Research Article

On Generalized Jordan Triple (σ, τ) -Higher Derivations in Prime Rings

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Let R be a ring and let U be a Lie ideal of R . Suppose that σ, τ are endomorphisms of R , and $\mathbb{N}$ is the set of all nonnegative integers. A family $F = \{f_n\}_{n \in \mathbb{N}}$ of mappings $f_n : R \rightarrow R$ is said to be a generalized (σ, τ) -higher derivation (resp., generalized Jordan triple (σ, τ) -higher derivation) of R if there exists a (σ, τ) -higher derivation $D = \{d_n\}_{n \in \mathbb{N}}$ of R such that $f_0 = I_R$, the identity map on R , $f_n(a + b) = f_n(a) + f_n(b)$, and $f_n(ab) = \sum_{i+j=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$ (resp., $f_n(aba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a))$) hold for all $a, b \in R$ and for every $n \in \mathbb{N}$. If the above conditions hold for all $a, b \in U$, then F is said to be a generalized (σ, τ) -higher derivation (resp., generalized Jordan triple (σ, τ) -higher derivation) of U into R . In the present paper it is shown that if U is a noncentral square closed Lie ideal of a prime ring R of characteristic different from two, then every generalized Jordan triple (σ, τ) -higher derivation of U into R is a generalized (σ, τ) -higher derivation of U into R .

1. Introduction

Let R be an associative ring with center $Z(R)$ (may be without identity element). For any $x, y, z \in R$, $[x, y] = xy - yx$ will denote the usual Lie product and the element $xyz - zyx$ will be denoted by $[x, y, z]$. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. A Lie ideal U of R is said to be square closed Lie ideal of R if $u^2 \in U$ for all $u \in U$. An additive mapping $d : R \rightarrow R$ is said to be a derivation on R if $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. Also, an additive mapping $d : R \rightarrow R$ is called a Jordan triple derivation if $d(aba) = d(a)ba + ad(b)a + abd(a)$ for all $a, b \in R$. Herstein [1, Lemma 3.5] showed that on a 2-torsion free ring every derivation is Jordan triple derivation, but the converse need not be true in general, whereas Brešar [2] proved that on a 2-torsion free semiprime ring every Jordan triple derivation is a derivation.

The concept of derivations was further extended to (σ, τ) -derivation. Let σ, τ be the endomorphisms of R . An additive mapping $d : R \rightarrow R$ is said to be a (σ, τ) -derivation of R if $d(xy) = d(x)\tau(y) + \sigma(x)d(y)$ holds for all $x, y \in R$. Moreover, an additive mapping $d : R \rightarrow R$ is said to be a

Jordan triple (σ, τ) -derivation of R if $d(xyx) = d(x)\tau(yx) + \sigma(a)d(b)\tau(a) + \sigma(ab)d(a)$ holds for all $x, y \in R$. Brešar [3] introduced the notion of generalized derivation as follows: an additive mapping $f : R \rightarrow R$ is said to be a generalized derivation on R if there exists a derivation d on R such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $f : R \rightarrow R$ is said to be a generalized (σ, τ) -derivation on R if there exists a (σ, τ) -derivation d on R such that $f(xy) = f(x)\tau(y) + \sigma(x)d(y)$ for all $x, y \in R$ (for reference see [4]). Correspondingly, the concept of generalized Jordan triple derivations was defined by Wu and Lu [5]. An additive mapping $f : R \rightarrow R$ is said to be a generalized Jordan triple derivation on R if there exists a Jordan triple derivation d on R such that $f(aba) = f(a)ba + ad(b)a + abd(a)$ for all $a, b \in R$. They also showed that every generalized Jordan triple derivation on a 2-torsion free prime ring is a generalized derivation. Further, Liu and Shiue [6] extended this result for generalized Jordan triple (σ, τ) -derivation. An additive mapping $f : R \rightarrow R$ is said to be a generalized Jordan triple (σ, τ) -derivation on R if there exists a Jordan triple (σ, τ) -derivation d on R such that $f(aba) = f(a)\tau(ba) + \sigma(a)d(b)\tau(a) + \sigma(ab)d(a)$ for all $a, b \in R$. Liu and

Shiue proved that on a 2-torsion free semiprime ring every generalized Jordan triple (σ, τ) -derivation is a generalized (σ, τ) -derivation.

Various results proved for derivations and generalized derivations were shown to be true in case of higher derivations which can be found in [7–10], and so forth. Let $\mathbb{N}$ be the set of all nonnegative integers. Following Hasse and Schmidt [11], a family of additive mappings $D = \{d_n\}_{n \in \mathbb{N}}$ on R is said to be a higher derivation on R if $d_0 = I_R$ (the identity map on R) and $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ holds for all $a, b \in R$ and for each $n \in \mathbb{N}$.

