Research Article

Differential Subordination with Generalized Derivative Operator of Analytic Functions

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Motivated by generalized derivative operator defined by the authors (El-Yagubi and Darus, 2013) and the technique of differential subordination, several interesting properties of the operator $\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b}$ are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

Also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ ($0 \leq \alpha < 1$) the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike of order α and convex of order α in $\mathbb{U}$.

$$\begin{aligned} \mathcal{S}^*(\alpha) &= \left\{ f \in \mathcal{A} : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in \mathbb{U} \right\}, \\ \mathcal{C}(\alpha) &= \left\{ f \in \mathcal{A} : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, z \in \mathbb{U} \right\}. \end{aligned} \quad (2)$$

Let $\mathcal{H}(\mathbb{U})$ be the class of holomorphic function in unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H}(\mathbb{U}), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, (z \in \mathbb{U}) \right\}. \quad (3)$$

Let two functions given by $f(z) = \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathbb{U}$. Then the Hadamard product (or convolution) $f * g$ of the two functions f, g is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (4)$$

Recall that the function f is subordinate to g if there exists the Schwarz function ω , analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. We denote this subordination by $f(z) < g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \rightarrow \mathbb{C}$ and h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (second order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) < h(z), \quad (z \in \mathbb{U}), \quad (5)$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p < q$ for all p satisfying (5).

A dominant $\bar{q}$ that satisfies $\bar{q} < q$ for all dominants q of (5) is said to be the best dominant of (5) (note that the best dominant is unique up to a rotation of $\mathbb{U}$).

In order to prove the original results we need the following lemmas.

Lemma 1 (see [1]). Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C} - \{0\}$ be a complex number with $\Re\{\gamma\} \geq 0$. If $p \in H[a; n]$ and

$$p(z) + \frac{zp'(z)}{\gamma} < h(z), \quad (z \in \mathbb{U}), \quad (6)$$

then

$$p(z) < q(z) < h(z), \quad (z \in \mathbb{U}), \quad (7)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{(\gamma/n)-1} dt, \quad (z \in \mathbb{U}). \quad (8)$$

The function q is convex and is the best dominant.

Lemma 2 (see [2]). Let g be a convex function in $\mathbb{U}$ and let

$$h(z) = g(z) + \alpha z g'(z), \quad (9)$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad (z \in \mathbb{U}) \quad (10)$$

is analytic in $\mathbb{U}$ and

$$p(z) + \alpha z p'(z) < h(z), \quad (z \in \mathbb{U}), \quad (11)$$

then

$$p(z) < g(z), \quad (12)$$

and this result is sharp.

Lemma 3 (see [3]). Let $f \in \mathcal{A}$; if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\frac{1}{2}, \quad (13)$$

then

$$\frac{2}{z} \int_0^z f(t) dt, \quad (z \in \mathbb{U}, z \neq 0) \quad (14)$$

belongs to the class of convex functions.

We now state the following generalized derivative operator [4]:

$$\begin{aligned} \mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \\ = z + \sum_{n=2}^{\infty} \left[\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{1 + \lambda_2(n-1) + b} \right]^m \mathcal{C}(\delta, n) a_n z^n, \end{aligned} \quad (15)$$

where $\lambda_2 \geq \lambda_1 \geq 0$, $\mathcal{C}(\delta, n) = (\delta + 1)_{n-1}/(n-1)!$, for $\delta, m, b \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $(x)_n$ is the Pochhammer symbol defined by

$$\begin{aligned} (x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)} \\ &= \begin{cases} 1, & n = 0, \\ x(x+1) \cdots (x+n-1), & n = \{1, 2, 3, \dots\}. \end{cases} \end{aligned} \quad (16)$$

Here $\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)$ can also be written in terms of convolution as

$$\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) = \mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \mathcal{D}_{\lambda_2}^{m, b}(z). \quad (17)$$

To prove our results, we need the following inclusion relation:

$$\begin{aligned} (1+b) \mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m+1, b} f(z) \\ = (1 - (\lambda_1 + \lambda_2) + b) \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right) \\ + (\lambda_1 + \lambda_2) z \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)', \end{aligned} \quad (18)$$

where $\varphi_{\lambda_2}^b(z)$ is analytic function given by $\varphi_{\lambda_2}^b(z) = z + \sum_{n=2}^{\infty} (z^n / (1 + \lambda_2(n-1) + b))$.

