Research Article

Periodic Boundary Value Problems for a Class of Impulsive Functional Differential Equations of Hybrid Type

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This paper is concerned with the existence of extreme solutions of periodic boundary value problems for a class of first-order impulsive functional differential equations of hybrid type. We obtain the sufficient conditions for existence of extreme solutions by using upper and lower solutions method coupled with monotone iterative technique.

1. Introduction

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses but also represents a more natural framework for mathematical modeling of many real-world phenomena [1–3]. Significant progress has been made in the theory of systems of impulsive differential equations in recent years (see [4–18] and the references cited therein). It is well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations; for details, see [19] and the references therein. There also exist several works devoted to the applications of this technique to periodic boundary value problems of impulsive differential equations; see [20–26]. In [27, 28], the authors introduce a new concept of upper and lower solutions for periodic boundary value problems of a class of first-order functional differential equations. In paper [23], the authors applied this new concept to study the periodic boundary value problems for first-order impulsive functional differential equations. Motivated by [23, 27, 28], we will study periodic boundary value problem for the first-order impulsive functional differential equation of hybrid type

$$\begin{aligned} x'(t) &= f(t, x(t), x(\theta(t)), (Kx)(t), (Hx)(t)), \\ t &\neq t_k, \quad t \in J = [0, T], \quad T > 0, \end{aligned}$$

$$\begin{aligned} \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) &= x(T), \end{aligned} \quad (1)$$

where $f \in C(J \times R^4, R)$, $\theta \in C(J, J)$, $I_k \in C(R, R)$, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Denote $J' = J \setminus \{t_1, t_2, \dots, t_m\}$. The integral part in (1) is defined by

$$\begin{aligned} (Kx)(t) &= \int_0^{\gamma(t)} k(t, s) x(\delta(s)) ds, \\ (Hx)(t) &= \int_0^T h(t, s) x(\sigma(s)) ds, \end{aligned} \quad (2)$$

where $\gamma, \delta, \sigma \in C(J, J)$, $k \in C(D, R^+)$, $D = \{(t, s) \in J \times J : 0 \leq s \leq \gamma(t) \leq T\}$, $h \in C(J \times J, R^+)$, $R^+ = [0, \infty)$, $K_0 = \max\{k(t, s) : (t, s) \in D\}$, and $H_0 = \max\{h(t, s) : (t, s) \in J \times J\}$.

Let $PC(J) = \{u : J \rightarrow R, u \text{ is continuous for } t \in J, t \neq t_k, u(t_i^+), \text{ and } u(t_i^-) \text{ exist, and } u(t_i^-) = u(t_i), i = 1, 2, \dots, m\}$. $PC^1(J) = \{u \in PC(J) : u \text{ is continuously differentiable for}$

$t \in J, t \neq t_k$. $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\begin{aligned} \|u\|_{PC(J)} &= \sup \{|u(t)| : t \in J\}, \\ \|u\|_{PC^1(J)} &= \max \left\{ \|u\|_{PC(J)}, \|u'\|_{PC(J)} \right\}. \end{aligned} \quad (3)$$

By a solution of (1), we mean a $u \in PC^1(J)$ for which problem (1) is satisfied.

Note that (1) has a very general form; as special instances resulting from (1), one can have impulsive differential equations with deviating arguments and impulsive differential equations with the Volterra or Fredholm operators. For example, if f does not include Kx and Hx , then (1) reduces to

$$\begin{aligned} x'(t) &= f(t, x(t), x(\theta(t))), \\ t \neq t_k, \quad t \in J = [0, T], \quad T > 0, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) &= x(T), \end{aligned} \quad (4)$$

periodic boundary problem for impulsive differential equations with deviating arguments, which is discussed in [22, 23]; when $\theta(t) = \gamma(t) = \delta(t) = \sigma(t) \equiv t$, (1) is the following periodic boundary problem for impulsive integrodifferential equations of mixed type:

$$\begin{aligned} x'(t) &= f(t, x(t), (Kx)(t), (Hx)(t)), \\ t \neq t_k, \quad t \in J = [0, T], \quad T > 0, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) &= x(T); \end{aligned} \quad (5)$$

similar problems are also discussed in [24–26].

2. Preliminaries

To apply the method of upper and lower solutions, we need the concept of lower solution and upper solution for (1).

