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Review Article

A Survey on Seifert Fiber Space Theorem

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We review the history of the proof of the Seifert fiber space theorem, as well as its motivations in 3-manifold topology and its generalizations.

1. Introduction

The reader is supposed to be familiar with the topology of 3-manifolds. Basic courses can be found in reference books (cf. [1–3]).

In the topology of low-dimensional manifolds (dimension at most 3) the fundamental group, or π_1, plays a central key role. On the one hand several of the main topological properties of 2 and 3-manifolds can be rephrased in term of properties of the fundamental group and on the other hand in the generic cases the π_1 fully determines their homeomorphic type. That the π_1 generally determines their homotopy type follows from the fact that they are generically $K(\pi, 1)$ (i.e. their universal covering space is contractible); that the homotopy type determines generically the homeomorphism type appears as a rigidity property, or informally because the lack of dimension prevents the existence of too many manifolds, which contrasts with higher dimensions.

This has been well known since a long time for surfaces; advances in the study of 3-manifolds have shown that it remains globally true in dimension 3. For example, one can think of the Poincaré conjecture, the Dehn and sphere theorems of Papakyriakopoulos, the torus theorem, the rigidity theorem for Haken manifold, or Mostow's rigidity theorem for hyperbolic 3-manifolds, and so forth. It provides a somehow common paradigm for their study, much linked to combinatorial and geometrical group theory, which has developed into an independent discipline: *low-dimensional topology*, among the more general *topology of manifolds*.

The first reference book on the subject originated in the annotated notes that the young student Seifert took during the courses of algebraic topology given by Threlfall. In 1933, Seifert introduces a particular class of 3-manifolds, known as *Seifert manifolds* or *Seifert fiber spaces*. They have been since widely studied, well understood, and have given a great impact on the modern understanding of 3-manifolds. They suit many nice properties, most of them being already known since the deep work of Seifert.

Nevertheless one of their main properties, the so-called *Seifert fiber space theorem*, has been a long standing conjecture before its proof was completed by a huge collective work involving Waldhausen, Gordon and Heil, Jaco and Shalen, Scott, Mess, Tukia, Casson and Jungreis, and Gabai, for about twenty years. It has become another example of the characteristic meaning of the π_1 for 3-manifolds.

The Seifert fiber space conjecture characterizes the Seifert fiber spaces with infinite π_1 in the class of orientable irreducible 3-manifolds in terms of a property of their fundamental groups: they contain an infinite cyclic normal subgroup. It has now become a theorem of major importance in the understanding of (compact) 3-manifolds as we further explain.

We review here the motivations and applications for the understanding of 3-manifolds of the Seifert fiber space theorem, its generalizations for nonorientable 3-manifolds and PD(3) groups, and the various steps in its proof.

2. The Seifert Fiber Space Theorem and Its Applications

2.1. Reviews on Seifert Fiber Space. Seifert fibered spaces originally appeared in a paper of Seifert [4]; they constitute a large class of 3-manifolds and are totally classified by means of a finite set of invariants. They have since widely appeared in the literature for playing a central key role in the topology of compact 3-manifolds and being nowadays (and since the original paper of Seifert) very well known and understood. They have allowed the development of central concepts in 3-manifold topology such as the JSJ-decomposition and Thurston's geometrization conjecture.

2.1.1. Seifert Fiber Spaces. Let M and N be two 3-manifolds, each being a disjoint union of a collection of simple closed curves called *fibers*. A *fiber-preserving homeomorphism* from M to N is a homeomorphism which sends each fiber of M onto a fiber of N; in such case M and N are said to be *fiber-preserving homeomorphic*.

Let $\mathbb{D}^2 = \{z \in \mathbb{C}; |z| \leqslant 1\}$ denotes the unit disk in the complex plane. A *fibered solid torus* is obtained from $\mathbb{D}^2 \times [0,1]$ by identifying the bottom and top disks $\mathbb{D}^2 \times 0$ and $\mathbb{D}^2 \times 1$ via a homeomorphism $(z,0) \mapsto (r(z),1)$ for some rotation $r : z \mapsto z \times \exp(i\theta)$ with angle θ commensurable with π. Therefore r has a finite order, say $n+1$, so that the images after identification of the union of intervals: $\bigcup_{i=0}^{n} r^i(z) \times [0,1]$ for $z \in \mathbb{D}^2$, yield a collection of fibers. When r has order > 1 the core fiber, image of $0 \times [0,1]$, is said to be exceptional and otherwise to be regular; all other fibers are said to be regular.

In the original definition of Seifert (the English version of the original paper of H. Seifert introducing Seifert fibered spaces (translated from German by W. Heil), among other things he classifies them and computes their fundamental group [5]), a fibration by circles of a closed 3-manifold M is a decomposition into a disjoint union of simple closed curves such that each fiber has a neighborhood fiber-preserving homeomorphic to a fibered solid torus. For a fiber, being regular or exceptional does not depend neither on the fiber-preserving homeomorphism nor on the neighborhood involved.

The definition naturally extends to compact 3-manifolds with nonempty boundary. It requires that the boundary be a disjoint union of fibers; therefore all components in the boundary are closed with Euler characteristic 0 and can only consist of tori and Klein bottles.