The concept of higher derivation was extended to (σ, τ) -higher derivation by the authors together with Haetinger [7]. Let $D = \{d_n\}_{n \in \mathbb{N}}$ be a family of maps $d_n : R \rightarrow R$. Then D is said to be a (σ, τ) -higher derivation on R if $d_0 = I_R$, $d_n(a + b) = d_n(a) + d_n(b)$ and $d_n(ab) = \sum_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$ hold for all $a, b \in R$ and for each $n \in \mathbb{N}$. If U is a Lie ideal of R , then D is said to be a (σ, τ) -higher derivation of U into R if the above conditions hold for all $a, b \in U$. Further, in [8] the authors introduced the notion of generalized (σ, τ) -higher derivation in rings. A family $F = \{f_n\}_{n \in \mathbb{N}}$ of mappings $f_n : R \rightarrow R$ is said to be a generalized (σ, τ) -higher derivation of R if there exists a (σ, τ) -higher derivation $D = \{d_n\}_{n \in \mathbb{N}}$ of R such that $f_0 = I_R$, $f_n(a + b) = f_n(a) + f_n(b)$ and $f_n(ab) = \sum_{i+j=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$ for all $a, b \in R$ and for each $n \in \mathbb{N}$.

Motivated by the concepts of Jordan triple derivation, generalized derivation, and (σ, τ) -higher derivation, we introduce generalized Jordan triple (σ, τ) -higher derivation as follows: a family $F = \{f_n\}_{n \in \mathbb{N}}$ of mappings $f_n : R \rightarrow R$ is said to be a generalized Jordan triple (σ, τ) -higher derivation of R if there exists a (σ, τ) -higher derivation $D = \{d_n\}_{n \in \mathbb{N}}$ of R such that $f_n(a + b) = f_n(a) + f_n(b)$ and $f_n(aba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k \tau^i(b))d_k(\tau^{n-k}(a))$ hold for all $a, b \in R$ and every $n \in \mathbb{N}$. Further, if D is a (σ, τ) -higher derivation of U into R and the above conditions hold for all $a, b \in U$, then F is said to be a generalized Jordan triple (σ, τ) -higher derivation of U into R .

It can be easily seen that, on a 2-torsion free ring R , every generalized (σ, τ) -higher derivation of a square closed Lie ideal U into R is a generalized Jordan triple (σ, τ) -higher derivation of U into R . In fact, if $F = \{f_n\}_{n \in \mathbb{N}}$ is generalized (σ, τ) -higher derivation of a square closed Lie ideal U into R , then $f_n(uv) = \sum_{i+j=n} f_i(\sigma^{n-i}(u))d_j(\tau^{n-j}(v))$ for all $u, v \in U$ and for each $n \in \mathbb{N}$. Replacing v with $2vu$ and using the fact that R is 2-torsion free, we obtain $f_n(uvu) = \sum_{i+j=n} f_i(\sigma^{n-i}(u))d_j(\sigma^k \tau^i(v))d_k(\tau^{n-k}(u))$ for all $u, v \in U$ and for each $n \in \mathbb{N}$; that is, $F = \{f_n\}_{n \in \mathbb{N}}$ is generalized Jordan triple (σ, τ) -higher derivation of U into R but the converse need not be true in general.

In the present paper, our objective is to find the conditions on R under which every generalized Jordan triple (σ, τ) -higher derivation of U into R is a generalized (σ, τ) -higher derivation of U into R . In fact our result generalizes the main theorem obtained in [4, 7–9, 12–14].

2. Main Results

Throughout, σ, τ will denote the endomorphisms of R such that τ is one-one and onto and $\sigma\tau = \tau\sigma$. Note that if a Lie ideal U of R is square closed, then for all $u, v \in U$; $uv + vu = (u + v)^2 - u^2 - v^2 \in U$ and since $uv - vu \in U$; yields $2uv \in U$. Let $F = \{f_n\}_{n \in \mathbb{N}}$ be a generalized Jordan triple (σ, τ) -higher derivation of U into R with associated (σ, τ) -higher derivation $D = \{d_n\}_{n \in \mathbb{N}}$ of U into R . For every fixed $n \in \mathbb{N}$ and each $u, v, w \in U$, we denote by $\Phi_n(u, v, w)$ the element of R as

$$\begin{aligned} \Phi_n(u, v, w) &= f_n(uvw) \\ &\quad - \sum_{i+j+k=n} f_i(\sigma^{n-i}(u))d_j(\sigma^k \tau^i(v))d_k(\tau^{n-k}(w)). \end{aligned} \quad (1)$$

Then, by using similar arguments as used in Lemma 2.2(iii) of [8], it can easily be seen that $\Phi_n(u, v, w)$ is additive in each argument and $\Phi_n(u, v, w) = -\Phi_n(w, v, u)$.

Following lemmas are essential for developing the proof of our main result. Proof of Lemma 1 can be seen in [15] whereas Lemma 2 was obtained in [9].

