Research Article

Exact Solution of Thermoelastic Problem for a One-Dimensional Bar without Energy Dissipation

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We consider a homogeneous isotropic thermoelastic half-space in the context of the theory of thermoelasticity without energy dissipation. There are no body forces or heat source acting on the half-space. The surface of the half-space is affected by a time dependent thermal shock and is traction free. The Laplace transform with respect to time is used. The inverse transforms are obtained in an exact manner for the temperature, thermal stress, and displacement distributions. These solutions are represented graphically and discussed for several cases of the applied heating. Comparison is made between the predictions here and those of the theory of thermoelasticity with one relaxation time.

1. Introduction

In 1967, Lord and Shulman [1] introduced the theory of generalized thermoelasticity with one relaxation time for an isotropic body. This theory corrects the unrealistic conclusions of the older theories (the uncoupled and the coupled theories of thermoelasticity) that heat waves travel with infinite speeds. The Heat conduction law of this theory is the Cattaneo law which is different from Fourier's law utilized in both the coupled and the uncoupled theories. Among the contribution to the subject are the works in [2–6]. In 1972 Green and Lindsay [7] developed the theory of generalized thermoelasticity with two relaxation times, based on a generalized inequality of thermodynamics. In this theory both the equations of motion and of heat conduction are hyperbolic. The heat conduction law is the same as Fourier's law when the system has a centre of symmetry. Among the contributions to this theory are the works in [8, 9].

Green and Naghdi [10–12] have formulated three new models of thermoelasticity. In one of these models Green and Naghdi [12] predict that the internal rate of production of entropy is identically zero; that is, there is no dissipation of thermal energy. This theory (GN theory) is known as thermoelasticity without energy dissipation theory. In the

development of this theory the thermal displacement gradient is considered as a constitutive variable, whereas in the conventional development of a thermoelasticity theory, the temperature gradient is taken as a constitutive variable [12]. A couple of uniqueness theorems have been proved in [13, 14], and one-dimensional waves in a half-space and in an unbounded body have been studied in [15, 16]. In view of some experimental evidence available in favour of finiteness of heat propagation speed, generalized thermoelasticity theories are considered to be more realistic than the conventional theory in dealing with practical problems involving very large heat fluxes at short intervals, like those occurring in laser units and energy channels [17].

2. Formulation of the Problem

In this work, we consider a homogeneous isotropic and thermoelastic half-space in the context of the theory of thermoelasticity without energy dissipation. There are no body forces or heat source acting on the half-space taken as $x \geq 0$. The surface of the half-space is affected by a time dependent thermal shock and is traction free.

From the physics of the problem, all functions will depend on the distance x and the time t only. The problem is thus one-dimensional. The displacement components have the form

$$\underline{u} = (u(x, t), 0, 0). \quad (1)$$

The basic equations due to Green and Nagdhi [12] in the absence of body forces and heat sources for isotropic elastic medium are given by the following:

(i) equation of motion:

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad (2)$$

(ii) the equation of heat conduction:

$$k^* \frac{\partial^2 T}{\partial x^2} = \rho C_E \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^3 u}{\partial t^2 \partial x}, \quad (3)$$

(iii) the constitutive equation:

$$\sigma_{xx} = \sigma = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \gamma (T - T_0), \quad (4)$$

where λ and μ are Lamé's moduli, T is the absolute temperature of the medium, and γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$, where α_t is the coefficient of linear thermal expansion and T_0 is a reference temperature assumed to be such that $|(T - T_0)/T_0| \ll 1$. k^* is a material constant, characteristic of the theory and C_E is the specific heat at constant strain.