A Seifert fibration is a foliation by circles; the converse is true for orientable 3-manifolds: it is a result of Epstein [6] that the orientable Seifert fiber spaces are characterized among all compact orientable 3-manifolds as those that admit a foliation by circles.

For nonorientable 3-manifolds the two notions do not agree because of the possibility for a foliation by circles to be locally fiber-preserving homeomorphic to a fibered solid Klein bottle. A *fibered solid Klein bottle* is obtained from $\mathbb{D}^2 \times [0,1]$ by identifying $\mathbb{D}^2 \times 0$ and $\mathbb{D}^2 \times 1$ via the homeomorphism $(z,0) \mapsto (\overline{z},1)$. The manifold obtained is nonorientable, with boundary a Klein bottle, and is naturally foliated by circles. The exceptional fibers are the images of $z \times [0,1]$ for all $z \in [-1,1]$; they are nonisolated and lie on the exceptional 1-sided annulus image of $\{x \in \mathbb{R}; |x| \leqslant 1\} \times [0,1]$. The regular fibers are the images of all $\{z, \overline{z}\} \times [0,1]$ for all $z \notin [-1,1]$; they wrap 2 times around the exceptional fibers (cf. Figure 1).

Modern considerations have pointed out a need to enlarge the original definition of Seifert in order to englobe this phenomenon for nonorientable 3-manifolds. In a more modern terminology, a compact 3-manifold is a Seifert fiber space whenever it admits a foliation by circles. We will talk of a Seifert bundle to emphasize the nuance. It is a consequence of Epstein's result that such manifolds are those that decompose into disjoint unions of fibers with all fibers having a neighborhood fiber-preserving homeomorphic to a fibered solid torus or to a fibered solid Klein bottle, the latter appearing only for nonorientable manifolds.

As above, given a Seifert fibration of 3-manifold, all fibers fall in two parts, depending only on the fibration: the regular and the exceptional fibers; the latter can be either isolated or nonisolated. Let M be a Seifert bundle; given a Seifert fibration of M, the space of fibers, obtained by identifying each fiber to a point, is a surface B, called the *base* of the fibration. This surface is naturally endowed with a structure of orbifold. Its singularities consists only of a finite number of cone points (which are the images of the isolated exceptional fibers) and of a finite number of reflector circles and lines (which are the images of the nonisolated exceptional fibers). This base orbifold is an invariant of the Seifert fibration.

In a modern terminology involving orbifolds, Seifert bundles are those 3-manifolds in the category of 3-orbifolds which are circle bundles over a 2-dimensional orbifold without corner reflectors singularity. This yields a second invariant e, the Euler number associated to the bundle (cf. [7]).

2.1.2. Some Topological Properties of Seifert Fiber Spaces. A Seifert bundle M is a circle bundle over a 2-dimensional orbifold, and it follows that the fibration lifts to the universal covering space $\widetilde{M}$ of M into a fibration by circles or lines over an orbifold $\widetilde{B}$ without proper covering. When the fibers are circles, it can only be S^2 with at most 2 cone points. In such case M is obtained by gluing on their boundary two fibered solid tori, and therefore it is a lens space, so that its universal covering space can only be S^3 or $S^2 \times \mathbb{R}$. When the fibers are lines $\widetilde{B}$ can only be one of S^2 or $\mathbb{R}^2$ and $\widetilde{M}$ is homeomorphic to $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ (see [7] for details).

Proposition 1. *The universal cover of a Seifert bundle has total space homeomorphic to either $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or S^3. Moreover, the Seifert fibration lifts in the universal cover to a foliation by lines (in both two former cases) or circles (in the latter case).*

Recall that a 3-manifold M is *irreducible* when every sphere embedded in M bounds a ball. An irreducible 3-manifold which does not contain any embedded 2-sided $\mathbb{P}^2$ is said to be $\mathbb{P}^2$-irreducible. Now by Alexander's theorems S^3 and $\mathbb{R}^3$ are irreducible, and it follows that any 3-manifold covered by S^3 or $\mathbb{R}^3$ is $\mathbb{P}^2$-irreducible. So that the only

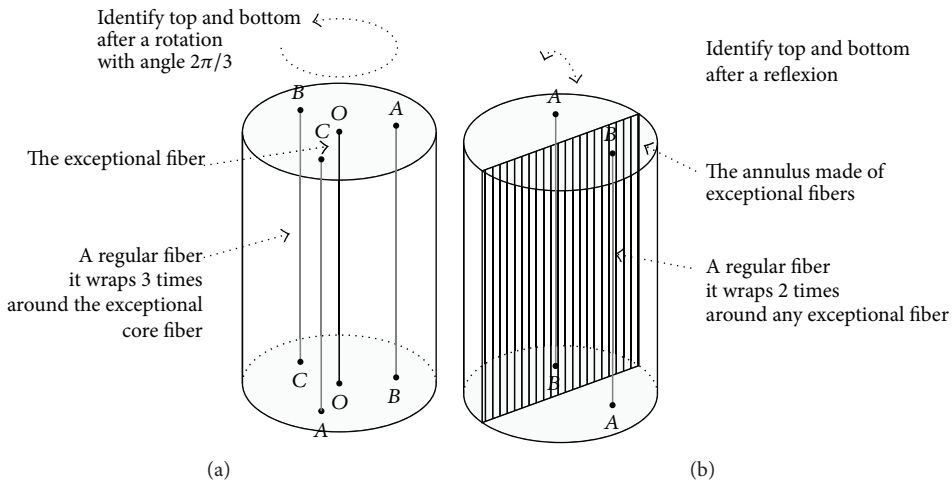

Figure 1: On the left, a fibered solid torus. On the right, a fibered solid Klein bottle.

non-$\mathbb{P}^2$-irreducible Seifert bundles are covered by $S^2 \times \mathbb{R}$, and there are very few such manifolds (cf. [8]); one obtains the following.