Definition 1. A function $\alpha \in PC^1(J)$ is a lower solution of (1) if there exist $M_1 > 0$, $M_i \geq 0$, $i = 2, 3, 4$, and $0 \leq L_k < 1$, $k = 1, 2, \dots, m$, such that

$$\begin{aligned} \alpha'(t) &\leq f(t, \alpha(t), \alpha(\theta(t)), (K\alpha)(t), (H\alpha)(t)) - a_\alpha(t), \\ t \neq t_k, \quad t \in J, \\ \Delta \alpha(t_k) &\leq I_k(\alpha(t_k)) - L_k a_{\alpha k}, \quad k = 1, 2, \dots, m; \end{aligned} \quad (6)$$

similarly, a function $\beta \in PC^1(J)$ is an upper solution of (1) if

$$\begin{aligned} \beta'(t) &\geq f(t, \beta(t), \beta(\theta(t)), (K\beta)(t), (H\beta)(t)) + a_\beta(t), \\ t \neq t_k, \quad t \in J, \\ \Delta \beta(t_k) &\geq I_k(\beta(t_k)) + L_k a_{\beta k}, \quad k = 1, 2, \dots, m, \end{aligned} \quad (7)$$

where

$$\begin{aligned} a_\alpha(t) &= \begin{cases} 0, & \text{if } \alpha(0) \leq \alpha(T), \\ \left(M_1 t + M_2 \theta(t) \right. \\ \quad \left. + M_3 K_0 \int_0^{\gamma(t)} \delta(s) ds \right. \\ \quad \left. + M_4 H_0 \int_0^T \sigma(s) ds + 1 \right) \\ \quad \times (T)^{-1} [\alpha(0) - \alpha(T)], & \text{if } \alpha(0) > \alpha(T), \end{cases} \\ a_{\alpha k} &= \begin{cases} 0, & \text{if } \alpha(0) \leq \alpha(T), \\ \frac{t_k}{T} [\alpha(0) - \alpha(T)], & \text{if } \alpha(0) > \alpha(T), \end{cases} \\ a_\beta(t) &= \begin{cases} 0, & \text{if } \beta(0) \geq \beta(T), \\ \left(M_1 t + M_2 \theta(t) \right. \\ \quad \left. + M_3 K_0 \int_0^{\gamma(t)} \delta(s) ds \right. \\ \quad \left. + M_4 H_0 \int_0^T \sigma(s) ds + 1 \right) \\ \quad \times (T)^{-1} [\beta(T) - \beta(0)], & \text{if } \beta(0) < \beta(T), \end{cases} \\ a_{\beta k} &= \begin{cases} 0, & \text{if } \beta(0) \geq \beta(T), \\ \frac{t_k}{T} [\beta(T) - \beta(0)], & \text{if } \beta(0) < \beta(T). \end{cases} \end{aligned} \quad (8)$$

In what follows, we define the set

$$[u, v] = \{w \in PC(J) : u(t) \leq w(t) \leq v(t), t \in J\} \quad (9)$$

for $u, v \in PC(J)$ and $u(t) \leq v(t)$, $t \in J$.

For the sake of convenience, we list the following conditions.

(H₁) Assume that $\alpha(t) \leq \beta(t)$ for $t \in J$, and there exist $M_1 > 0$ and $M_i \geq 0$, $i = 2, 3, 4$, such that

$$\begin{aligned} &f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4) \\ &\geq -M_1(x_1 - y_1) - M_2(x_2 - y_2) \\ &\quad - M_3(x_3 - y_3) - M_4(x_4 - y_4), \end{aligned} \quad (10)$$

for $\alpha(t) \leq y_1 \leq x_1 \leq \beta(t)$, $\alpha(\theta(t)) \leq y_2 \leq x_2 \leq \beta(\theta(t))$, $(K\alpha)(t) \leq y_3 \leq x_3 \leq (K\beta)(t)$, and $(H\alpha)(t) \leq y_4 \leq x_4 \leq (H\beta)(t)$, $t \in J$.

(H₂) There exists L_k with $0 \leq L_k < 1$, $k = 1, 2, \dots, m$, such that

$$\begin{aligned} &I_k(x) - I_k(u) \geq -L_k(x - u), \\ &\text{for } \alpha(t_k) \leq u(t_k) \leq x(t_k) \leq \beta(t_k), \\ &k = 1, 2, \dots, m. \end{aligned} \quad (11)$$