Let us introduce the following nondimension variables:

$$\begin{aligned} x^* &= c_1 \eta x, & u^* &= c_1 \eta u, & t^* &= c_1^2 \eta t, \\ \sigma^* &= \frac{\sigma}{\lambda + 2\mu}, & \theta &= \frac{\gamma (T - T_0)}{(\lambda + 2\mu)}. \end{aligned} \quad (5)$$

The governing equations (2)–(4) in nondimensional form become (dropping the asterisks for convenience)

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} &= \frac{\partial^2 u}{\partial t^2}, \\ C_t^2 \frac{\partial^2 \theta}{\partial x^2} &= \frac{\partial^2 \theta}{\partial t^2} + \varepsilon \frac{\partial^3 u}{\partial t^2 \partial x}, \\ \sigma &= \frac{\partial u}{\partial x} - \theta, \end{aligned} \quad (6)$$

where $C_t^2 = k^*/\rho C_E c_1^2$, and $\varepsilon = \gamma^2 T_0 / \rho C_E (\lambda + 2\mu)$.

3. Solution of the Problem in the Laplace Transform Domain

Applying the Laplace transform with parameter s defined by the relation [14]

$$\bar{f}(s) = \int_0^\infty f(t) e^{-st} dt, \quad (7)$$

to both sides of (6), we obtain

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial \bar{\theta}}{\partial x} = s^2 \bar{u}, \quad (8)$$

$$C_t^2 \frac{\partial^2 \bar{\theta}}{\partial x^2} = s^2 \bar{\theta} + \varepsilon s^2 \frac{\partial \bar{u}}{\partial x}, \quad (9)$$

$$\bar{\sigma} = \frac{\partial \bar{u}}{\partial x} - \bar{\theta}. \quad (10)$$

Eliminating $\bar{u}$ from (12) and (14), we get

$$\left\{ D^4 - \frac{(C_t^2 + 1 + \varepsilon)}{C_t^2} s^2 D^2 + \frac{s^4}{C_t^2} \right\} \bar{\theta} = 0, \quad (11)$$

where $D = \partial/\partial x$.

Equation (11) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)\bar{\theta} = 0, \quad (12)$$

where k_1^2 and k_2^2 are the roots of the characteristic equation

$$k^4 - \frac{(C_t^2 + 1 + \varepsilon)}{C_t^2} s^2 k^2 + \frac{s^4}{C_t^2} = 0. \quad (13)$$

k_1^2 and k_2^2 are given by

$$k_1^2 = s^2 (B + A), \quad k_2^2 = s^2 (B - A), \quad (14)$$

where $B = (C_t^2 + 1 + \varepsilon)/2C_t^2$ and $A^2 = ((C_t^2 + 1 + \varepsilon)/2C_t^2)^2 - 1/C_t^2$.

The solutions of (12), bounded as x tends to ∞ have the form

$$\bar{\theta} = D_1 e^{-k_1 x} + D_2 e^{-k_2 x}, \quad (15)$$

where D_1 and D_2 are some parameters depending on s only.

To calculate $\bar{u}$ using (9), we obtain

$$\frac{\partial \bar{u}}{\partial x} = \frac{C_t^2}{\varepsilon s^2} \left[D_1 \left(k_1^2 - \frac{s^2}{C_t^2} \right) e^{-k_1 x} + D_2 \left(k_2^2 - \frac{s^2}{C_t^2} \right) e^{-k_2 x} \right]. \quad (16)$$

Integrating both sides of (16) with respect to x , we get

$$\bar{u} = \frac{-C_t^2}{\varepsilon s^2} \left[\frac{D_1}{k_1} \left(k_1^2 - \frac{s^2}{C_t^2} \right) e^{-k_1 x} + \frac{D_2}{k_2} \left(k_2^2 - \frac{s^2}{C_t^2} \right) e^{-k_2 x} \right]. \quad (17)$$

Substituting (15) and (17) into (10), we obtain

$$\begin{aligned} \bar{\sigma} &= \frac{C_t^2}{\varepsilon s^2} \left[D_1 \left(k_1^2 - \frac{s^2 (1 + \varepsilon)}{C_t^2} \right) e^{-k_1 x} \right. \\ &\quad \left. + D_2 \left(k_2^2 - \frac{s^2 (1 + \varepsilon)}{C_t^2} \right) e^{-k_2 x} \right]. \end{aligned} \quad (18)$$

