Research Article

On the Preservation of Infinite Divisibility under Length-Biasing

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The law $L(X)$ of $X \geq 0$ has distribution function F and first moment $0 < m < \infty$. The law $L(\widehat{X})$ of the length-biased version of X has by definition the distribution function $m^{-1} \int_0^x y dF(y)$. It is known that $L(X)$ is infinitely divisible if and only if $\widehat{X} \stackrel{d}{=} X + Z$, where Z is independent of X . Here we assume this relation and ask whether $L(Z)$ or $L(\widehat{X})$ is infinitely divisible. Examples show that both, neither, or exactly one of the components of the pair $(L(X), L(\widehat{X}))$ can be infinitely divisible. Some general algorithms facilitate exploring the general question. It is shown that length-biasing up to the fourth order preserves infinite divisibility when $L(X)$ has a certain compound Poisson law or the Lambert law. It is conjectured for these examples that this extends to all orders of length-biasing.

1. Introduction

Let X be a nonnegative random variable whose distribution function (DF) and Laplace-Stieltjes transform (LST) is denoted by $F(x)$ and $\phi(s)$, respectively. If the law of X , $L(X)$, is infinitely divisible (infdiv), then $\phi(s) = \exp(-c(s))$, where the Laplace exponent (or cumulant function) $c(s)$ is a Bernstein function, denoted by $c \in \mathcal{B}$. This means that there is a measure (called the Lévy measure) λ on $(0, \infty)$ satisfying $\int_0^\infty (x \wedge 1) \lambda(dx) < \infty$ and a constant $l \geq 0$ such that

$$c(s) = ls + \int_0^\infty (1 - e^{-sx}) \lambda(dx). \quad (1)$$

In this case $l = \inf \text{supp}(F)$ is the left-extremity of the support of F and hence we will set $l = 0$, thereby losing no generality. Differentiation of ϕ yields $-\phi'(s) = \phi(s)c'(s)$ which in turn is equivalent to the convolution identity

$$\int_0^x y dF(y) = \int_0^x F(x-y) K(dy), \quad (2)$$

where $K(dx) = x\lambda(dx)$. Conversely, if a DF F satisfies (2) where K is a measure having a finite LST, then F is the DF of an infdiv law. See Theorem 4.10 in [1] and the reference there to Steutel's original formulation of this result.

Suppose now that the first moment $m = E(X) \in (0, \infty)$ and let $\widehat{F}(x) = m^{-1} \int_0^x y dF(y)$ denote the DF of the length-biased (or size-biased) version of F . Clearly $m = c'(0) = \int_0^\infty x\lambda(dx)$, and hence $G(x) = m^{-1}K[0, x]$ is the DF of a random variable, Z say. So if $\widehat{X}$ denotes a random variable having the DF $\widehat{F}$ and $L(X)$ is infdiv, then (2) has the random variable formulation

$$\widehat{X} \stackrel{d}{=} X + Z, \quad (3)$$

where $\stackrel{d}{=}$ denotes equality in law and the random variables on the right-hand side are independent. (Note that it is always assumed that random variables occurring on the right-hand side of in-law equalities are independent.) The equality (3) underlines the fact that length-biasing is an increasing operation with respect to the stochastic order; $P(\widehat{X} > x) \geq P(X > x)$ if $x > 0$. Conversely, if (3) holds for some positive defect random variable Z , then $L(X)$ is infdiv. See [2] for more on this result.

The primary question we address is that if (3) holds, then is $L(\widehat{X})$ also infdiv? A secondary question is whether $L(Z)$ is infdiv? If it is, then clearly $L(\widehat{X})$ is infdiv. The primary question can be extended to asking whether arbitrary order length-biasing of an infdiv law $L(X)$ also is infdiv. Here, if

$r > 0$, then we define the order- r length-biased version of F by $\hat{F}_r(x) = \mathcal{L}_r(F)(x) = m_r^{-1} \int_0^x y^r dF(y)$, where $m_r := E(X^r) \in (0, \infty)$. Thus $\hat{F} = \hat{F}_1$ and $\mathcal{L}_r$ denotes the length-bias operator of order r acting on distribution functions having a finite moment of order r . We extend this notation by writing $\hat{X}_r = \mathcal{L}_r X$ for a random variable whose DF is $\hat{F}_r$ and we denote the corresponding LST by $\phi_r(s)$. See [3] for various properties of $\mathcal{L}_r$.