Theorem 2. *With the exceptions of $\mathbb{P}^3 \# \mathbb{P}^3$, $S^2 \times S^1$, and $S^2 \tilde{\times} S^1$, a Seifert bundle is irreducible. The only irreducible and non-$\mathbb{P}^2$-irreducible Seifert fiber bundle is $\mathbb{P}^2 \times S^1$.*

A 3-manifold M is said to be *Haken* if M is $\mathbb{P}^2$-irreducible and either M is B^3 or M contains a 2-sided properly embedded incompressible surface F. Consider a two-sided simple curve in the base orbifold that consists only of nonsingular points and is closed or has its both extremities lying in the boundary. Whenever that curve does not bound on one of its sides a disk with at most one cone point, its preimage yields a properly embedded two-sided incompressible surface.

Theorem 3. *With the exception of lens spaces, $\mathbb{P}^3 \# \mathbb{P}^3$, $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, and $\mathbb{P}^2 \times S^1$, a Seifert bundle is either Haken or has base S^2 and exactly 3 exceptional fibers; in this last case M is Haken if and only if $H_1(M, \mathbb{Z})$ is infinite.*

Note that a $\mathbb{P}^2$-irreducible 3-manifold M whose first homology group $H_1(M, \mathbb{Z})$ is infinite does contain a 2-sided properly embedded incompressible surface and therefore is Haken; nevertheless, there exists Haken 3-manifolds $\neq B^3$ with finite first homology group (examples can be constructed by performing Dehn obturations on knots).

2.1.3. The Cyclic Normal Subgroup of the π_1 of a Seifert Fiber Space. Existence of a Seifert fibration on a manifold M has a strong consequence on its fundamental group. We have seen that the fibration lifts to the universal cover space $\widetilde{M}$ into a foliation by circles or lines with basis $\widetilde{B}$. The action of $\pi_1(M)$ onto $\widetilde{M}$ preserves the circles or lines of the foliation, so that it induces an action of $\pi_1(M)$ onto $\widetilde{B}$ that defines a homomorphism ϕ from $\pi_1(M)$ onto the orbifold fundamental group $\pi_1^{\mathrm{orb}}(B)$ of B. The kernel $\ker \phi$ consists of those elements that act as the identity on $\widetilde{B}$ or equivalently that preserve all the fibers of $\widetilde{M}$. Therefore $\ker \phi$ acts freely on each fiber. In case $\widetilde{M}$ is fibered by lines, $\ker \phi$ is infinite cyclic. In case $\widetilde{M}$ is $\mathbb{S}^3$, $\ker \phi$ is finite cyclic and $\pi_1(M)$ is finite. Finally, since the covering restricts onto the fibers of $\widetilde{M}$ to covers of the fibers of M, $\ker \phi$ is generated by any regular fiber.

Theorem 4. *Let M be a Seifert bundle with base orbifold B. Then one has the short exact sequence:*

$$1 \longrightarrow N \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1, \tag{1}$$

where N is the cyclic subgroup generated by any regular fiber. Moreover N is infinite whenever $\pi_1(M)$ is infinite.

In particular from $\pi_1^{\mathrm{orb}}(B)$ and the homomorphism $\pi_1^{\mathrm{orb}}(B) \to \mathrm{Aut}(N)$ (note that $\mathrm{Aut}(N)$ has order 1 or 2), one deduces a finite presentation for any Seifert bundle, strongly related to its Seifert fibration (see [5, 7, 9]).

2.2. The Seifert Fiber Space Theorem

2.2.1. Statement of the Seifert Fiber Space Theorem. As seen above an infinite fundamental group of a Seifert bundle contains a normal infinite cyclic subgroup. The Seifert fiber space theorem (or S.f.s.t.) uses this property of the π_1 to characterize Seifert fiber spaces in the class of orientable irreducible 3-manifolds with infinite π_1. It can be stated as follows.