Lemma 2 (see [5]). Assume that $s \in [0, T]$, $x \in PC^1(J)$, $p, q \in PC(J)$, $a_i \geq 0$, b_i , $i = 1, 2, \dots, m$, are constants such that

$$\begin{aligned} x'(t) &\leq p(t)x(t) + q(t), \quad t \in [s, T], \quad t \neq t_k, \\ x(t_k^+) &\leq a_k x(t_k) + b_k, \quad t_k \in [s, T], \quad k = 1, 2, \dots, m; \end{aligned} \quad (12)$$

then

$$\begin{aligned} x(t) &\leq x(s^+) \left(\prod_{s < t_k < t} a_k \right) \exp \left(\int_s^t p(u) du \right) \\ &+ \sum_{s < t_k < t} \left(\prod_{t_k < t_i < t} a_i \right) \exp \left(\int_{t_k}^t p(\tau) d\tau \right) b_k \\ &+ \int_s^t \left(\prod_{u < t_k < t} a_k \right) \exp \left(\int_u^t p(\tau) d\tau \right) q(u) du, \\ &t \in [s, T]. \end{aligned} \quad (13)$$

Lemma 3. Assume that $x \in PC^1(J)$, $M_1 > 0$, $M_i \geq 0$, $i = 2, 3, 4$, and $0 \leq L_k < 1$, ($k = 1, 2, \dots, m$) such that

$$\begin{aligned} x'(t) &\leq -M_1 x(t) - M_2 x(\theta(t)) - M_3 (Kx)(t) \\ &- M_4 (Hx)(t) - a_x(t), \quad t \in J, \quad t \neq t_k, \end{aligned} \quad (14)$$

$$\Delta x(t_k) \leq -L_k x(t_k) - L_k a_{xk}, \quad k = 1, 2, \dots, m,$$

where

$$\begin{aligned} a_x(t) &= \begin{cases} 0, & \text{if } x(0) \leq x(T), \\ \left(M_1 t + M_2 \theta(t) \right. \\ \quad \left. + M_3 K_0 \int_0^{\gamma(t)} \delta(s) ds \right. \\ \quad \left. + M_4 H_0 \int_0^{\sigma(t)} \sigma(s) ds + 1 \right) \\ \quad \times (T)^{-1} [x(0) - x(T)], & \text{if } x(0) > x(T), \end{cases} \\ a_{xk} &= \begin{cases} 0, & \text{if } x(0) \leq x(T), \\ \frac{t_k}{T} [x(0) - x(T)], & \text{if } x(0) > x(T). \end{cases} \end{aligned} \quad (15)$$

If

$$\begin{aligned} &M_2 \int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\ &+ M_3 \int_0^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\ &\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\ &+ M_4 \int_0^T \left(\int_0^{\sigma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\ &\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt \\ &\leq e^{-M_1 T} \prod_{j=1}^m (1 - L_k)^2, \end{aligned} \quad (16)$$

then $x(t) \leq 0$ on J .

Proof.

Case 1 ($x(0) \leq x(T)$). Let $v(t) = x(t)e^{M_1 t}$; then $v \in PC^1(J)$ and

$$\begin{aligned} v'(t) &\leq -M_2 e^{M_1(t-\theta(t))} v(\theta(t)) - M_3 (K_1 v)(t) \\ &- M_4 (H_1 v)(t), \quad t \in J', \\ \Delta v(t_k) &\leq -L_k v(t_k), \quad k = 1, 2, \dots, m, \\ v(0) &\leq v(T) e^{-M_1 T}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} (K_1 v)(t) &= \int_0^{\gamma(t)} k_1(t, s) v(\delta(s)) ds, \\ (H_1 v)(t) &= \int_0^{\sigma(t)} h_1(t, s) v(\sigma(s)) ds, \\ k_1(t, s) &= e^{M_1(t-\delta(s))} k(t, s), \\ h_1(t, s) &= e^{M_1(t-\sigma(s))} h(t, s). \end{aligned} \quad (18)$$

Obviously, $v(t) \leq 0$ implies that $x(t) \leq 0$. To show $v(t) \leq 0$, suppose, on the contrary, that $v(\tilde{t}) > 0$ for some $\tilde{t} \in J$. Then there are two possible cases:

(a) $v(t) \geq 0$ for all $t \in J$;

(b) there exists $t_1^* \in J$ such that $v(t_1^*) < 0$.

In case (a), (17) implies that $v'(t) \leq 0$ for $t \in J'$ and $\Delta v(t_k) \leq 0$ ($k = 1, 2, \dots, m$); hence, $v(t)$ is nonincreasing in J ; then $v(0) \geq v(\tilde{t}) > 0$ and $v(0) \geq v(T)$. On the other hand, $v(0) = x(0) \leq x(T) < x(T)e^{M_1 T} = v(T)$, which is a contradiction.

In case (b), let $t^* = \max\{t \in J, \inf_{s \in J} v(s) = v(t) = -\lambda, \lambda > 0\}$. We only consider $t^* \in (t_{p-1}, t_p]$ for some $p \in \{1, 2, \dots, m+1\}$, as the proof is similar for the case $t^* = t_{p-1}^+$. Next, we consider two subcases.