2. Main Results

In the present paper, we will use the method of differential subordination to derive certain properties of generalised derivative operator $\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)$. Note that differential subordination has been studied by various authors, and here we follow similar works done by Oros [5] and G. Oros and G. I. Oros [6].

Definition 4. For $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $0 \leq \alpha < 1$, let $\mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m, b}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \right)' > \alpha, \quad (z \in \mathbb{U}). \quad (19)$$

Also, let $\mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m, b}(\beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' > \beta, \quad (z \in \mathbb{U}). \quad (20)$$

Remark 5. It is clear that $\mathcal{R}_{\lambda_1, 0, 0}^{1, 0}(\alpha) \equiv \mathcal{R}(\lambda_1, \alpha)$, and the class of functions $f \in \mathcal{A}$ satisfy

$$\Re \left(\lambda_1 z f''(z) + f'(z) \right) > \alpha, \quad (z \in \mathbb{U}), \quad (21)$$

studied by Ponnusamy [7] and others.

Theorem 6. Let

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in \mathbb{U}), \quad (22)$$

be convex in $\mathbb{U}$, with $h(0) = 1$ and $0 \leq \alpha < 1$.

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m+1, b} f(z) \right)' < h(z), \quad (z \in \mathbb{U}), \quad (23)$$

then

$$\begin{aligned} \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ < q(z) \\ = 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1 + \lambda_2) z^{(1+b)/(\lambda_1 + \lambda_2)}} \sigma \left(\frac{1+b}{\lambda_1 + \lambda_2} \right), \end{aligned} \quad (24)$$

where σ is given by

$$\sigma(x) = \int_0^x \frac{t^{x-1}}{1+t} dt, \quad (z \in \mathbb{U}). \quad (25)$$

The function q is convex and is the best dominant.

Proof. By differentiating (18), with respect to z , we obtain

$$\begin{aligned} & (1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m+1, b} f(z) \right)' \\ &= (1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &+ (\lambda_1 + \lambda_2) z \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)'' \end{aligned} \quad (26)$$

Using (26) in (23), the differential subordination (23) becomes

$$\begin{aligned} & \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &+ \frac{(\lambda_1 + \lambda_2)}{1+b} z \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)'' \\ &< h(z) = \frac{1 + (2\alpha - 1)z}{1+z}. \end{aligned} \quad (27)$$

Let

$$\begin{aligned} & p(z) \\ &= \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{(1 + (\lambda_1 + \lambda_2)(n-1) + b)^m}{(1 + \lambda_2(n-1) + b)^{m+1}} \right) \mathcal{C}(\delta, n) a_n z^n \right)' \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1, 1], z \in \mathbb{U}). \end{aligned} \quad (28)$$

Using (28) in (27), the differential subordination becomes

$$p(z) + \frac{(\lambda_1 + \lambda_2)}{1+b} z p'(z) < h(z) = \frac{1 + (2\alpha - 1)z}{1+z}. \quad (29)$$

By using Lemma 1, we have

$$\begin{aligned} & p(z) < q(z) \\ &= \frac{1+b}{(\lambda_1 + \lambda_2) z^{(1+b)/(\lambda_1 + \lambda_2)}} \int_0^z h(t) t^{((1+b)/(\lambda_1 + \lambda_2)) - 1} dt \\ &= \frac{1+b}{(\lambda_1 + \lambda_2) z^{(1+b)/(\lambda_1 + \lambda_2)}} \\ &\quad \times \int_0^z \left(\frac{1 + (2\alpha - 1)t}{1+t} \right) t^{((1+b)/(\lambda_1 + \lambda_2)) - 1} dt \\ &= 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1 + \lambda_2) z^{(1+b)/(\lambda_1 + \lambda_2)}} \sigma \left(\frac{1+b}{\lambda_1 + \lambda_2} \right), \end{aligned} \quad (30)$$

where σ is given by (25); that is,

$$\begin{aligned} & \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &< q(z) \\ &= 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1 + \lambda_2) z^{(1+b)/(\lambda_1 + \lambda_2)}} \sigma \left(\frac{1+b}{\lambda_1 + \lambda_2} \right). \end{aligned} \quad (31)$$