Theorem 5 (Seifert Fiber Space Theorem, 1990s). *Let M be an orientable irreducible 3-manifold whose π_1 is infinite and contains a nontrivial normal cyclic subgroup. Then M is a Seifert fiber space.*

It has become a theorem with the common work of a large number of topologists: for the Haken case: Waldhausen (Haken manifolds whose group has a nontrivial center

are Seifert fibered [10]), Gordon and Heil (S.f.s.t partially obtained in the Haken case: either M is a Seifert fiber space or M is obtained by gluing two copies of a twisted I-bundle [11]), and Jaco and Shalen (the end of the proof of S.f.s.t. in the Haken case; but almost the famous Jaco-Shalen-Johansen theorem [12]); for the non-Haken case: the rigidity theorem for Seifert fiber spaces Scott [13], Mess (reduces the proof of the S.f.s.t. to prove that convergence groups of the circle are virtually surface groups [14], unpublished), Tukia (solves partially the convergence groups conjecture [15]), Casson and Jungreis (provide an alternate solution independently by giving solutions to the case remaining in Tukia; they deduce the S.f.s.t. [16]), and Gabai (proves that convergence groups of the circle are fuchsian groups; he deduces the S.f.s.t., torus theorem and geometrization in that case [17]); for the orientable non-Haken case, one can also cite Maillot (showed that groups quasi-isometric to a simply connected complete Riemannian surface are virtually surface groups [18] and extends the strategy of Mess et al. to recover the S.f.s.t. in case of 3-orbifolds, and in corollary S.f.s.t. for 3-manifolds [19]) and Bowditch (gives an independent proof of the S.f.s.t.; it generalizes to PD(3) groups [20]) who give an alternate proof including the unpublished key result of Mess.

It has been generalized to the nonorientable case by Whitten.

Theorem 6 (Seifert Bundle Theorem, (S.f.s.t. generalized to the nonorientable case) [21]). *Let M be a $\mathbb{P}_2$-irreducible 3-manifold whom π_1 is infinite and contains a nontrivial cyclic normal subgroup. Then M is a Seifert bundle.*

Further Heil and Whitten have generalized the theorem to the nonorientable irreducible 3-manifolds (cf. the case nonorientable irreducible non-$\mathbb{P}^2$-irreducible of the S.f.s.t. They deduce the torus theorem and geometrization in that case (modulo fake $\mathbb{P}_2 \times S^1$, that is, modulo the Poincaré conjecture.) [22]); see Theorem 9.

We can also note that those two theorems can be generalized in several ways: for open 3-manifolds and 3-orbifolds [19], and for PD(3) groups [20], or by weakening the condition of existence of a normal $\mathbb{Z}$ by the existence of a nontrivial finite conjugacy class (the condition in S.f.s.t. that the group contains a normal $\mathbb{Z}$ can be weakened by the condition that it contains a nontrivial finite conjugacy class, or equivalently that its Von-Neumann algebra is not a factor of type $II-1$. It applies also to PD(3) groups [23]).

2.2.2. Explanations of the Hypotheses Involved. Let us now discuss the hypotheses of the theorems.

(i) Using the sphere theorem together with classical arguments of algebraic topology one sees that a $\mathbb{P}_2$-irreducible manifold with infinite π_1 has a torsion free fundamental group. So that in both conjectures one can replace:

"...whose π_1 is infinite and contains a nontrivial normal cyclic subgroup." by:

"...whose π_1 contains a normal infinite cyclic subgroup."

(ii) What about nonirreducible orientable 3-manifolds with $\mathbb{Z} \triangleleft \pi_1(M)$? An orientable Seifert fiber space is either irreducible or homeomorphic to $S^1 \times S^2$ or to $\mathbb{P}^3 \# \mathbb{P}^3$. As a consequence of the Kneser-Milnor theorem, an orientable nonirreducible 3-manifold M whose π_1 contains a nontrivial normal cyclic subgroup is either $S^1 \times S^2$, or $M' \# C$ with M' irreducible and C simply connected, or its π_1 is the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$. With the Poincaré conjecture (now stated by Perelman) M is obtained from $S^1 \times S^2$, $\mathbb{P}^3 \# \mathbb{P}^3$ or from an irreducible 3-manifold by removing a finite number of balls. Therefore the result becomes as follows.

Theorem 7. *Let M be an orientable nonirreducible 3-manifold whose π_1 contains an infinite cyclic normal subgroup. Then the manifold obtained by filling all spheres in ∂M with balls is a Seifert fiber space.*

(i) What about irreducible and non-$\mathbb{P}^2$-irreducible 3-manifolds with $\mathbb{Z} \triangleleft \pi_1(M)$? A nonorientable Seifert fiber space is either $\mathbb{P}^2$-irreducible or $S^1 \tilde{\times} S^2$ or $\mathbb{P}^2 \times S^1$. A large class of irreducible, non-$\mathbb{P}^2$-irreducible 3-manifolds does not admit a Seifert fibration whereas their π_1 contains $\mathbb{Z}$ as a normal subgroup. We will see that nevertheless the result generalizes also in that case by considering what Heil and Whitten call a Seifert bundle mod $\mathbb{P}$.

(ii) What about 3-manifolds with finite π_1? According to the Kneser-Milnor decomposition, such manifolds, after eventually filling all spheres in the boundary with balls and replacing all homotopy balls with balls, become irreducible. Note that as a consequence of the Dehn's lemma, their boundary can only consist of $\mathbb{S}^2$ and $\mathbb{P}^2$. The only nonorientable irreducible 3-manifold with finite π_1 is $\mathbb{P}^2 \times I$ (a result of D. Epstein, cf. [3] Theorems 9.5 and 9.6). All known orientable irreducible 3-manifolds with finite π_1 turn to be elliptic manifolds, which are all Seifert fibered, and the orthogonalization conjecture (cf. section 2.3.3 proved by the work of Perelman) has made a statement of that, one of its consequences being the Poincaré conjecture that all homotopy balls are real balls. Thus the long standing conjecture has become a theorem.