The boundary conditions of the problem can be written as follows:

$$\sigma = 0, \quad \theta = F(t), \quad \text{at } x = 0. \quad (19)$$

Applying the Laplace transform with parameter s to both sides of (19), we get

$$\bar{\theta} = \bar{F}(s), \quad \bar{\sigma} = 0, \quad \text{at } x = 0. \quad (20)$$

Substituting (15) and (18) into (20), we obtain the following system of linear equations in the unknown parameters D_1 and D_2 :

$$\begin{aligned} D_1 + D_2 &= \bar{F}(s), \\ D_1 \left(k_1^2 - \frac{s^2(1+\varepsilon)}{C_t^2} \right) + D_2 \left(k_2^2 - \frac{s^2(1+\varepsilon)}{C_t^2} \right) &= 0. \end{aligned} \quad (21)$$

The solution of the above equations is

$$D_1 = E\bar{F}(s), \quad D_2 = (1-E)\bar{F}(s), \quad (22)$$

where $E = (1 + \varepsilon - C_t^2(B-A))/2C_t^2A$.

Substituting (22) into (15), (17), and (18), we obtain

$$\begin{aligned} \bar{\theta} &= \bar{F}(s) \left(Ee^{-\sqrt{B+A}xs} + (1-E)e^{-\sqrt{B-A}xs} \right), \\ \bar{u} &= \frac{-\bar{F}(s)C_t^2}{\varepsilon s} \left[\frac{E}{\sqrt{B+A}} \left(B+A - \frac{1}{C_t^2} \right) e^{-\sqrt{B+A}xs} \right. \\ &\quad \left. + \frac{1-E}{\sqrt{B-A}} \left(B-A - \frac{1}{C_t^2} \right) e^{-\sqrt{B-A}xs} \right], \quad (23) \\ \bar{\sigma} &= \frac{\bar{F}(s)C_t^2}{\varepsilon} \left[E \left(B+A - \frac{1+\varepsilon}{C_t^2} \right) e^{-\sqrt{B+A}xs} \right. \\ &\quad \left. + (1-E) \left(B-A - \frac{1+\varepsilon}{C_t^2} \right) e^{-\sqrt{B-A}xs} \right]. \end{aligned}$$

The above equations can be written as

$$\bar{\theta} = \bar{F}(s) (Ee^{-\mu_1 s} + (1-E)e^{-\mu_2 s}), \quad (24)$$

$$\bar{u} = \frac{-\bar{F}(s)}{s} [C_1 e^{-\mu_1 s} + C_2 e^{-\mu_2 s}], \quad (25)$$

$$\bar{\sigma} = \bar{F}(s) [C_3 e^{-\mu_1 s} + C_4 e^{-\mu_2 s}], \quad (26)$$

where

$$\begin{aligned} C_1 &= \frac{C_t^2(1-E)}{\varepsilon\sqrt{B-A}} \left(B-A - \frac{1}{C_t^2} \right), \\ C_2 &= \frac{EC_t^2}{\varepsilon\sqrt{B+A}} \left(B+A - \frac{1}{C_t^2} \right), \\ C_3 &= \frac{C_t^2 E (B+A - (1+\varepsilon)/C_t^2)}{\varepsilon}, \\ C_4 &= \frac{C_t^2 (1-E) (B-A - (1+\varepsilon)/C_t^2)}{\varepsilon}, \\ \mu_1 &= \sqrt{B+A}, \quad \mu_2 = \sqrt{B-A}. \end{aligned} \quad (27)$$

This completes the solution of the problem in the Laplace transform domain.

