tu, v\rangle \tag{49}$$

for all $u \in X$, $v \in X'$. Then we set

$$\int_{\Sigma\times\Xi} \langle f, d(m\otimes\nu)\rangle = \int_\Sigma \langle T, dm\rangle . \tag{50}$$

Because $\mathfrak{C}_1(X)$ is a lattice ideal, for each $A \in \mathscr{B}$, there exists a positive operator $\int_A f\, d\nu \in \mathfrak{C}_1(X)$ such that

$$\left\langle \left(\int_A f\, d\nu \right) u, v \right\rangle = \int_A \langle f(\xi)u, v\rangle\, d\nu \le \langle Tu, v\rangle , \tag{51}$$

for all $u \in X_+$, $v \in X'_+$.

Remark 8. For each $u \in X$, $v \in X'$, the tensor product $u \otimes v$ and $T \mapsto \int_\Sigma \langle T, dm\rangle$ are continuous linear functionals on $\mathfrak{C}_1(X)$, so it is natural to assume that both (48) and (49) hold.

The following statement is a consequence of the definitions.

Proposition 9. *Let $f : \Xi \to \mathscr{L}_+(X)$ be a positive operator valued function such that f is $(m \otimes \nu)$-integrable.*

Then $f(\xi) \in \mathfrak{C}_1(X)$ for ν-almost all $\xi \in \Xi$, the scalar valued function $\xi \mapsto \int_\Sigma \langle f(\xi), dm\rangle$ is ν-integrable, and the equalities

$$\int_{\Sigma\times\Xi} \langle f, d(m\otimes\nu)\rangle = \int_\Sigma \left\langle \int_\Xi f\, d\nu, dm \right\rangle \tag{52}$$

$$= \int_\Xi \int_\Sigma \langle f(\xi), dm\rangle\, d\nu(\xi) \tag{53}$$

hold. Moreover, $\int_\Sigma \langle \int_A f\, d\nu, dm\rangle = \int_A \int_\Sigma \langle f(\xi), dm\rangle\, d\nu(\xi)$ for every $A \in \mathscr{B}$.

Proof. Equation (52) is the definition of $\int_{\Sigma\times\Xi} \langle f, d(m \otimes \nu)\rangle$ and (53) is a reformulation of assumption (48). For ν-almost all $\xi \in \Xi$, we can find a martingale $\mathscr{F}_\xi$ and a regularisation $k_\xi(x, y)$, $x, y \in \Sigma$, of the kernel associated with $f(\xi)$ such that

$$\left\langle \left(\int_A f\, d\nu \right) u, v \right\rangle = \int_A \int_\Sigma \int_\Sigma k_\xi(x,y) u(x)\overline{v}(y)\, d\mu(x)\, d\mu(y)\, d\nu(\xi) \tag{54}$$

for all $A \in \mathscr{B}$ and $u \in X$, $v \in X'$. Then, for each $A \in \mathscr{B}$, we have

$$\begin{aligned} \int_\Sigma \left\langle \int_A f\, d\nu, dm \right\rangle &= \int_A \int_\Sigma k_\xi(x,x)\, d\mu(x)\, d\nu(\xi) \\ &= \int_A \int_\Sigma \langle f(\xi), dm\rangle\, d\nu(\xi) \end{aligned} \tag{55}$$

by the scalar Fubini-Tonelli Theorem. □

The following result follows from the observation in Theorem 1 that $\mathfrak{C}_1(X)$ is a lattice ideal and an application of monotone convergence.

Proposition 10. *Let $M : \mathscr{B} \to \mathscr{L}_+(X)$ be a positive operator valued measure on a measurable space $(\Xi, \mathscr{B})$. If $\int_\Sigma \langle M(\Xi), dm\rangle < \infty$, then the set function $\langle M, m\rangle : A \mapsto \int_\Sigma \langle M(A), dm\rangle$, $A \in \mathscr{B}$, is a finite measure such that*

$$\int_\Sigma \langle M(A), dm\rangle \le \int_\Sigma \langle M(\Xi), dm\rangle , \quad A \in \mathscr{B},$$

$$\int_\Sigma \langle M(f), dm\rangle = \int_\Xi f\, d\langle M, m\rangle \le \|f\|_\infty \int_\Sigma \langle M(\Xi), dm\rangle \tag{56}$$

for all $\mathscr{B}$-measurable $f : \Xi \to [0, \infty]$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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