Subcase I ($v(T) \geq 0$). From (17), we get

$$\begin{aligned} v'(t) &\leq \lambda M_2 e^{M_1(t-\theta(t))} + \lambda M_3 \int_0^{\gamma(t)} k_1(t, s) ds \\ &+ \lambda M_4 \int_0^{\sigma(t)} h_1(t, s) ds. \end{aligned} \quad (19)$$

From Lemma (14), we have

$$\begin{aligned}
v(T) &\leq v(t^*) \prod_{t^* < t_k < T} (1 - L_k) \\
&+ \lambda M_2 \int_{t^*}^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
&+ \lambda M_3 \int_{t^*}^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
&+ \lambda M_4 \int_{t^*}^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt \\
&\leq -\lambda \prod_{k=p}^m (1 - L_k) \\
&+ \lambda M_2 \int_{t^*}^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
&+ \lambda M_3 \int_{t^*}^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
&+ \lambda M_4 \int_{t^*}^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt;
\end{aligned} \tag{20}$$

hence,

$$\begin{aligned}
v(T) &< -\lambda \left(\prod_{k=1}^m (1 - L_k) \right) \\
&+ \lambda M_2 \int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
&+ \lambda M_3 \int_0^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt
\end{aligned}$$

$$\begin{aligned}
&+ \lambda M_4 \int_0^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
\end{aligned} \tag{21}$$

This yields

$$\begin{aligned}
\prod_{k=1}^m (1 - L_k) &< M_2 \int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
&+ M_3 \int_0^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
&+ M_4 \int_0^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
\end{aligned} \tag{22}$$

It contradicts (16).

Subcase II ($v(T) < 0$). If $t^* < \tilde{t}$, similar to the subcase I, it also yields a contradiction. We assume without loss of generality that $t^* > \tilde{t}$ and $\tilde{t} \in (t_q, t_{q+1}]$ for some $q \in \{0, 1, 2, \dots, m-1\}$. By Lemma 2, we have

$$\begin{aligned}
v(\tilde{t}) &\leq v(0) \prod_{0 < t_k < \tilde{t}} (1 - L_k) \\
&+ \lambda M_2 \int_0^{\tilde{t}} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
&+ \lambda M_3 \int_0^{\tilde{t}} \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
&+ \lambda M_4 \int_0^{\tilde{t}} \left(\int_0^T \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right. \\
&\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
\end{aligned} \tag{23}$$

Noting that $v(0) \leq v(T)e^{-M_1 T}$, we get

$$\begin{aligned}
 0 < v(\tilde{t}) &\leq v(T) e^{-M_1 T} \prod_{K=1}^q (1 - L_k) \\
 &+ \lambda M_2 \int_0^{\tilde{t}} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
 &+ \lambda M_3 \int_0^{\tilde{t}} \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
 &+ \lambda M_4 \int_0^{\tilde{t}} \left(\int_0^T \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
 \end{aligned} \tag{24}$$

Hence,

$$\begin{aligned}
 v(T) \prod_{K=1}^q (1 - L_k) \\
 &\geq -\lambda M_2 e^{M_1 T} \int_0^{\tilde{t}} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
 &- \lambda M_3 e^{M_1 T} \int_0^{\tilde{t}} \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
 &- \lambda M_4 e^{M_1 T} \int_0^{\tilde{t}} \left(\int_0^T \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
 \end{aligned} \tag{25}$$

From (20) and (25), we obtain

$$\begin{aligned}
 \prod_{k=p}^m (1 - L_k) \left(\prod_{j=1}^q (1 - L_k) \right) \\
 &\leq M_2 e^{M_1 T} \int_0^{\tilde{t}} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
 &+ M_3 e^{M_1 T} \int_0^{\tilde{t}} \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < \tilde{t}} (1 - L_k) \right) \right.
 \end{aligned}$$

$$\begin{aligned}
 &\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt \\
 &+ M_2 \prod_{j=1}^q (1 - L_k) \int_{t^*}^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
 &+ M_3 \prod_{j=1}^q (1 - L_k) \int_{t^*}^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
 &+ M_4 \prod_{j=1}^q (1 - L_k) \int_{t^*}^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
 \end{aligned} \tag{26}$$