Lemma 1. Assume that R is a prime ring of characteristic different from two and U is a noncentral square closed Lie ideal of R . Let $G_1, G_2, \dots, G_n$ be additive groups, $S : G_1 \times G_2 \times \dots \times G_n \rightarrow R$ and let $T : G_1 \times G_2 \times \dots \times G_n \rightarrow R$ be the mappings which are additive in each argument. If $S(a_1, a_2, \dots, a_n) \times T(a_1, a_2, \dots, a_n) = \{0\}$ for every $x \in U$, $a_i \in G_i$, $i = 1, 2, \dots, n$, then $S(a_1, a_2, \dots, a_n) = \{0\}$ for every $a_i \in G_i$, $i = 1, 2, \dots, n$ or $T(b_1, b_2, \dots, b_n) = \{0\}$ for every $b_i \in G_i$, $i = 1, 2, \dots, n$.

Lemma 2. Let R be a prime ring of characteristic different from two and let U be a square closed Lie ideal of R such that $U \not\subseteq Z(R)$. Then there exist elements $u, v, w \in U$ such that $[u, v, w] \neq 0$.

Now, we prove the following lemma which plays a crucial role in developing the proof of our main result.

Lemma 3. Let R be a 2-torsion free ring and let U be a square closed Lie ideal of R . If $F = \{f_n\}_{n \in \mathbb{N}}$ is a generalized Jordan triple (σ, τ) -higher derivation of U into R and $\Phi_m(u, v, w) = 0$ for all $u, v, w \in U$ and every $m < n$, then $\Phi_n(u, v, w)\tau^n(x)\tau^n[w, v, u] = 0$ for all $u, v, w, x \in U$ and every $n \in \mathbb{N}$.

Proof. Let $\zeta = 4(uvwxvuvu + wvuxuvuv)$. Then

$$\begin{aligned} f_n(\zeta) &= f_n(u(4vwxvuv)u) + f_n(w(4vuxuv)w) \\ &= 4 \sum_{i+j+k=n} f_i(\sigma^{n-i}(u)) \\ &\quad \times d_j(\sigma^k \tau^i(v(wxw)v))d_k(\tau^{n-k}(u)) \\ &\quad + 4 \sum_{i+j+k=n} f_i(\sigma^{n-i}(w)) \end{aligned}$$

$$\begin{aligned}
& \times d_j(\sigma^k \tau^i(vuxu)v) d_k(\tau^{n-k}(w)) \\
& = 4 \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(u)) \\
& \quad \times d_l(\sigma^{y+p+q+s+k} \tau^i(v)) d_y(\sigma^{p+q+s+k} \tau^{i+l}(w)) \\
& \quad \times d_p(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+p}(w)) \\
& \quad \times d_s(\sigma^k \tau^{i+l+y+p+q}(v)) d_k(\tau^{n-k}(u)) \\
& + 4 \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(w)) \\
& \quad \times d_l(\sigma^{y+p+q+s+k} \tau^i(v)) \\
& \quad \times d_y(\sigma^{p+q+s+k} \tau^{i+l}(u)) \\
& \quad \times d_p(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+p}(u)) \\
& \quad \times d_s(\sigma^k \tau^{i+l+y+p+q}(v)) d_k(\tau^{n-k}(w)).
\end{aligned} \tag{2}$$

From our hypothesis we have

$$\begin{aligned}
f_n(uvu) &= \sum_{i+j+k=n} f_i(\sigma^{n-i}(u)) d_j(\sigma^k \tau^i(v)) d_k(\tau^{n-k}(u)), \\
&\quad \forall u, v \in U, \quad n \in \mathbb{N}.
\end{aligned} \tag{3}$$

Linearizing the above equation we obtain

$$\begin{aligned}
& f_n(uvw + wvu) \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(u)) d_j(\sigma^k \tau^i(v)) d_k(\tau^{n-k}(w)) \\
& \quad + \sum_{i+j+k=n} f_i(\sigma^{n-i}(w)) d_j(\sigma^k \tau^i(v)) d_k(\tau^{n-k}(u)) \\
&\quad \forall u, v \in U, \quad n \in \mathbb{N}.
\end{aligned} \tag{4}$$

Again consider $f_n(\zeta) = f_n((2uvw)x(2wvu) + (2wvu)x(2uvw))$. Using relation (4) we have

$$\begin{aligned}
& f_n(\zeta) \\
&= 4 \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(uvw)) \\
&\quad \times d_p(\sigma^\gamma \tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(wvu)) \\
& \quad + f_\alpha(\sigma^{n-\alpha}(wvu)) d_p(\sigma^\gamma \tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(uvw)).
\end{aligned} \tag{5}$$

Combining the two equations (2) and (5)

$$\begin{aligned}
0 &= 4 \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+p+q+s+k} \tau^i(v)) \\
&\quad \times d_y(\sigma^{p+q+s+k} \tau^{i+l}(w)) \\
&\quad \times d_p(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+p}(w)) \\
&\quad \times d_s(\sigma^k \tau^{i+l+y+p+q}(v)) d_k(\tau^{n-k}(u)) \\
&\quad - 4 \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(uvw)) d_p(\sigma^\gamma \tau^\alpha(x)) \\
&\quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
&\quad + 4 \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(w)) d_l(\sigma^{y+p+q+s+k} \tau^i(v)) \\
&\quad \times d_y(\sigma^{p+q+s+k} \tau^{i+l}(u)) \\
&\quad \times d_p(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+p}(u)) \\
&\quad \times d_s(\sigma^k \tau^{i+l+y+p+q}(v)) d_k(\tau^{n-k}(w)) \\
&\quad - 4 \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(wvu)) d_p(\sigma^\gamma \tau^\alpha(x)) \\
&\quad \times d_\gamma(\tau^{n-\gamma}(uvw)).
\end{aligned} \tag{6}$$