Theorem 8 (Consequence of the Orthogonalization Conjecture, 2000s). *A 3-manifold with finite π_1 containing no sphere in its boundary is a closed Seifert fibered space when orientable and $\mathbb{P}^2 \times I$ when nonorientable.*

2.3. Motivations. Three important questions in 3-dimensional topology have motivated the birth of the Seifert fiber space conjecture (S.f.s.c. for short). At first was the center conjecture (1960s), then the torus theorem conjecture (1978), and finally Thurston's geometrization conjecture (1980s).

2.3.1. The Center Conjecture. It is the problem 3.5 in Kirby's list of problems in 3-dimensional topology [24] that has been attributed to Thurston.

Center Conjecture. Let M be an orientable irreducible 3-manifold with infinite π_1 having a nontrivial center. Then M is a Seifert fiber space.

The conjecture is an immediate corollary of the Seifert fiber space theorem since any cyclic subgroup in the center of a group G is normal in G.

It has first been observed and proved for knot complements: Murasugi [25] and Neuwirth (proof of the center conjecture for alternated knots [26]) in 1961 for alternated knots and then Burde and Zieschang (proof of the center conjecture for knot complements: those whose π_1 has center $\neq 1$ are the toric knot complements [27]) in 1966 for all knots. In 1967 Waldhausen [10] has proved the more general case of Haken manifolds. The proof has been achieved in the nineties with that of the Seifert fiber space theorem.

From a presentation of a Seifert fiber space one shows that an infinite cyclic normal subgroup in the π_1 of an orientable Seifert fiber space is central if and only if the base is orientable. A Seifert fiber space with infinite π_1 and with a nonorientable base has a centerless π_1 containing a nontrivial normal cyclic subgroup. The S.f.s.c. generalizes the center conjecture in that sense.

2.3.2. The Torus Theorem Conjecture. The long-standing so-called "torus theorem conjecture" asserts the following.

Torus Theorem Conjecture. Let M be an orientable irreducible 3-manifold with $\pi_1(M)$ containing $\mathbb{Z}\oplus\mathbb{Z}$ as a subgroup. Then either M contains an incompressible torus or M is a (small) Seifert fiber space.

In has been proven to be true for Haken manifolds by Waldhausen in 1968 (announced in [28] (announcement of the torus theorem for Haken manifolds.)), written and published by Feustel in [29, 30]). Note that together with the theory of *sufficiently large* 3-manifolds developed by Haken during the 60s, this partial result has finally led to the *Jaco-Shalen-Johansen* decomposition of Haken 3-manifolds (1979, cf. [12]).

The proof in full generality of the conjecture reduces to that of the S.f.s.c. help to the "strong torus theorem" proved by Scott in 1978 (the "strong torus theorem" [31]).

Strong Torus Theorem. Let M be an orientable irreducible 3-manifold, with $\pi_1(M)$ containing $\mathbb{Z}\oplus\mathbb{Z}$ as a subgroup. Then either M contains an incompressible torus, or $\pi_1(M)$ contains a nontrivial normal cyclic subgroup.

With the strong torus theorem, the torus theorem conjecture follows as a corollary from the Seifert fiber space theorem.

2.3.3. The Geometrization Conjecture. A major problem in 3-dimensional topology is the classification of 3-manifolds. Surfaces have been classified at the beginning of the nineteen's century help to the Euler characteristic. The Poincaré's duality (cf. [2]) has consequence that for any closed 3-manifold the Euler characteristic vanishes to 0; the techniques used in dimension 2 cannot be applied in dimension 3.

Topological surfaces inherit more structure from Riemannian geometry. The uniformization theorem of Riemann shows that they can be endowed with a complete Riemannian metric of constant curvature, so that they all arise as a quotient space of one of the three 2-dimensional geometrical spaces: euclidian $\mathbb{E}^2$, elliptic $\mathbb{S}^2$, and hyperbolic $\mathbb{H}^2$, by a discrete subgroup of isometry that acts freely.

Such property cannot generalize to dimension 3, for there are known obstructions: for example, a connected sum of nonsimply connected 3-manifolds cannot be modeled on one of the geometrical spaces $\mathbb{E}^3$, $\mathbb{S}^3$, and $\mathbb{H}^3$; otherwise an essential sphere would lift to the universal cover into an essential sphere in $\mathbb{E}^3$ or $\mathbb{H}^3$ ($\mathbb{S}^3$ cannot arise since the π_1 is infinite), but $\mathbb{E}^3$ and $\mathbb{H}^3$ are both homeomorphic to $\mathbb{R}^3$ and do not contain any essential sphere by Alexander's theorem.

Nevertheless, this fact generalizes to dimension 3 help to the topological decomposition of 3-manifolds. Any orientable 3-manifold can be cut canonically along essential surfaces: spheres (Kneser-Milnor), discs (Dehn's lemma), and tori (Jaco-Shalen-Johanson) so that the pieces obtained are all irreducible with incompressible boundary and are either Seifert fiber spaces or do not contain any incompressible torus.

Thurston's geometrization conjecture asserts that all the pieces obtained in the topological decomposition of an orientable 3-manifold can be endowed in their interior with a complete locally homogeneous Riemannian metric or equivalently that their interior are quotients of an homogeneous Riemannian simply connected 3-manifold by a discrete subgroup of isometry acting freely.