The paper is structured as follows. Definitions of some important classes of infdiv laws are collected in Section 2. Generalities about length-biasing and infinite divisibility are addressed in Section 3. First, several examples show that both, neither, or exactly one member of the pair $(L(X), L(\hat{X}))$ can be infdiv. Second, if $L(\hat{X}_n)$ is infdiv, then there is a law $L(Z_n)$ such that $\hat{X}_{n+1} \stackrel{d}{=} \hat{X}_n + Z_n$. If also $L(Z_n)$ is infdiv, then so is $L(\hat{X}_{n+1})$. This is the approach taken for the particular cases discussed in Sections 4 and 5. Theorem 7 provides a convenient recursive method of computing the LSTs of the defect laws $L(Z_n)$. This section ends with a general additive representation for $L(\hat{X}_n)$, assuming that it is infdiv, and also Lemma 8 giving a general formula for the result of applying $\mathcal{L}_n$ to the sum of two independent random variables.

In Section 4 we examine the case where $L(X)$ has the unit rate compound Poisson law whose jump law is the Exp(35) law and Section 5 deals with the case which first motivated this paper; that is, $L(X)$ is the Lambert law introduced by Pakes [4]. As mentioned, the approach in both cases is, having established that $L(\hat{X}_n)$ is infdiv, to investigate as far as possible the infdiv status of $L(Z_n)$. We thereby establish in both cases that $L(\hat{X}_n)$ is infdiv for $n = 1, \dots, 4$ and we conjecture that this is the case for all n . Our algebraic manipulations become more involved as n increases, suggesting other methods are needed to resolve our conjectures one way or the other.

We end this introduction by reminding the reader that length-biasing is important in contexts such as random sampling of regions in spatial problems, for example, the distribution of the volume or surface area of spheres given the distribution of radii; see page 170 in [5]. Another context is measuring income inequality via the Lorenz curve, that is, the set $\{(\hat{F}(x), F(x)) : x \geq 0\}$; see page 40 in [6]. Many probability laws can be characterized through relations involving length-biasing, as in [3, 7]. The first of this pair of references cites other occurrences of length-biasing.

2. Classes of Infdiv Laws

Recall that a positive-valued function $B(s)$ defined and real-analytic on $(0, \infty)$ is completely monotone, written $B \in \mathcal{CM}$, if $(-1)^n B^{(n)}(s) \geq 0$ for all $n = 1, 2, \dots$. Bernstein's theorem asserts that $B \in \mathcal{CM}$ if and only if there is a positive measure μ such that $B(s) = \int_0^\infty e^{-sx} \mu(dx)$. In particular, if $c(s) \in \mathcal{B}$, then its derivative $c'(s) \in \mathcal{CM}$.

We are principally concerned with infdiv laws $L(X)$ where λ has a density $\ell(x)$. We say that $L(X)$ is self-decomposable, written $L(X) \in SD$, if $x\ell(x)$ is nonincreasing.

Self-decomposable laws comprise the limit laws of affine transformations of sums of independent random variables.

We say that $L(X)$ is an exponential mixture law, written $L(X) \in \mathcal{EM}$, if there is a possibly defective DF $\mathcal{M}$ on $[0, \infty)$ (the mixing DF) such that the corresponding DF $F(x) = \int_0^\infty (1 - e^{-xy}) d\mathcal{M}(y)$. This is equivalent to the random variable equality $X \stackrel{d}{=} \varepsilon/M$, where ε has the standard exponential law Exp(35) and M has the DF $\mathcal{M}$. Thus, $P(X = 0) = P(M = \infty)$. In addition, $L(X)$ is infdiv and its Laplace exponent has the form

$$c(s) = \int_0^\infty \left(\frac{1}{y} - \frac{1}{y+s} \right) v(y) dy, \quad (4)$$

where $0 \leq v(y) \leq 1$.

Following terminology in [8], say that the Laplace exponent $c \in \mathcal{B}$ is a complete Bernstein function, written $c \in \mathcal{CB}$, if $\ell \in \mathcal{CM}$. The corresponding class of laws are termed Bondesson (B0) laws in [8] and $\mathcal{T}_2$ laws in [9], which term replaces the original descriptive designation GCMED, meaning “generalised convolution of mixtures of exponential distributions.” This last term underlines the fact that the B0 laws comprise the smallest class which is closed under convolutions and weak limits and contains $\mathcal{EM}$. Finally, we observe that, like $\mathcal{B}$, $\mathcal{CB}$ is closed under composition.