Considering the first term,

$$\begin{aligned}
& \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+p+q+s+k} \tau^i(v)) \\
&\quad \times d_y(\sigma^{p+q+s+k} \tau^{i+l}(w)) d_p(\sigma^{q+s+k} \tau^{i+l+y}(x)) \\
&\quad \times d_q(\sigma^{s+k} \tau^{i+l+y+p}(w)) d_s(\sigma^k \tau^{i+l+y+p+q}(v)) d_k(\tau^{n-k}(u)) \\
&= \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+p+q+s+k} \tau^i(v)) \\
&\quad \times d_y(\sigma^{q+s+k} \tau^{i+l}(w)) (\sigma^{q+s+k} \tau^{i+l+y}(x)) \\
&\quad \times d_q(\sigma^{s+k} \tau^{i+l+y}(w)) d_s(\sigma^k \tau^{i+l+y+q}(v)) d_k(\tau^{n-k}(u)) \\
&\quad + \sum_{i+l+y+p+q+s+k=n-1} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+1+q+s+k} \tau^i(v)) \\
&\quad \times d_y(\sigma^{1+q+s+k} \tau^{i+l}(w)) \\
&\quad \times d_1(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+1}(w)) \\
&\quad \times d_s(\sigma^k \tau^{i+l+y+1+q}(v)) d_k(\tau^{n-k}(u)) \\
&\quad + \dots
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i+l+y+q+s+k=1} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+n-1+q+s+k} \tau^i(v)) \\
& \times d_y(\sigma^{n-1+q+s+k} \tau^{i+l}(w)) \\
& \times d_{n-1}(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+n-1}(w)) \\
& \times d_s(\sigma^k \tau^{i+l+y+n-1+q}(v)) d_k(\tau^{n-k}(u)) \\
& + \sum_{i+l+y+q+s+k=0} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+n+q+s+k} \tau^i(v)) \\
& \times d_y(\sigma^{n+q+s+k} \tau^{i+l}(w)) \\
& \times d_n(\sigma^{q+s+k} \tau^{i+l+y}(x)) \\
& \times d_q(\sigma^{s+k} \tau^{i+l+y+n}(w)) d_s(\sigma^k \tau^{i+l+y+n+q}(v)) d_k(\tau^{n-k}(u)) \\
& = \sum_{i+l+y=n} f_i(\sigma^{n-i}(u)) d_l(\sigma^y \tau^i(v)) \\
& \quad \times d_y(\tau^{n-y}(w)) \tau^n(x) \tau^n(wvu) \\
& + \sum_{q+s+k=n} \sigma^n(uvw) \sigma^n(x) d_q(\sigma^{n-q}(w)) \\
& \quad \times d_s(\sigma^k \tau^q(v)) d_k(\tau^{n-k}(u)) \\
& + \sum_{\substack{0 < i+l+y, q+s+k \leq n-1 \\ i+l+y+q+s+k=n}} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+q+s+k} \tau^i(v)) \\
& \times d_y(\sigma^{q+s+k} \tau^{i+l}(w)) (\sigma^{q+s+k} \tau^{i+l+y}(x)) \\
& \times d_q(\sigma^{s+k} \tau^{i+l+y}(w)) d_s(\sigma^k \tau^{i+l+y+q}(v)) d_k(\tau^{n-k}(u)) \\
& + \sum_{i+l+y+q+s+k=n-1} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+q+s+k+1} \tau^i(v)) \\
& \times d_y(\tau^{i+l} \sigma^{q+s+k+1}(w)) \\
& \times d_1(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+1}(w)) \\
& \times d_s(\sigma^k \tau^{i+l+y+q+1}(v)) d_k(\tau^{n-k}(u)) \\
& + \cdots + f_1(\sigma^{n-1}(u)) (\sigma^{n-1} \tau(v)) (\sigma^{n-1} \tau(w)) \\
& \times d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(u) d_1(\sigma^{n-1}(v)) (\sigma^{n-1} \tau(w)) \\
& \times d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(uv) d_1(\sigma^{n-1}(w)) d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(w)) \tau^n(vu) \\
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) (\sigma \tau^{n-1}(w)) d_1(\tau^{n-1}(v)) \tau^n(u) \\
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) (\sigma \tau^{n-1}(w))
\end{aligned}$$

$$\begin{aligned}
& \times (\sigma \tau^{n-1}(v)) d_1(\tau^{n-1}(u)) \\
& + \sigma^n(uvw) d_n(x) \tau^n(wvu).
\end{aligned} \tag{7}$$