The function q is convex and is the best dominant. The proof is complete. $\square$

Theorem 7. If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $0 \leq \alpha < 1$, then one has

$$\mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m+1, b}(\alpha) \subset \mathcal{K}_{\lambda_1, \lambda_2, \delta}^{m, b}(\beta), \quad (32)$$

where

$$\beta = 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1 + \lambda_2)} \sigma \left(\frac{1+b}{\lambda_1 + \lambda_2} \right), \quad (33)$$

and σ is given by (25).

Proof. Let $f \in \mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m+1, b}(\alpha)$, and then from (19) we have

$$\Re \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m+1, b} f(z) \right)' > \alpha, \quad (z \in \mathbb{U}), \quad (34)$$

which is equivalent to

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m+1, b} f(z) \right)' < h(z) = \frac{1 + (2\alpha - 1)z}{1+z}. \quad (35)$$

Using Theorem 6, we have

$$\begin{aligned} & \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &< q(z) \\ &= 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1 + \lambda_2) z^{(1+b)/(\lambda_1 + \lambda_2)}} \sigma \left(\frac{1+b}{\lambda_1 + \lambda_2} \right). \end{aligned} \quad (36)$$

Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis, we deduce that

$$\begin{aligned} & \Re \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &> \Re q(1) \\ &= \beta(\alpha, \lambda_1, \lambda_2, b) \\ &= 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1 + \lambda_2)} \sigma \left(\frac{1+b}{\lambda_1 + \lambda_2} \right), \end{aligned} \quad (37)$$

for which we deduce $\mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m+1, b}(\alpha) \subset \mathcal{K}_{\lambda_1, \lambda_2, \delta}^{m, b}(\beta)$. The proof is complete. $\square$

Theorem 8. Let q be a convex function in $\mathbb{U}$, with $q(0) = 1$, and let

$$h(z) = q(z) + \frac{(\lambda_1 + \lambda_2)}{1+b} z q'(z), \quad (z \in \mathbb{U}). \quad (38)$$

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m+1, b} f(z) \right)' < h(z), \quad (39)$$

then

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' < q(z), \quad (z \in \mathbb{U}), \quad (40)$$

and the result is sharp.

Proof. Using (28) in (26), the differential subordination (39) becomes

$$p(z) + \frac{(\lambda_1 + \lambda_2)}{1+b} z p'(z) < h(z) = q(z) + \frac{(\lambda_1 + \lambda_2)}{1+b} z q'(z) \quad (z \in \mathbb{U}). \quad (41)$$

Using Lemma 2, we have

$$p(z) < q(z), \quad (z \in \mathbb{U}); \quad (42)$$

that is,

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' < q(z), \quad (z \in \mathbb{U}), \quad (43)$$

and the result is sharp. The proof of Theorem 8 is complete. $\square$

Example 9. For $m = 1$, $\delta = 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $b \in \mathbb{N}_0$, $q(z) = (1+z)/(1-z)$, $f \in \mathcal{A}$, and $z \in \mathbb{U}$, by applying Theorem 8, we have

$$\begin{aligned} h(z) &= \frac{1+z}{1-z} + \frac{(\lambda_1 + \lambda_2)}{1+b} z \left(\frac{1+z}{1-z} \right)' \\ &= \frac{(1+b) + 2(\lambda_1 + \lambda_2)z - (1+b)z^2}{(1+b)(1-z)^2}. \end{aligned} \quad (44)$$

By using equality (18) we find that

$$\begin{aligned} (1+b) \mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} &= (1 - (\lambda_1 + \lambda_2) + b) \\ &\quad \times \left(f(z) * \varphi_{\lambda_2}^b(z) \right) + (\lambda_1 + \lambda_2) z \left(f(z) * \varphi_{\lambda_2}^b(z) \right)'. \end{aligned} \quad (45)$$