4. Exact Solution in the Physical Domain

We shall derive the temperature and stress distributions in an exact manner; by applying the inverse Laplace transform to both sides of (24) and (26), we get the following expressions:

$$\begin{aligned} \theta(x, t) &= Ef(t - \mu_1 x) H(t - \mu_1 x) \\ &\quad + (1-E)f(t - \mu_2 x) H(t - \mu_2 x), \end{aligned} \quad (28)$$

$$\begin{aligned} \sigma(x, t) &= C_3 f(t - \mu_1 x) H(t - \mu_1 x) \\ &\quad + C_4 f(t - \mu_2 x) H(t - \mu_2 x). \end{aligned} \quad (29)$$

In obtaining the above equations, we have used the following property of the Laplace Transform:

$$L^{-1} [\bar{F}(s) e^{-\alpha s}] = f(t - \alpha) H(t - \alpha), \quad (30)$$

where $H(t - \mu)$ is the Heaviside unit step function; we have

$$\begin{aligned} L^{-1} \left[\frac{\bar{F}(s) e^{-\alpha s}}{s} \right] &= \int_0^t [L^{-1} \bar{F}(s) e^{-\alpha s}]_{t=\tau} d\tau \\ &= \int_0^t f(\tau - \alpha) H(\tau - \alpha) d\tau. \end{aligned} \quad (31)$$

According to the definition of Heaviside unit step function, we have two cases:

- (i) if $\mu < t$, then the integral right hand side of (31) becomes $\int_\mu^t f(\tau - \mu) d\tau$;
- (ii) while if $\mu > t$ in this case the integral vanishes; thus (31) becomes

$$L^{-1} \left[\frac{\bar{F}(s) e^{-\alpha s}}{s} \right] = H(t - \alpha) \int_\mu^t f(\tau - \alpha) d\tau. \quad (32)$$

Now, using (32) and (25), we obtain the exact form of the displacement distribution in the form

$$\begin{aligned} u(x, t) &= - \left[C_1 H(t - \mu_1 x) \int_{\mu_1 x}^t f(\tau - \mu_1 x) d\tau \right. \\ &\quad \left. + C_2 H(t - \mu_2 x) \int_{\mu_2 x}^t f(\tau - \mu_2 x) d\tau \right]. \end{aligned} \quad (33)$$

This completes the solution of the problem in the physical domain.

5. Numerical Results and Discussion

We will consider the solution of the problem for different choices for the function $f(t)$ as shown in Table 1, where θ_0 and a are constants.

Substituting $f(t)$ into (29), (30), and (33), we obtain the following.

TABLE 1

Case	$f(t)$
1	$\theta_0 H(t)$
2	e^{-t}
3	$H(t-a) \sin\left(\frac{\pi t}{a}\right)$

FIGURE 1: Temperature distribution for Case 1.

Case 1. Consider the following:

$$\begin{aligned}
 \theta(x, t) &= EH(t - \mu_1 x) + (1 - E)H(t - \mu_2 x), \\
 \sigma(x, t) &= C_3 H(t - \mu_1 x) + C_4 H(t - \mu_2 x), \\
 u(x, t) &= -[C_1(t - \mu_1 x)H(t - \mu_1 x) \\
 &\quad + C_2(t - \mu_2 x)H(t - \mu_2 x)].
 \end{aligned} \quad (34)$$

Case 2. Consider the following:

$$\begin{aligned}
 \theta(x, t) &= Ee^{-(t-\mu_1 x)}H(t - \mu_1 x) \\
 &\quad + (1 - E)e^{-(t-\mu_2 x)}H(t - \mu_2 x), \\
 \sigma(x, t) &= C_3 e^{-(t-\mu_1 x)}H(t - \mu_1 x) \\
 &\quad + C_4 e^{-(t-\mu_2 x)}H(t - \mu_2 x), \\
 u(x, t) &= -C_1 H(t - \mu_1 x) [1 - e^{-(t-\mu_1 x)}] \\
 &\quad - C_2 H(t - \mu_2 x) [1 - e^{-(t-\mu_2 x)}].
 \end{aligned} \quad (35)$$

Case 3. Consider the following:

$$\begin{aligned}
 \theta &= EH(t - \mu_1 x)H(a - t) \sin\left(\frac{\pi(t - \mu_1 x)}{a}\right) \\
 &\quad + (1 - E)H(t - \mu_2 x)H(a - t) \sin\left(\frac{\pi(t - \mu_2 x)}{a}\right), \\
 \sigma &= C_3 H(t - \mu_1 x)H(a - t) \sin\left(\frac{\pi(t - \mu_1 x)}{a}\right) \\
 &\quad + C_4 H(t - \mu_2 x)H(a - t) \sin\left(\frac{\pi(t - \mu_2 x)}{a}\right),
 \end{aligned}$$

FIGURE 2: Displacement distribution for Case 1.