Calculating the second term,

$$\begin{aligned}
& \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(uvw)) d_p(\sigma^\gamma \tau^\alpha(x)) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& = \sum_{\alpha+\gamma=n} f_\alpha(\sigma^{n-\alpha}(uvw)) (\sigma^\gamma \tau^\alpha(x)) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& + \sum_{\alpha+\gamma=n-1} f_\alpha(\sigma^{n-\alpha}(uvw)) d_1(\sigma^\gamma \tau^\alpha(x)) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& + \cdots \\
& + \sum_{\alpha+\gamma=1} f_\alpha(\sigma^{n-\alpha}(uvw)) d_{n-1}(\sigma^\gamma \tau^\alpha(x)) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& + \sum_{\alpha+\gamma=0} f_\alpha(\sigma^{n-\alpha}(uvw)) d_n(\sigma^\gamma \tau^\alpha(x)) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& = f_n(uvw) \tau^n(x) \tau^n(wvu) \\
& + \sigma^n(uvw) \sigma^n(x) d_n(wvu) \\
& + \sum_{\substack{0 < \alpha, \gamma \leq n-1 \\ \alpha+\gamma=n}} f_\alpha(\sigma^{n-\alpha}(uvw)) \sigma^\gamma \tau^\alpha(x) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& + \sum_{\alpha+\gamma=n-1} f_\alpha(\sigma^{n-\alpha}(uvw)) d_1(\sigma^\gamma \tau^\alpha(x)) \\
& \quad \times d_\gamma(\tau^{n-\gamma}(wvu)) \\
& + \cdots + f_1(\sigma^{n-1}(uvw)) d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(wvu)) \\
& + \sigma^n(uvw) d_n(x) \tau^n(wvu).
\end{aligned} \tag{8}$$

Using the hypothesis that $\Phi_m(u, v, w) = 0$ for all $m < n$, we have

$$\begin{aligned}
& \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(uvw)) d_p(\sigma^\gamma \tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(wvu)) \\
& = f_n(uvw) \tau^n(x) \tau^n(wvu) + \sigma^n(uvw) \sigma^n(x) d_n(wvu)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{0 < i+l+y, q+s+k \leq n-1 \\ i+l+y+q+s+k=n}} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+q+s+k} \tau^i(v)) \\
& \times d_y(\sigma^{q+s+k} \tau^{i+l}(w)) \sigma^{q+s+k} \tau^{i+l+y}(x) \\
& \times d_q(\sigma^{s+k} \tau^{i+l+y}(w)) d_s(\sigma^k \tau^{i+l+y+q}(v)) d_k(\tau^{n-k}(u)) \\
& + \sum_{i+l+y+q+s+k=n-1} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+q+s+k+1} \tau^i(v)) \\
& \times d_y(\sigma^{q+s+k+1} \tau^{i+l}(w)) \\
& \times d_1(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+1}(w)) \\
& \times d_s(\sigma^k \tau^{i+l+y+q+1}(v)) d_k(\tau^{n-k}(u)) \\
& + \dots \\
& + \left(\sum_{i+l+y=1} f_i(\sigma^{1-i} \sigma^{n-1}(u)) d_l(\sigma^y \tau^i \sigma^{n-1}(v)) \right. \\
& \quad \times d_y(\tau^{1-y} \sigma^{n-1}(w)) d_{n-1}(\tau(x)) \tau^n(wvu) \Big) \\
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) \\
& \times \left(\sum_{q+u+k=1} d_q(\sigma^{1-q} \tau^{n-1}(w)) \right. \\
& \quad \times d_u(\sigma^k \tau^q \tau^{n-1}(v)) d_k(\sigma^{1-k} \tau^{n-1}(u)) \Big) \\
& + \sigma^n(uvw) d_n(x) \tau^n(wvu) \\
& = f_n(uvw) \tau^n(x) \tau^n(wvu) + \sigma^n(uvw) \sigma^n(x) d_n(wvu) \\
& + \sum_{\substack{0 < i+l+y, q+s+k \leq n-1 \\ i+l+y+q+s+k=n}} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+q+s+k} \tau^i(v)) \\
& \times d_y(\sigma^{q+s+k} \tau^{i+l}(w)) \sigma^{q+s+k} \tau^{i+l+y}(x) \\
& \times d_q(\sigma^{s+k} \tau^{i+l+y}(w)) d_s(\sigma^k \tau^{i+l+y+q}(v)) d_k(\tau^{n-k}(u)) \\
& + \sum_{i+l+y+q+s+k=n-1} f_i(\sigma^{n-i}(u)) d_l(\sigma^{y+q+s+k+1} \tau^i(v)) \\
& \times d_y(\tau^{i+l} \sigma^{q+s+k+1}(w)) \\
& \times d_1(\sigma^{q+s+k} \tau^{i+l+y}(x)) d_q(\sigma^{s+k} \tau^{i+l+y+1}(w)) \\
& \times d_s(\sigma^k \tau^{i+l+y+q+1}(v)) d_k(\tau^{n-k}(u)) \\
& + \dots + \sigma^n(uvw) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(w)) \tau^n(vu) \\
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) (\sigma \tau^{n-1}(w)) d_1(\tau^{n-1}(v)) \tau^n(u)
\end{aligned}$$