Now,

$$\begin{aligned} (1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} * \varphi_{\lambda_2}^b(z) \right) &= (1 - (\lambda_1 + \lambda_2) + b) \left(z + \sum_{n=2}^{\infty} \frac{a_n z^n}{(1 + \lambda_2(n-1) + b)^2} \right) \\ &\quad + (\lambda_1 + \lambda_2) \left(z + \sum_{n=2}^{\infty} \frac{n a_n z^n}{(1 + \lambda_2(n-1) + b)^2} \right). \end{aligned} \quad (46)$$

A straightforward calculation gives the following:

$$\begin{aligned} &\left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &= 1 + \sum_{n=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{(1 + \lambda_2(n-1) + b)^2} \right) n a_n z^{n-1} \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{n(1 + (\lambda_1 + \lambda_2)(n-1) + b)}{(1 + \lambda_2(n-1) + b)^2} \right) a_n z^n \right) \\ &\quad \times (z)^{-1} \\ &= \left(\left(f(z) * \varphi_{\lambda_2}^b(z) \right) * \left(z + \sum_{n=2}^{\infty} \left(\frac{n(1 + (\lambda_1 + \lambda_2)(n-1) + b)}{(1 + \lambda_2(n-1) + b)} \right) a_n z^n \right) \right) \\ &\quad \times (z)^{-1}. \end{aligned} \quad (47)$$

Similarly, using (18), we see that

$$\begin{aligned} (1+b) \mathcal{D}_{\lambda_1, \lambda_2, 0}^{2, b} f(z) &= (1 - (\lambda_1 + \lambda_2) + b) \\ &\quad \times \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) * \varphi_{\lambda_2}^b(z) \right) \\ &\quad + (\lambda_1 + \lambda_2) z \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) * \varphi_{\lambda_2}^b(z) \right)', \end{aligned} \quad (48)$$

and then

$$\begin{aligned} (1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{2, b} f(z) \right)' &= (1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) * \varphi_{\lambda_2}^b(z) \right)' \\ &\quad + (\lambda_1 + \lambda_2) z \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) * \varphi_{\lambda_2}^b(z) \right)'' . \end{aligned} \quad (49)$$

By using (47) we have

$$\begin{aligned} &\left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) * \varphi_{\lambda_2}^b(z) \right)'' \\ &= \sum_{n=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{(1 + \lambda_2(n-1) + b)^2} \right) n(n-1) a_n z^{n-2}; \end{aligned} \quad (50)$$

we deduce

$$\begin{aligned} &(1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{2, b} f(z) \right)' \\ &= (1+b) \left(1 + \sum_{n=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{(1 + \lambda_2(n-1) + b)^2} \right) n a_n z^{n-1} \right) \\ &\quad + (\lambda_1 + \lambda_2) \\ &\quad \times \sum_{n=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{(1 + \lambda_2(n-1) + b)^2} \right) n(n-1) a_n z^{n-1}; \end{aligned} \quad (51)$$

that is,

$$\begin{aligned}
 & \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{2, b} f(z) \right)' \\
 &= 1 + \sum_{n=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{1 + \lambda_2(n-1) + b} \right)^2 n a_n z^{n-1} \\
 &= \left((f(z) * \varphi_{\lambda_2}^b(z)) \right. \\
 &\quad \left. * \left(z + \sum_{n=2}^{\infty} \left(\frac{n(1 + (\lambda_1 + \lambda_2)(n-1) + b)^2}{(1 + \lambda_2(n-1) + b)} \right) a_n z^n \right) \right) \\
 &\quad \times (z)^{-1}.
 \end{aligned} \tag{52}$$

From Theorem 8 we get

$$\begin{aligned}
 & \left((f(z) * \varphi_{\lambda_2}^b(z)) \right. \\
 &\quad \left. * \left(z + \sum_{n=2}^{\infty} \left(\frac{n(1 + (\lambda_1 + \lambda_2)(n-1) + b)^2}{(1 + \lambda_2(n-1) + b)} \right) a_n z^n \right) \right) \times (z)^{-1} \\
 &\quad < \frac{(1+b) + 2(\lambda_1 + \lambda_2)z - (1+b)z^2}{(1+b)(1-z)^2},
 \end{aligned} \tag{53}$$

which implies that

$$\begin{aligned}
 & \left((f(z) * \varphi_{\lambda_2}^b(z)) \right. \\
 &\quad \left. * \left(z + \sum_{n=2}^{\infty} \left(\frac{n(1 + (\lambda_1 + \lambda_2)(n-1) + b)}{(1 + \lambda_2(n-1) + b)} \right) a_n z^n \right) \right) \times (z)^{-1} \\
 &\quad < \frac{1+z}{1-z}, \quad (z \in \mathbb{U}).
 \end{aligned} \tag{54}$$