By multiplying $\prod_{j=q+1}^m (1 - L_k)$ on both sides of (26), we have

$$\begin{aligned}
 \prod_{k=p}^m (1 - L_k) \left(\prod_{j=1}^m (1 - L_k) \right) \\
 &\leq M_2 e^{M_1 T} \int_0^{\tilde{t}} \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
 &+ M_3 e^{M_1 T} \int_0^{\tilde{t}} \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
 &+ M_4 e^{M_1 T} \int_0^{\tilde{t}} \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt \\
 &+ M_2 \prod_{j=1}^m (1 - L_k) \int_{t^*}^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
 &+ M_3 \prod_{j=1}^m (1 - L_k) \int_{t^*}^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
 &\quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt
 \end{aligned}$$

$$\begin{aligned}
& + M_4 \prod_{j=1}^m (1 - L_k) \int_{t^*}^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
& \quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
\end{aligned} \tag{27}$$

Noting that $\prod_{j=1}^m (1 - L_k) < 1$, $e^{M_1 T} > 1$, we get

$$\begin{aligned}
& \prod_{k=p}^m (1 - L_k) \left(\prod_{j=1}^m (1 - L_k) \right) \\
& < M_2 e^{M_1 T} \int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
& + M_3 e^{M_1 T} \int_0^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
& \quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
& + M_4 e^{M_1 T} \int_0^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
& \quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt.
\end{aligned} \tag{28}$$

This implies

$$\begin{aligned}
& e^{-M_1 T} \prod_{j=1}^m (1 - L_k)^2 \\
& < M_2 \int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) e^{M_1(t-\theta(t))} dt \\
& + M_3 \int_0^T \left(\int_0^{\gamma(t)} \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
& \quad \left. \times e^{M_1(t-\delta(s))} k(t, s) ds \right) dt \\
& + M_4 \int_0^T \left(\int_0^T \left(\prod_{t < t_k < T} (1 - L_k) \right) \right. \\
& \quad \left. \times e^{M_1(t-\sigma(s))} h(t, s) ds \right) dt,
\end{aligned} \tag{29}$$

which contradicts (16). This completes the proof for the case $x(0) \leq x(T)$.

Case 2 ($x(0) > x(T)$). Let $v(t) = x(t) + (t/T)[x(0) - x(T)]$; then $v(0) = x(0) = v(T)$, and

$$\begin{aligned}
& v'(t) + M_1 v(t) + M_2 v(\theta(t)) \\
& + M_3 (Kv)(t) + M_4 (Hv)(t) \\
& \leq x'(t) + M_1 x(t) + M_2 x(\theta(t)) \\
& + M_3 (Kx)(t) + M_4 (Hx)(t) \\
& + \left(M_1 t + M_2 \theta(t) + M_3 K_0 \int_0^{\gamma(t)} \delta(s) ds \right. \\
& \quad \left. + M_4 H_0 \int_0^T \sigma(s) ds + 1 \right) \times (T)^{-1} \\
& \times [x(0) - x(T)] \leq 0, \quad t \neq t_k, \quad t \in J, \\
& \Delta v(t_k) = \Delta x(t_k) \leq -L_k x(t_k) - L_k \frac{t_k}{T} [x(0) - x(T)] \\
& = -L_k v(t_k), \quad k = 1, 2, \dots, m.
\end{aligned} \tag{30}$$

In view of Case 1, we have $v(t) \leq 0$, for $t \in J$. Therefore, $x(t) \leq 0$ for $t \in J$. The proof of Lemma 3 is complete. $\square$

Lemma 4. Assume that $x \in PC^1(J)$, $M_1 > 0$, $M_i \geq 0$, $i = 2, 3, 4$, $\eta \in PC(J)$, and $0 \leq L_k < 1$, $k = 1, 2, \dots, m$. If

$$\left(M_2 T + M_3 T^2 K_0 + M_4 T^2 H_0 + \sum_{k=1}^m L_k \right) \frac{e^{M_1 T}}{e^{M_1 T} - 1} < 1, \tag{31}$$

then the impulsive differential problem

$$\begin{aligned}
& x'(t) = -M_1 x(t) - M_2 x(\theta(t)) - M_3 (Kx)(t) \\
& \quad - M_4 (Hx)(t) + \eta(t), \quad t \in J, \quad t \neq t_k, \\
& \Delta x(t_k) = -L_k x(t_k) + d_k, \quad d_k \in R, \quad k = 1, 2, \dots, m, \\
& x(0) = x(T),
\end{aligned} \tag{32}$$

has a unique solution.