$$\begin{aligned}
& + \sigma^n(uvw) d_{n-1}(\sigma(x)) (\sigma \tau^{n-1}(w)) (\sigma \tau^{n-1}(v)) \\
& \times d_1(\tau^{n-1}(u)) \\
& + f_1(\sigma^{n-1}(u)) (\sigma^{n-1} \tau(v)) (\tau \sigma^{n-1}(w)) \\
& \times d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(u) d_1(\sigma^{n-1}(v)) (\sigma^{n-1} \tau(w)) d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(uv) d_1(\sigma^{n-1}(w)) d_{n-1}(\tau(x)) \tau^n(wvu) \\
& + \sigma^n(uvw) d_n(x) \tau^n(wvu).
\end{aligned} \tag{9}$$

Now, subtracting the two terms so obtained we find that

$$\begin{aligned}
& f_n(uvw) \tau^n(x) \tau^n(wvu) \\
& - \sum_{i+l+s=n} f_i(\sigma^{n-i}(u)) d_l(\sigma^s \tau^i(v)) \\
& \quad \times d_s(\tau^{n-s}(w)) \tau^n(x) \tau^n(wvu) \\
& = \left\{ f_n(uvw) \right. \\
& \quad \left. - \sum_{i+l+s=n} f_i(\sigma^{n-i}(u)) d_l(\sigma^s \tau^i(v)) d_s(\tau^{n-s}(w)) \right\} \\
& \quad \times \tau^n(x) \tau^n(wvu) \\
& = \Phi_n(u, v, w) \tau^n(x) \tau^n(wvu).
\end{aligned} \tag{10}$$

Similarly, the difference of the last two terms yields

$$\begin{aligned}
& \sum_{i+l+y+p+q+s+k=n} f_i(\sigma^{n-i}(w)) d_l(\sigma^{y+p+q+s+k} \tau^i(v)) \\
& \times d_y(\sigma^{p+q+s+k} \tau^{i+l}(u)) d_p(\sigma^{q+s+k} \tau^{i+l+s}(x)) \\
& \times d_q(\sigma^{s+k} \tau^{i+l+y+p}(u)) \\
& \times d_s(\sigma^k \tau^{i+l+y+p+q}(v)) d_k(\tau^{n-k}(w)) \\
& - \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(wvu)) d_p(\sigma^\gamma \tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(uvw)) \\
& = \Phi_n(w, v, u) \tau^n(x) \tau^n(uvw).
\end{aligned} \tag{11}$$

Thus, (6) reduces to

$$\begin{aligned}
& 4(\Phi_n(u, v, w) \tau^n(x) \tau^n(wvu) \\
& + \Phi_n(w, v, u) \tau^n(x) \tau^n(uvw)) = 0.
\end{aligned} \tag{12}$$

Since the characteristic of R is not two, we find that $\Phi_n(u, v, w) \tau^n(x) \tau^n[w, v, u] = 0$. $\square$

Lemma 4. Let R be a prime ring of characteristic different from 2 and let U be a noncentral square closed Lie ideal of R . If $F = \{f_n\}_{n \in \mathbb{N}}$ is a generalized Jordan triple (σ, τ) -higher derivation on U into R , then $\Phi_n(u, v, w) = 0$, for all $u, v, w \in U$ and each $n \in \mathbb{N}$.

Proof. Trivially for $n = 0$, $\Phi_0(u, v, w) = 0$. By induction let us assume that $\Phi_m(u, v, w) = 0$, for every $u, v, w \in U$ and for all $m < n$. By Lemma 3, we have $\Phi_n(u, v, w)\tau^n(x)\tau^n[w, v, u] = 0$ or $\tau^{-n}(\Phi_n(u, v, w))x[w, v, u] = 0$. Again, by Lemma 1, we have $\Phi_n(u, v, w) = 0$ for all $u, v, w \in U$ or $[w, v, u] = 0$ for all $u, v, w \in U$. If $[w, v, u] = 0$ for all $u, v, w \in U$ holds, then this contradicts Lemma 2. Hence, in either case $\Phi_n(u, v, w) = 0$ for all $u, v, w \in U, n \in \mathbb{N}$. $\square$

Theorem 5. Let R be a prime ring of characteristic different from 2 and let U be a square closed Lie ideal of R such that $U \not\subseteq Z(R)$. Then, every generalized Jordan triple (σ, τ) -higher derivation of U into R is a generalized (σ, τ) -higher derivation of U into R .

Proof. Let $F = \{f_n\}_{n \in \mathbb{N}}$ be generalized Jordan triple (σ, τ) -higher derivation.