Thurston has proved that there are eight homogeneous (nonnecessarily isotropic) geometries in dimension 3: the three isotropic geometries, elliptic $\mathbb{S}^3$, euclidian $\mathbb{E}^3$, and hyperbolic $\mathbb{H}^3$, the two product geometries, $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, and the three twisted geometries, *Nil*, *Sol*, and the universal cover $\widetilde{SL_2\mathbb{R}}$ of $SL_2\mathbb{R}$ (cf. [7]).

It turns out that the orientable 3-manifolds modeled on one of the six geometries $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, *Nil*, and $\widetilde{SL_2\mathbb{R}}$ are precisely the Seifert fiber spaces (and the geometry involved only depends on the Euler characteristic of the base and on the Euler number e of the bundle, see Theorem 5.3.ii in [7]). It follows from that all discrete subgroups of their isometry group preserve a foliation by lines or circles of the underlying topological spaces S^3, $S^2\times\mathbb{R}$ or $\mathbb{R}^3$.

The strategy for proving the geometrization conjecture has naturally felt into three conjectures.

Conjectures. Let M be an orientable irreducible 3-manifolds.

(1) (Orthogonalization conjecture). If $\pi_1(M)$ is finite, then M is elliptic.

(2) (S.f.s.c.). If $\pi_1(M)$ is infinite and contains a nontrivial normal cyclic subgroup then M is a Seifert fiber space (S.f.s.c.)

(3) If $\pi_1(M)$ is infinite and contains no nontrivial normal cyclic subgroup then M is geometrizable.

With the strong torus theorem [31] the third conjecture reduces to conjecture 2 and also to consider the two cases when $\pi_1(M)$ does not contain $\mathbb{Z}\oplus\mathbb{Z}$ and when M contains an incompressible torus. In the latter case, the JSJ decomposition theorem applies. A manifold modeled on one of the seven nonhyperbolic geometries has, with a few exceptions well understood, a π_1 containing $\mathbb{Z}\oplus\mathbb{Z}$. Finally, the conjecture 3 reduces to the conjecture 2 and to the following conjecture.

(4) (Hyperbolization conjecture). If $\pi_1(M)$ is infinite and contains no $\mathbb{Z}\oplus\mathbb{Z}$ then M is hyperbolic.

Thurston has proved the conjecture to be true for Haken manifolds by proving the following.

Thurston's Hyperbolization Theorem. Any Haken manifold with infinite π_1 that contains no incompressible torus is hyperbolic.

Indeed a Haken 3-manifold with finite π_1 is a ball and it follows from Thurston's hyperbolization theorem together with the Seifert fiber space theorem for Haken manifolds that the geometrization conjecture is true for Haken 3-manifolds.

To prove the geometrization conjecture in all cases it suffices to prove the orthogonalization conjecture, the S.f.s.c., and the hyperbolization conjecture for non-Haken orientable 3-manifolds.

So that the S.f.s.c. appears as one of three parts of the geometrization conjecture that the conjecture is true for orientable irreducible 3-manifolds whose π_1 contains an infinite cyclic normal subgroup. It is the one usually considered as the easiest one and has been the first to be proved. The two other parts have been proved by the work of Perelman et al. following the Hamilton program making use of the Ricci flow (cf. [32]).

3. History of the Proof of the S.f.s. Theorems

The progress in proving the conjecture has been made as follows.

3.1. The Haken Orientable Case: 1967–1979. In *1967*, Waldhausen and Feustel [10, 29, 30] show that a Haken 3-manifold has a π_1 with nontrivial center if and only if it is a Seifert fiber space with an orientable base. It motivates the S.f.s.c. and solves the Haken case when the cyclic normal subgroup is central. The result is announced by Waldhausen in [10] and a proof appears in two papers of Feustel [29, 30].

In *1975*, Gordon and Heil [11] show partially the S.f.s.c. in the Haken case: a Haken manifold M whose π_1 contains an infinite cyclic normal subgroup is either a Seifert space or is obtained by gluing two copies of a twisted I-bundle along a nonorientable surface. So that they reduce the remaining Haken cases to this case of 3-manifolds.

In *1979*, Jaco and Shalen [12] and independently MacLachlan (unpublished) achieve the proof for the remaining Haken 3-manifolds.

3.2. The Non-Haken Orientable Case: 1979–1994. Using the Haken case already proved, it suffices to restrict oneself to the closed 3-manifolds. The proof has proceeded in several main steps.

In *1983*, Scott [13], by generalizing a result of Waldhausen in the Haken case, shows a strong result of rigidity for Seifert fiber spaces.

Let M and N be two closed orientable irreducible 3-manifolds, with N a Seifert fiber space with infinite π_1. If $\pi_1(M)$ and $\pi_1(N)$ are isomorphic, then M and N are homeomorphic.

Note that with the sphere theorem and classical arguments of algebraic topology, M and N are $K(\pi,1)$ and the conditions "$\pi_1(M)$ and $\pi_1(N)$ which are isomorphic" can be replaced by "M and N which have the same homotopy type."

That reduces the proof of the Seifert fiber space theorem to a similar statement involving the fundamental group. The conjecture becomes as follows.