Proof. Define a map $A : PC(J) \rightarrow PC(J)$ by

$$\begin{aligned}
[Ax](t) &= \int_0^T G(t, s) (\eta(s) - M_2 x(\theta(s)) \\
& \quad - M_3 (Kx)(s) - M_4 (Hx)(s)) ds \\
& + \sum_{k=1}^m G(t, t_k) (-L_k x(t_k) + d_k),
\end{aligned} \tag{33}$$

where

$$G(t, s) = \begin{cases} \frac{e^{-M_1(t-s)}}{1 - e^{-M_1 T}}, & 0 \leq s < t \leq T, \\ \frac{e^{-M_1(T+t-s)}}{1 - e^{-M_1 T}}, & 0 \leq t \leq s \leq T. \end{cases} \quad (34)$$

It is easy to verify that x is a solution of (32) if and only if x is a fixed point of A . For any $x, y \in PC(J)$, we have

$$\begin{aligned} & |[Ax](t) - [Ay](t)| \\ & \leq \left| M_2 \int_0^T G(t, s) (y(\theta(s)) - x(\theta(s))) ds \right| \\ & \quad + \left| M_3 \int_0^T \int_0^{\gamma(s)} G(t, s) k(s, u) \right. \\ & \quad \quad \left. \times (y(\delta(u)) - x(\delta(u))) du ds \right| \\ & \quad + \left| M_4 \int_0^T \int_0^T G(t, s) h(s, u) \right. \\ & \quad \quad \left. \times (y(\sigma(u)) - x(\sigma(u))) du ds \right| \\ & \quad + \left| \sum_{k=1}^m G(t, t_k) L_k(y(t_k) - x(t_k)) \right| \\ & \leq \left(M_2 T + M_3 T^2 K_0 + M_4 T^2 H_0 + \sum_{k=1}^m L_k \right) \\ & \quad \times \frac{e^{M_1 T}}{e^{M_1 T} - 1} \|x - y\|; \end{aligned} \quad (35)$$

this implies

$$\begin{aligned} & \|[Ax] - [Ay]\| \\ & \leq \left(M_2 T + M_3 T^2 K_0 + M_4 T^2 H_0 + \sum_{k=1}^m L_k \right) \\ & \quad \times \frac{e^{M_1 T}}{e^{M_1 T} - 1} \|x - y\|. \end{aligned} \quad (36)$$

Condition (31) implies that A is a contraction mapping. Banach's fixed point theorem implies that A has a unique fixed point, and so (32) has a unique solution. The proof is complete. $\square$

3. Main Results

Theorem 5. Assume that α and β are the lower solution and upper solution for (1) with $\alpha \leq \beta$ on J , respectively. Further, suppose that (H_1) , (H_2) , (16), and (31) are satisfied. Then, (1) possesses in $[\alpha, \beta]$ the minimal and maximal solution $u, v \in PC^1(J)$, respectively. Moreover, there exist the monotone sequences $\{u_n\} \nearrow u$ and $\{v_n\} \searrow v$ uniformly on J , respectively.

Proof. For all $\psi \in [\alpha, \beta]$, we consider linear impulsive differential equation

$$\begin{aligned} y'(t) &= -M_1 y(t) - M_2 y(\theta(t)) - M_3 (Ky)(t) \\ &\quad - M_4 (Hy)(t) + \eta(t), \quad t \neq t_k, \quad t \in J, \\ \Delta y(t_k) &= I_k(\psi(t_k)) - L_k(y(t_k) - \psi(t_k)), \\ &\quad k = 1, 2, \dots, m, \\ y(0) &= y(T), \end{aligned} \quad (37)$$

where

$$\begin{aligned} \eta(t) &= f(t, \psi(t), \psi(\theta(t)), (K\psi)(t), (H\psi)(t)) + M_1 \psi(t) \\ &\quad + M_2 \psi(\theta(t)) + M_3 (K\psi)(t) + M_4 (H\psi)(t). \end{aligned} \quad (38)$$

By Lemma 4, one can see that (37) has a unique solution $y \in PC^1(J)$. Denote that $A\psi$ is a unique solution of (37).

Let $u_0 = \alpha$, $v_0 = \beta$ and $u_n = Au_{n-1}$, $v_n = Av_{n-1}$. We will show that

- (a) $u_0 \leq Au_0$, $Av_0 \leq v_0$,
- (b) A is nondecreasing in $[u_0, v_0]$.

The Proof of Property (a). Let $p = u_0 - u_1$; since $u_1(0) = u_1(T)$, it follows that $p(0) - p(T) = u_0(0) - u_0(T)$, and

$$\begin{aligned} p'(t) &= u'_0(t) - u'_1(t) \\ &\leq f(t, u_0(t), u_0(\theta(t)), (Ku_0)(t), (Hu_0)(t)) \\ &\quad - a_{u_0}(t) + M_1(u_1(t)) + M_2(u_1(\theta(t))) \\ &\quad + M_3(Ku_1)(t) + M_4(Hu_1)(t) \\ &\quad - f(t, u_0(t), u_0(\theta(t)), (Ku_0)(t), (Hu_0)(t)) \\ &\quad - M_1 u_0(t) - M_2 u_0(\theta(t)) \\ &\quad - M_3 (Ku_0)(t) - M_4 (Hu_0)(t) \\ &= -M_1 p(t) - M_2 p(\theta(t)) - M_3 (Kp)(t) \\ &\quad - M_4 (Hp)(t) - a_p(t), \quad t \neq t_k, \quad t \in J, \end{aligned}$$