It can be easily seen that $f_0 = I_R$. By induction, assume that $f_m(uv) = \sum_{i+j=m} f_i(\sigma^{n-i}(u))d_j(\tau^{n-j}(v))$ holds for all $m < n$.

For $u, v, x \in U$, take $\xi = 4uvxuv \in U$ and use Lemma 4 to get

$$\begin{aligned}
 & f_n(\xi) \\
 &= 4f_n(u(vxu)v) \\
 &= 4 \sum_{i+j+k=n} f_i(\sigma^{n-i}(u))d_j(\sigma^k\tau^i(vxu))d_k(\tau^{n-k}(v)) \\
 &= 4 \sum_{i+s+p+q+k=n} f_i(\sigma^{n-i}(u))d_s(\sigma^{p+q+k}\tau^i(v)) \\
 &\quad \times d_p(\sigma^{q+k}\tau^{i+s}(x))d_q(\sigma^k\tau^{s+p+i}(u)) \\
 &\quad \times d_k(\tau^{n-k}(v)) \\
 &= 4 \left(\sum_{i+s+q+k=n} f_i(\sigma^{n-i}(u))d_s(\sigma^{q+k}\tau^i(v)) \right. \\
 &\quad \times d_0(\sigma^{q+k}\tau^{i+s}(x))d_q(\sigma^k\tau^{s+i}(u))d_k(\tau^{n-k}(v)) \\
 &\quad + \sum_{i+s+q+k=n-1} f_i(\sigma^{n-i}(u))d_s(\sigma^{1+q+k}\tau^i(v)) \\
 &\quad \times d_1(\sigma^{q+k}\tau^{i+s}(x))d_q(\sigma^k\tau^{1+s+i}(u))d_k(\tau^{n-k}(v)) \\
 &\quad + \cdots + f_1(\sigma^{n-1}(u))(\sigma^{n-1}\tau(v)) \\
 &\quad \times d_{n-1}(\tau(x))(\tau^n(uv)) \\
 &\quad + \sigma^n(u)d_1(\sigma^{n-1}(v))d_{n-1}(\tau(x))(\tau^n(uv)) \\
 &\quad + \sigma^n(uv)d_{n-1}(\sigma(x))d_1(\tau^{n-1}(u))(\tau^n(v)) \\
 &\quad + \sigma^n(uv)d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(u))d_1(\tau^{n-1}(v)) \\
 &\quad \left. + \sigma^n(uv)d_n(x)\tau^n(uv) \right). \tag{13}
 \end{aligned}$$