(S.f.s.c.) If M is a closed orientable 3-manifold and $\Gamma = \pi_1(M)$ contains an infinite cyclic normal subgroup, then Γ is the π_1 of a closed orientable Seifert fiber space.

Scott remarks also that Γ is the group of a closed orientable Seifert fiber space if and only if $\Gamma/\mathbb{Z}$ is the fundamental group of a 2-dimensional closed orbifold (eventually nonorientable); an alternate proof is given in the Lemma 15.3 of [20]. So finally, he reduces the proof of the Seifert fiber space theorem to the proof of the following conjecture.

(S.f.s.c.) If M is a closed orientable 3-manifold and $\Gamma = \pi_1(M)$ contains an infinite cyclic normal subgroup, then $\Gamma/\mathbb{Z}$ is the fundamental group of a (closed) 2-orbifold.

Late *1980s*, Mess [14] proves in an unpublished paper.

Let M be closed orientable and irreducible with $\Gamma = \pi_1(M)$ containing an infinite cyclic normal subgroup C. Then

(i) the covering of M associated to C is homeomorphic to the open manifold $\mathbb{D}^2 \times S^1$;

(ii) the covering action of Γ/C on $\mathbb{D}^2 \times S^1$ gives rise to an almost everywhere defined action on $\mathbb{D}^2\times 1$: following its terminology Γ/C is coarse quasi-isometric to the Euclidean or hyperbolic plane;

(iii) in the case where Γ/C is coarse quasi-isometric to the Euclidean plane, then it is the group of a 2-orbifold; hence, (with the result of Scott) M is a Seifert fiber space;

(iv) in the remaining case where Γ/C is coarse quasi-isometric to the hyperbolic plane, Γ/C induces an action on the circle at infinity, which makes Γ/C a convergence group.

A *convergence group* is a group G acting by orientation preserving homeomorphism on the circle, in such a way that if T denotes the set of ordered triples: $(x, y, z) \in S^1 \times S^1 \times S^1$, $x \neq y \neq z$, with x, y, z appearing in that order on S^1 in the direct sense, the action induced by G on T is free and properly discontinuous.

With the work of Mess the proof of the S.f.s.c. reduces to that of:

(Convergence groups conjecture). Convergence groups are fundamental groups of 2-orbifolds.

In *1988*, Tukia shows (see [15]) that some of the convergence groups (when T/Γ is noncompact and Γ has no torsion elements with order >3) are Fuchsian groups and in particular are π_1^{orb} of 2-orbifolds.

In *1992*, Gabai shows (see [17]) independently from other works and in full generality that convergence groups are Fuchsian groups. They act on S^1, up to conjugacy in Homeo(S^1), as the restrictions on $S^1 = \partial\mathbb{H}^2$ of their natural action on $\overline{\mathbb{H}}^2$. That proves the S.f.s.t.

In *1994*, At the same time Casson and Jungreis show the cases left remained by Tukia (see [16]). That proves independently from Gabai the S.f.s.t.

The Seifert fiber space theorem is proved and then follow the center conjecture, the torus theorem, and one of three of the geometrization conjectures.

In *1999*, later, Bowditch obtains a different proof of the S.f.s.c. (see [20]). Its proof generalizes to other kinds of group as PD(3) groups (groups of type FP with cohomological dimension $\leqslant 3$ whose cohomology groups satisfy a relation somewhat analogous to Poincaré duality for 3-manifolds; it is conjectured that they coincide with π_1 of closed $\mathbb{P}^2$-irreducible 3-manifolds with infinite π_1; see [20]).

In *2000*, in his thesis Maillot extends the techniques of Mess to establish a proof of the S.f.s.c. in the more general cases of open 3-manifolds and of 3-orbifolds. This result already known as a consequence of the S.f.s.c. (as proved by Mess et al.) and of the Thurston orbifolds theorem, has the merit to be proved by using none of these results and to state as a corollary a complete proof of the S.f.s.c. and of the techniques of Mess.

In *2003*, the argument of Mess is pursued by Maillot in [19] where he reduces to groups quasi-isometric to a Riemannian plane, and he shows in [18] that they are virtually surface groups (and hence Fuchsian groups).

3.3. The Nonorientable Case: 1992–94. The solution to the S.f.s.c. in the nonorientable case goes back to Whitten, improved by Whitten and Heil.

In *1992*, Whitten proves in [21] the S.f.s.t. for nonorientable and irreducible 3-manifolds, which are not a fake $\mathbb{P}_2 \times S^1$, and whose π_1 does not contain any $\mathbb{Z}_2 * \mathbb{Z}_2$. He obtains in particular the S.f.s.t. in the $\mathbb{P}_2$-irreducible case.