$$\begin{aligned} \Delta p(t_k) &= \Delta u_0(t_k) - \Delta u_1(t_k) \\ &\leq I_k(u_0(t_k)) - a_{u_0 k} - I_k(u_0(t_k)) \\ &\quad + L_k(u_1(t_k) - u_0(t_k)) \\ &= -L_k p(t_k) - a_{pk}, \end{aligned} \quad (39)$$

where

$$a_{u_0}(t) = a_p(t) = \begin{cases} 0, & \text{if } p(0) \leq p(T), \\ \left(M_1 t + M_2 \theta(t) + M_3 K_0 \int_0^{\gamma(t)} \delta(s) ds + M_4 H_0 \int_0^T \sigma(s) ds + 1 \right) \times (T)^{-1} [p(0) - p(T)], & \text{if } p(0) > p(T), \end{cases}$$

$$a_{u_0 k} = a_{pk} = \begin{cases} 0, & \text{if } p(0) \leq p(T), \\ \frac{t_k}{T} [p(0) - p(T)], & \text{if } p(0) > p(T). \end{cases} \quad (40)$$

By Lemma 3, we have $p(t) \leq 0$, for $t \in J$; that is, $u_0 \leq Au_0$. Analogously, it is proved that $Av_0 \leq v_0$.

Next, we prove the property (b). Let $\psi_1, \psi_2 \in [u_0, v_0]$ with $\psi_1 \leq \psi_2$, $\omega_1 = A\psi_1$, $\omega_2 = A\psi_2$, and $p = \omega_1 - \omega_2$; then, from (H_1) and (H_2) , we have that, for $t \neq t_k$,

$$\begin{aligned} p'(t) &= \omega_1'(t) - \omega_2'(t) = -M_1 \omega_1(t) - M_2 \omega_1(\theta(t)) \\ &\quad - M_3 K(\omega_1)(t) - M_4 H(\omega_1)(t) \\ &\quad + f(t, \psi_1(t), \psi_1(\theta(t)), (K\psi_1)(t), (H\psi_1)(t)) \\ &\quad + M_1 \psi_1(t) + M_2 \psi_1(\theta(t)) + M_3 (K\psi_1)(t) \\ &\quad + M_4 (H\psi_1)(t) + M_1 \omega_2(t) + M_2 \omega_2(\theta(t)) \\ &\quad + M_3 K(\omega_2)(t) + M_4 H(\omega_2)(t) \\ &\quad - f(t, \psi_2(t), \psi_2(\theta(t)), (K\psi_2)(t), (H\psi_2)(t)) \\ &\quad - M_1 \psi_2(t) - M_2 \psi_2(\theta(t)) \\ &\quad - M_3 (K\psi_2)(t) - M_4 (H\psi_2)(t) \\ &\leq M_1 (\psi_2(t) - \psi_1(t)) \\ &\quad + M_2 (\psi_2(\theta(t)) - \psi_1(\theta(t))) \\ &\quad + M_3 ((K\psi_2)(t) - (K\psi_1)(t)) \\ &\quad + M_4 ((H\psi_2)(t) - (H\psi_1)(t)) - M_1 \omega_1(t) \\ &\quad - M_2 \omega_1(\theta(t)) - M_3 K(\omega_1)(t) - M_4 H(\omega_1)(t) \\ &\quad + M_1 \psi_1(t) + M_2 \psi_1(\theta(t)) + M_3 (K\psi_1)(t) \\ &\quad + M_4 (H\psi_1)(t) + M_1 \omega_2(t) + M_2 \omega_2(\theta(t)) \\ &\quad + M_3 K(\omega_2)(t) + M_4 H(\omega_2)(t) - M_1 \psi_2(t) \\ &\quad - M_2 \psi_2(\theta(t)) - M_3 (K\psi_2)(t) - M_4 (H\psi_2)(t) \\ &= -M_1 p(t) - M_2 p(\theta(t)) \\ &\quad - M_3 (Kp)(t) - M_4 (Hp)(t), \end{aligned} \quad (41)$$