$$\begin{aligned}
 & \times d_q(\sigma^k\tau^{s+n-1+i}(u))d_k(\tau^{n-k}(v)) \\
 & + \sum_{i+s+q+k=0} f_i(\sigma^{n-i}(u))d_s(\sigma^{n+q+k}\tau^i(v)) \\
 & \times d_n(\sigma^{q+k}\tau^{i+s}(x)) \\
 & \times d_q(\sigma^k\tau^{n+s+i}(u))d_k(\tau^{n-k}(v)) \Big) \\
 &= 4 \left(\sum_{i+s=n} f_i(\sigma^{n-i}(u))d_s(\tau^{n-s}(v))(\tau^n(xuv)) \right. \\
 & + \sum_{q+k=n} (\sigma^n(uvx))d_q(\sigma^{n-q}(u))d_k(\tau^{n-k}(v)) \\
 & + \sum_{\substack{0 < i+s, q+k \leq n-1 \\ i+s+q+k=n}} f_i(\sigma^{n-i}(u)) \\
 & \quad \times d_s(\sigma^{q+k}\tau^i(v))(\sigma^{q+k}\tau^{i+s}(x)) \\
 & \quad \times d_q(\sigma^k\tau^{s+i}(u))d_k(\tau^{n-k}(v)) \\
 & + \sum_{i+s+q+k=n-1} f_i(\sigma^{n-i}(u))d_s(\sigma^{1+q+k}\tau^i(v)) \\
 & \quad \times d_1(\sigma^{q+k}\tau^{i+s}(x))d_q(\sigma^k\tau^{1+s+i}(u))d_k(\tau^{n-k}(v)) \\
 & + \cdots + f_1(\sigma^{n-1}(u))(\sigma^{n-1}\tau(v)) \\
 & \quad \times d_{n-1}(\tau(x))(\tau^n(uv)) \\
 & + \sigma^n(u)d_1(\sigma^{n-1}(v))d_{n-1}(\tau(x))(\tau^n(uv)) \\
 & + \sigma^n(uv)d_{n-1}(\sigma(x))d_1(\tau^{n-1}(u))(\tau^n(v)) \\
 & + \sigma^n(uv)d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(u))d_1(\tau^{n-1}(v)) \\
 & \left. + \sigma^n(uv)d_n(x)\tau^n(uv) \right).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & f_n(\xi) \\
 &= 4f_n((uv)x(uv)) \\
 &= 4 \sum_{i+j+k=n} f_i(\sigma^{n-i}(uv))d_j(\sigma^k\tau^i(x))d_k(\tau^{n-k}(uv)) \\
 &= 4 \left(\sum_{i+k=n} f_i(\sigma^{n-i}(uv))d_0(\sigma^k\tau^i(x))d_k(\tau^{n-k}(uv)) \right.
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i+k=n-1} f_i(\sigma^{n-i}(uv)) d_1(\sigma^k \tau^i(x)) d_k(\tau^{n-k}(uv)) \\
& + \cdots + \sum_{i+k=1} f_i(\sigma^{n-i}(uv)) d_{n-1}(\sigma^k \tau^i(x)) d_k(\tau^{n-k}(uv)) \\
& + \sum_{i+k=0} f_i(\sigma^{n-i}(uv)) d_n(\sigma^k \tau^i(x)) d_k(\tau^{n-k}(uv)) \\
& = 4 \left(\sigma^n(uv) \sigma^n(x) d_n(uv) + f_n(uv) \tau^n(x) \tau^n(uv) \right. \\
& \quad + \sum_{\substack{0 < i, k \leq n-1 \\ i+k=n}} f_i(\sigma^{n-i}(uv)) (\sigma^k \tau^i(x)) d_k(\tau^{n-k}(uv)) \\
& \quad + \sum_{i+k=n-1} f_i(\sigma^{n-i}(uv)) d_1(\sigma^k \tau^i(x)) d_k(\tau^{n-k}(uv)) \\
& \quad + \cdots + f_1(\sigma^{n-i}(uv)) d_{n-1}(\tau(x)) (\tau^n(uv)) \\
& \quad + (\sigma^n(uv)) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(uv)) \\
& \quad \left. + (\sigma^n(uv)) d_n(x) \tau^n(uv) \right) \\
& = 4 \left(\sigma^n(uv) \sigma^n(x) d_n(uv) + f_n(uv) \tau^n(x) \tau^n(uv) \right. \\
& \quad + \sum_{\substack{0 < l+s, p+q \leq n-1 \\ l+s+p+q=n}} f_l(\sigma^{n-l}(u)) d_s(\tau^l \sigma^{q+p}(v)) \\
& \quad \quad \times (\sigma^{q+k} \tau^{l+s}(x)) \\
& \quad \quad \times d_q(\sigma^p \tau^{l+s}(u)) d_p(\tau^{n-p}(v)) \\
& \quad + \sum_{l+s+p+q=n-1} f_l(\sigma^{n-l}(u)) d_s(\tau^l \sigma^{q+p+1}(v)) \\
& \quad \quad \times d_1(\sigma^{q+k} \tau^{l+s}(x)) d_q(\sigma^p \tau^{l+s+1}(u)) \\
& \quad \quad \times d_p(\tau^{n-p}(v)) \\
& \quad + \cdots \\
& \quad + f_1(\sigma^{n-1}(u)) (\sigma^{n-1} \tau(v)) d_{n-1}(\tau(x)) (\tau^n(uv)) \\
& \quad + \sigma^n(u) d_1(\sigma^{n-1}(v)) d_{n-1}(\tau(x)) (\tau^n(uv)) \\
& \quad + (\sigma^n(uv)) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(u)) \tau^n(v) \\
& \quad + (\sigma^n(uv)) d_{n-1}(\sigma(x)) (\sigma \tau^{n-1}(u)) d_1(\tau^{n-1}(v)) \\
& \quad \left. + (\sigma^n(uv)) d_n(x) d_k(\tau^n(uv)) \right). \tag{14}
\end{aligned}$$

Comparing both equations, reordering the indices and as characteristic of R is different from two we finally obtain

$$\left\{ f_n(uv) - \sum_{i+j=n} f_i(\sigma^{n-i}(u)) \right. \\
\left. \times d_j(\tau^{n-j}(v)) \right\} \tau^n(x) \tau^n(uv) = 0 \tag{15}$$

$$\forall u, v \in U.$$

As τ is one-one and onto implementing Lemma 1, it can be easily seen that $f_n(uv) - \sum_{i+j=n} f_i(\sigma^{n-i}(u)) d_j(\tau^{n-j}(v)) = 0$ for all $u, v \in U$ or $u'v' = 0$ for all $u', v' \in U$. Now if $u'v' = 0$ for all $u', v' \in U$, then U is commutative and hence using similar arguments as used in the second paragraph of the proof of Lemma 1.3 of [1] U is central, a contradiction. Therefore, $f_n(uv) - \sum_{i+j=n} f_i(\sigma^{n-i}(u)) d_j(\tau^{n-j}(v)) = 0$ for all $u, v \in U$. This completes the proof of our theorem. $\square$

Remark 6. For $\sigma = \tau = I_R$ and $f_n = d_n$ for each $n \in \mathbb{N}$ our result reduces to the main result obtained by Ferrero and Haetinger [9]

Corollary 7 ([9, Theorem 1.2]). *Let R be a prime ring of characteristic different from two and let U be a noncentral square closed Lie ideal of R . Then, every Jordan triple higher derivation of U into R is a higher derivation of U into R .*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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