FIGURE 3: Stress distribution for Case 1.

FIGURE 4: Temperature distribution for Case 2.

$$\begin{aligned}
 u &= \frac{a}{\pi} \left(C_1 H(t - \mu_1 x) H(a - t) \right. \\
 &\quad \times \left[\cos\left(\frac{\pi(t - \mu_1 x)}{a}\right) - 1 \right] \\
 &\quad + C_2 H(t - \mu_2 x) H(a - t) \\
 &\quad \times \left[\cos\left(\frac{\pi(t - \mu_2 x)}{a}\right) - 1 \right] \Big). \quad (36)
 \end{aligned}$$

FIGURE 5: Displacement distribution for Case 2.

FIGURE 6: Stress distribution for Case 2.

Obviously the above exact solution verifies the boundary conditions introduced in (19) for all different cases.

We shall now compare, graphically, our results, where the theory of thermoelasticity without energy dissipation (GN theory) was used with the results of the solution of a similar problem in [18], where the generalized theory of thermoelasticity due to Lord-Shulman (LS theory) was used. Due to the nature of the characteristic roots of the LS theory, the authors of [18] could not obtain a closed form exact solution. They used asymptotic expansions valid for small values of time.

For purposes of numerical evaluations, the calculations were performed for a copper-like material for which [19]

$$\varepsilon = 0.0168, \quad C_t^2 = 40, \quad \theta_0 = 1, \quad a = 0.04. \quad (37)$$

The temperature, displacement, and stress distributions are plotted for one value of time, namely, for $t = 0.1$. In all figures, LS theory is represented by dotted lines, while GN theory is represented by solid lines.

For the three cases, the temperature distributions are shown in Figures 1, 4, and 7, the displacement distributions are shown in Figures 2, 5, and 8, while the stress distributions are shown in Figures 3, 6, and 9.

At the locations $x_1 = t/\mu_1$ and $x_2 = t/\mu_2$ the temperature and stress distributions have two finite jumps, while the displacement is continuous everywhere having a discontinuous first derivative at the above locations. The first

FIGURE 7: Temperature distribution for Case 3.

FIGURE 8: Displacement distribution for Case 3.

discontinuity is not shown in Figures 1, 4, and 7 because it is very small ($E \approx 4(10)^{-4}$).

x_1 and x_2 are the locations of the wave fronts. We thus have two waves propagating into the medium from the surface of the half space. The first is mainly mechanical and has velocity equal to $v_1 = 1/\mu_1$. The second is mainly thermal and has velocity equal to $v_2 = 1/\mu_2$.

For $x > x_2$ the solution for all functions is identically zero. Thus, the solution is confined to a finite region of space and does not extend to infinity. This is not the case for the coupled and the uncoupled theories of thermoelasticity, where the solution extends to infinity instantaneously signifying an infinite speed of propagation of waves. The values of the velocities and locations of the wave fronts are as follows:

$$\begin{aligned} v_1 &= 0.999784687321833, \\ v_2 &= 6.32591737054768, \\ x_1 &= 0.09997846873218333, \\ x_2 &= 0.632591737054768. \end{aligned} \quad (38)$$

The difference between the predictions of the theories of LS and GN is most apparent in the graphs of the temperature distribution. In the LS theory the temperature

FIGURE 9: Stress distribution for Case 3.

decreases monotonically signifying continuous dissipation of heat energy. This is not the case for GN theory.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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