Theorem 10. Let q be a convex function in $\mathbb{U}$, with $q(0) = 1$, and let

$$h(z) = q(z) + zp'(z), \quad (z \in \mathbb{U}). \tag{55}$$

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \right)' < h(z), \tag{56}$$

then

$$\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} < q(z), \quad (z \in \mathbb{U}), \tag{57}$$

and the result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z}, \quad (z \in \mathbb{U}). \tag{58}$$

Differentiating (58), with respect to z , we obtain

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \right)' = p(z) + zp'(z), \quad (z \in \mathbb{U}). \tag{59}$$

Using (58), the differential subordination (56) becomes

$$p(z) + zp'(z) < h(z) = q(z) + zq'(z), \quad (z \in \mathbb{U}). \tag{60}$$

Using Lemma 2, we deduce that

$$p(z) < q(z), \quad (z \in \mathbb{U}). \tag{61}$$

By using (58), we have

$$\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} < q(z), \quad (z \in \mathbb{U}). \tag{62}$$

The proof of Theorem 10 is complete. $\square$

Example 11. For $\delta = 0$, $m = 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $b \in \mathbb{N}_0$, $q(z) = 1/(1-z)$, $f \in \mathcal{A}$, and $z \in \mathbb{U}$, from Theorem 10 we obtain

$$h(z) = \frac{1}{1-z} + z \left(\frac{1}{1-z} \right)' = \frac{1}{(1-z)^2}. \tag{63}$$

From Example 9, we have

$$\begin{aligned}
 & (1+b) \mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) \\
 &= (1 - (\lambda_1 + \lambda_2) + b) (f(z) * \varphi_{\lambda_2}^b(z)) \\
 &\quad + (\lambda_1 + \lambda_2) z (f(z) * \varphi_{\lambda_2}^b(z))',
 \end{aligned} \tag{64}$$

and then

$$\begin{aligned}
 (1+b) \left(\mathcal{D}_{\lambda_1, \lambda_2, 0}^{1, b} f(z) \right)' &= (1+b) \left(f(z) * \varphi_{\lambda_2}^b(z) \right)' \\
 &\quad + (\lambda_1 + \lambda_2) z \left(f(z) * \varphi_{\lambda_2}^b(z) \right)''.
 \end{aligned} \tag{65}$$

From Theorem 10 we deduce that

$$\left(f(z) * \varphi_{\lambda_2}^b(z) \right)' + \frac{(\lambda_1 + \lambda_2)}{(1+b)} z \left(f(z) * \varphi_{\lambda_2}^b(z) \right)'' < \frac{1}{(1-z)^2} \tag{66}$$

implies that

$$\begin{aligned}
 & \frac{(1 - (\lambda_1 + \lambda_2) + b) (f(z) * \varphi_{\lambda_2}^b(z)) + (\lambda_1 + \lambda_2) z (f(z) * \varphi_{\lambda_2}^b(z))'}{z(1+b)} \\
 &\quad < \frac{1}{1-z}.
 \end{aligned} \tag{67}$$

Theorem 12. Let h be a convex function in $\mathbb{U}$, with $h(0) = 1$, $0 \leq \alpha < 1$, and let

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in \mathbb{U}). \quad (68)$$

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \right)' < h(z), \quad (69)$$

then

$$\begin{aligned} \frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} &< q(z) = 2\alpha - 1 \\ &+ \frac{2(1 - \alpha) \ln(1 + z)}{z}, \quad (z \in \mathbb{U}). \end{aligned} \quad (70)$$