and, for $t = t_k$,

$$\begin{aligned} \Delta p(t_k) &= \Delta \omega_1(t_k) - \Delta \omega_2(t_k) \\ &= I_k(\psi_1(t_k)) - L_k(\omega_1(t_k) - \psi_1(t_k)) - I_k(\psi_2(t_k)) \\ &\quad + L_k(\omega_2(t_k) - \psi_2(t_k)) \\ &\leq L_k(\psi_2(t_k) - \psi_1(t_k)) - L_k(\omega_1(t_k) - \psi_1(t_k)) \\ &\quad + L_k(\omega_2(t_k) - \psi_2(t_k)) \\ &= -L_k p(t_k); \end{aligned} \quad (42)$$

it is clear that $p(0) = p(T)$. It follows by Case 1 of Lemma 3 that $p(t) \leq 0$ for $t \in J$; that is, A is monotonely nondecreasing in $[u_0, v_0]$.

It follows, from the properties (a) and (b), that

$$\alpha \leq u_1 \leq \dots \leq u_n \leq v_n \leq \dots \leq v_1 \leq \beta, \quad \text{on } J. \quad (43)$$

By standard arguments, we conclude that there exist u and v such that

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= u(t), \quad \lim_{n \rightarrow \infty} v_n = v(t), \quad \text{uniformly on } J, \\ u_n(t) &\leq u(t) \leq v(t) \leq v_n(t). \end{aligned} \quad (44)$$

It is easy to show that u and v are solutions of (1) using u_n, v_n satisfy the relations

$$\begin{aligned} u_{n+1}'(t) &= -M_1(u_{n+1}(t)) - M_2(u_{n+1}(\theta(t))) \\ &\quad - M_3(Ku_{n+1})(t) - M_4(Hu_{n+1})(t) \\ &\quad + f(t, u_n(t), u_n(\theta(t)), (Ku_n)(t), (Hu_n)(t)) \\ &\quad + M_1 u_n(t) + M_2 u_n(\theta(t)) \\ &\quad + M_3 (Ku_n)(t) + M_4 (Hu_n)(t), \quad t \in J', \\ \Delta u_{n+1}(t_k) &= I_k(u_n(t_k)) - L_k(u_{n+1}(t_k) - u_n(t_k)), \\ &\quad k = 1, 2, \dots, m, \\ u_{n+1}(0) &= u_{n+1}(T), \\ v_{n+1}'(t) &= -M_1(v_{n+1}(t)) - M_2(v_{n+1}(\theta(t))) \\ &\quad - M_3(Kv_{n+1})(t) - M_4(Hv_{n+1})(t) \\ &\quad + f(t, v_n(t), v_n(\theta(t)), (Kv_n)(t), (Hv_n)(t)) \\ &\quad + M_1 v_n(t) + M_2 v_n(\theta(t)) \\ &\quad + M_3 (Kv_n)(t) + M_4 (Hv_n)(t), \quad t \in J', \\ \Delta v_{n+1}(t_k) &= I_k(v_n(t_k)) - L_k(v_{n+1}(t_k) - v_n(t_k)), \\ &\quad k = 1, 2, \dots, m, \\ v_{n+1}(0) &= v_{n+1}(T). \end{aligned} \quad (45)$$

Taking the limit as $n \rightarrow \infty$ on both sides of above relations, we have

$$\begin{aligned} u'(t) &= f(t, u(t), u(\theta(t)), (Ku)(t), (Hu)(t)), \\ &\quad t \neq t_k, \quad t \in J, \\ \Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) &= u(T), \\ v'(t) &= f(t, v(t), v(\theta(t)), (Kv)(t), (Hv)(t)), \\ &\quad t \neq t_k, \quad t \in J, \\ \Delta v(t_k) &= I_k(v(t_k)), \quad k = 1, 2, \dots, m, \\ v(0) &= v(T). \end{aligned} \quad (46)$$

Equation (46) show that u and v are solutions of (1). Finally, we prove that, if $z \in [\alpha, \beta]$ is any solution of (1), then $u(t) \leq z(t) \leq v(t)$ on J . To this end, we assume, without loss of generality, that $u_n(t) \leq z(t) \leq v_n(t)$ for some n , since $\alpha(t) \leq z(t) \leq \beta(t)$. From property (b), we can get

$$u_{n+1}(t) \leq z(t) \leq v_{n+1}(t), \quad t \in J. \quad (47)$$

Since $u_0(t) = \alpha(t) \leq z(t) \leq \beta(t) = v_0(t)$, $t \in J$, by induction, we can conclude that

$$u_n(t) \leq z(t) \leq v_n(t), \quad \forall n. \quad (48)$$

Passing to the limit as $n \rightarrow \infty$, we obtain

$$u(t) \leq z(t) \leq v(t), \quad t \in J. \quad (49)$$

This ends the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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