In 1994, Heil and Whitten in [22] characterise those nonorientable irreducible 3-manifolds which do not contain a fake $\mathbb{P}_2 \times I$ and whose π_1 contains a normal $\mathbb{Z}$ subgroup and possibly $\mathbb{Z}_2 * \mathbb{Z}_2$: they call them *Seifert bundles* mod $\mathbb{P}$; they constitute a larger class than Seifert bundles. Consider the manifold $\mathbb{P}$ obtained from two copies of $\mathbb{P}^2 \times I$ by a connected disk summand, that is, by gluing a $\mathbb{D}^2 \times I$ along $\mathbb{D}^2 \times 0$ and $\mathbb{D}^2 \times 1$ to one boundary components of each copies of $\mathbb{P}^2 \times I$ (see Figure 2). It is a nonorientable 3-manifold that is irreducible with a boundary made of two $\mathbb{P}^2$ and one Klein bottle. Its fundamental group is the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$, which contains a unique infinite cyclic normal subgroup. The Klein bottle in the boundary contains a "fibered" annulus that is made of two rectangular bands with vertices A, B, C, and D (as in Figure 2) whose π_1 embeds in $\pi_1(\mathbb{P})$ onto the normal

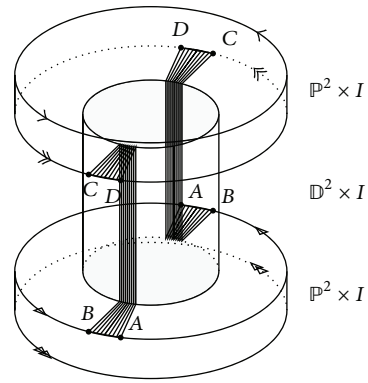

Figure 2: The nonorientable 3-manifold $\mathbb{P}$ and a fibered annulus in the Klein bottle component of $\partial\mathbb{P}$.

$\mathbb{Z}$, so that it can be fibered with fibers that generate the cyclic infinite normal subgroup.

A Seifert bundle mod $\mathbb{P}$ is either $\mathbb{P}$ or is obtained from a Seifert bundle by gluing on its boundary a finite number of copies of the manifold $\mathbb{P}$ along fibered annuli and such that the fibers map onto fibers. Therefore, the π_1 contains an infinite cyclic normal subgroup. From the irreducibility of $\mathbb{P}$ and the incompressibility of the fibered annuli follows that Seifert bundles mod $\mathbb{P}$, with the same exceptions as in Theorem 2, are all irreducible.

Theorem 9. *Let M be a nonorientable irreducible 3-manifold which does not contain a fake $\mathbb{P}_2 \times I$ and such that $\pi_1(M)$ contains a nontrivial cyclic normal subgroup. Then M is either $\mathbb{P}_2 \times I$ or a Seifert bundle* mod $\mathbb{P}$.

They deduce in the nonorientable case the torus theorem: "if M is an irreducible nonorientable 3-manifold with $\pi_1(M) \supset \mathbb{Z} \oplus \mathbb{Z}$, then M contains an incompressible torus or Klein bottle."

The orientation covering space N of a Seifert bundle mod $\mathbb{P}$, M, becomes a Seifert bundle $\overline{N}$ once all spheres in its boundary have been filled up with balls. The covering involution of N extends to $\overline{N}$ with isolated fixed points and quotient an orbifold $\overline{M}$, obtained from M by filling up all projective planes in its boundary with cones over $\mathbb{P}^2$. The orbifold $\overline{M}$ is modeled in the sense of orbifolds onto one of the 6 Seifertic geometries, and finally Seifert bundles mod $\mathbb{P}$ are precisely those nonorientable 3-manifolds that are said to be modeled (with a noncomplete metric) onto one of the 6 geometries: $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, *Nil*, and the universal cover of $\widetilde{SL_2\mathbb{R}}$.

3.4. With the Poincaré Conjecture. In *2003–06*, the work of Perelman (2003 et al.) proves the Poincaré conjecture as a corollary of the orthogonalization conjecture. Hence, each simply connected closed 3-manifold is a 3-sphere and there exists no fake ball or fake $\mathbb{P}_2 \times I$. As already said one can simply avoid in the S.f.s. theorem the hypothesis of irreducibility. The more general statement of the theorem becomes as follows.

Theorem 10. *Let M be a 3-manifold whose π_1 is infinite and contains a nontrivial cyclic normal subgroup. After filling up all spheres in ∂M with balls one obtains either a connected sum of $\mathbb{P}_2 \times I$ with itself or with $\mathbb{P}_3$, or a Seifert bundle* mod $\mathbb{P}$. *Moreover, M is a Seifert bundle exactly when $\pi_1(M)$ does not contain any $\mathbb{Z}_2$ or equivalently when ∂M does not contain any $\mathbb{P}_2$.*

3.5. 3-Manifold Groups with Infinite Conjugacy Classes. 2005. It is shown in [4] that in both cases of 3-manifold groups and of PD(3)-groups the hypothesis in the S.f.s. and Seifert bundles theorems: "*contains a nontrivial cyclic subgroup*" can be weakened into "*contains a nontrivial finite conjugacy class*" or equivalently into "*has a Von Neumann algebra which is not a factor of type II-1*". The results become:

(1) Let G be either an infinite π_1 of a $\mathbb{P}^2$-irreducible 3-manifold, or a PD(3)-group, that contains a nontrivial finite conjugacy class. Then G is the π_1 of a Seifert bundle.

(2) Let G be an infinite π_1 of a irreducible 3-manifold. If G contains a nontrivial finite conjugacy class, then G is the π_1 of a Seifert bundle mod $\mathbb{P}$.

The proof of these results makes use of the S.f.s. theorems for 3-manifolds as well as for PD(3) groups.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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