The function q is convex and is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{(1 + (\lambda_1 + \lambda_2)(n - 1) + b)^m}{(1 + \lambda_2(n - 1) + b)^m} \right) \right. \\ &\quad \left. \times \mathcal{E}(\delta, n) a_n z^n \right) \times (z)^{-1} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1, 1], z \in \mathbb{U}). \end{aligned} \quad (71)$$

Differentiating (71), with respect to z , we obtain

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \right)' = p(z) + zp'(z), \quad (z \in \mathbb{U}). \quad (72)$$

Using (72), the differential subordination (69) becomes

$$p(z) + zp'(z) < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in \mathbb{U}). \quad (73)$$

Using Lemma 1, we deduce that

$$\begin{aligned} p(z) &< q(z) = \frac{1}{z} \int_0^z h(t) dt \\ &= \frac{1}{z} \int_0^z \left(\frac{1 + (2\alpha - 1)t}{1 + t} \right) dt \\ &= \frac{1}{z} \left(\int_0^z \frac{1}{1 + t} dt + (2\alpha - 1) \int_0^z \frac{t}{1 + t} dt \right) \\ &= 2\alpha - 1 + \frac{2(1 - \alpha) \ln(1 + z)}{z}. \end{aligned} \quad (74)$$

By using (71), we have

$$\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} < q(z) = 2\alpha - 1 + \frac{2(1 - \alpha) \ln(1 + z)}{z}. \quad (75)$$

The proof of Theorem 12 is complete. $\square$

Corollary 13. If $f \in \mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m, b}(\alpha)$, then

$$\Re \left(\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} \right) > (2\alpha - 1) + 2(1 - \alpha) \ln 2, \quad (z \in \mathbb{U}). \quad (76)$$

Proof. Since $f \in \mathcal{R}_{\lambda_1, \lambda_2, \delta}^{m, b}(\alpha)$, from Definition 4 we have

$$\Re \left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(\alpha) \right)' > \alpha, \quad (z \in \mathbb{U}), \quad (77)$$

which is equivalent to

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(\alpha) \right)' < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}. \quad (78)$$

Using Theorem 12, we obtain

$$\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} < q(z) = 2\alpha - 1 + \frac{2(1 - \alpha) \ln(1 + z)}{z}. \quad (79)$$

Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis, we have that

$$\begin{aligned} \Re \left(\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} \right) &> \Re q(1) = (2\alpha - 1) \\ &+ 2(1 - \alpha) \ln 2, \quad (z \in \mathbb{U}). \end{aligned} \quad (80)$$

$\square$

Theorem 14. Let $h \in \mathcal{H}(\mathbb{U})$, with $h(0) = 1$, $h'(0) \neq 0$, which satisfies the inequality

$$\Re \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \quad (z \in \mathbb{U}). \quad (81)$$

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z) \right)' < h(z), \quad (82)$$

then

$$\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t) dt. \quad (83)$$

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m, b} f(z)}{z} \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{(1 + (\lambda_1 + \lambda_2)(n - 1) + b)^m}{(1 + \lambda_2(n - 1) + b)^m} \right) \right. \\ &\quad \left. \times \mathcal{E}(\delta, n) a_n z^n \right) \times (z)^{-1} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1, 1], z \in \mathbb{U}). \end{aligned} \quad (84)$$

Differentiating (84), with respect to z , we obtain

$$\left(\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m,b} f(z)\right)' = p(z) + zp'(z), \quad (z \in \mathbb{U}). \quad (85)$$

Using (85), the differential subordination (82) becomes

$$p(z) + zp'(z) < h(z), \quad (z \in \mathbb{U}). \quad (86)$$

Using Lemma 1, we deduce that

$$p(z) < q(z) = \frac{1}{z} \int_0^z h(t) dt; \quad (87)$$

by using (84), we have

$$\frac{\mathcal{D}_{\lambda_1, \lambda_2, \delta}^{m,b} f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t) dt. \quad (88)$$

From Lemma 3, we see that the function q is convex, and from Lemma 1, q is the best dominant for subordination (82). The proof of Theorem 14 is complete. $\square$

Note that other work related to differential operators and differential subordination can be seen in [8–13].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

Entisar El-Yagubi and Maslina Darus read and approved the final paper.

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