An important subclass of complete Bernstein Lévy exponents is the set of Thorin Bernstein functions $\mathcal{TB}$ defined by the requirement that $x\ell(x) \in \mathcal{CM}$. Thus, $\mathcal{TB} \subset \mathcal{CB} \cap SD$. Note that the composition of two functions in $\mathcal{TB}$ need not be in $\mathcal{TB}$, although it will be in $\mathcal{CB}$. Infdiv laws $L(X)$ having a Lévy exponent in $\mathcal{TB}$ comprise those laws which are the weak limits of sums of independent gamma distributed random variables, and hence they are called generalized gamma convolutions (GGCs). The class of GGCs is the smallest which contains gamma laws and is closed under convolutions and weak limits. They have become important because in recent years the self-decomposability of many familiar continuous laws has been demonstrated by showing that they are GGCs; see [9] or [1]. The Lévy exponent of a GGC has the form

$$c(s) = \int_0^\infty \log \left(1 + \frac{s}{y} \right) M_T(dy), \quad (5)$$

where the Thorin measure M_T satisfies $\int_{0+}^1 \log y^{-1} M_T(dy) < \infty$ and $\int_{1+}^\infty y^{-1} M_T(dy) < \infty$.

Computing the derivative of the Laplace exponent (5) and noting that $(y+s)^{-1} = \int_y^\infty (z+s)^{-2} dz$ yield the representation

$$c'(s) = \int_0^\infty (z+s)^{-2} M_T([0, z]) dz. \quad (6)$$

Comparing this with the derivative obtained from (4), we obtain the following known result (page 352 in [1]).

Lemma 1. *If $L(X) \in \text{GGC}$, then $L(X) \in \mathcal{EM}$ if and only if $M_T(\mathbb{R}_+) \leq 1$.*

Finally, we say that a positive-valued function $h(x)$ ($x \geq 0$) is hyperbolically completely monotone (HCM) if, for each

$u > 0$, the function of $v > 0$ equal to $h(uv)h(u/v)$ is a completely monotone function of $v + v^{-1}$. A key result is that a law which has an HCM density is a GGC and hence is self-decomposable. Example 3 below gives two examples and [9] presents many more. A recent summary with examples of the above ideas is [10].

We end this section with the following result about LSTs having the following form. Let $\{p_j : j = 1, 2, \dots\}$ be a discrete law, $\{c(j)\}$ a sequence of positive numbers, and $0 \leq a(j) \leq 2$ constants. The gamma mixture LST

$$\phi(s) = \sum_{j \geq 1} p_j \left(1 + \frac{s}{c(j)}\right)^{-a(j)} \quad (7)$$

will arise in examples below. Let $\beta(a, b)$ have the beta density $\propto u^{a-1}(1-u)^{b-1}$, where, conventionally, $a, b > 0$. It is consistent with the moment structure of the beta laws to specify $P(\beta(0, b) = 0) = 1$ and $P(\beta(a, 0) = 1) = 1$. Also, let γ_a be a random variable having the standard gamma law with density $\propto x^{a-1}e^{-x}$.

Lemma 2. *The LST (7) is infdiv with the scale mixture representation*

$$\phi(s) = E \left[\left(1 + \frac{s\beta(a(J), 2 - a(J))}{c(J)}\right)^{-2} \right], \quad (8)$$

where $P(J = j) = p_j$, and $\beta(a(\cdot), 2 - a(\cdot))$ and J are independent.

Proof. It follows from the well-known gamma-beta identity $\gamma_a \stackrel{d}{=} \beta(a, b)\gamma_{a+b}$ that

$$E \left[(1 + s\beta(a, 2 - a))^{-2} \right] = (1 + s)^{-a}. \quad (9)$$

Hence the form (7) follows by computing the expectation at (8). The infdiv assertion follows because scale mixtures of the gamma(2) law are infdiv; see page 344 in [1]. $\square$

We note that scale mixtures of the gamma(a) law may, or may not, be infdiv if $a > 2$; see page 409 in [1]. Finally, although we will not use this, it is worth observing that a limit argument based on (7) and Lemma 2 shows that if M is a DF on $[0, \infty)$ and $a(v)$ is a function on this set with values in $[0, 2]$, then

$$\phi(s) = \int_0^\infty \left(1 + \frac{s}{c(v)}\right)^{-a(v)} dM(v) \quad (10)$$

is an infdiv LST.

3. General Observations

We begin with examples which illustrate what is possible in relation to the infinite divisibility of $L(X)$ and its length-biased versions. The first simply reminds readers that there is a rich collection of infdiv laws whose length-biased versions of all orders also are infdiv.

Example 3. Suppose that $L(X)$ has a HCM density $f(x)$. If r is a real constant, then $x^r f(x)$ also is HCM; see page 68 in [9]. It follows that if m_r is finite, then $L(\widehat{X}_r)$ is infdiv. One example is the generalized inverse Gaussian laws (page 27 in [11]) whose densities $f(x; a, b, c) \propto x^{a-1} \exp[-(bx + c/x)]$, where $a, b > 0$ and $c \geq 0$. Clearly moments m_r are finite if $r > 0$. The case $c = 0$ yields the gamma laws, $X = \gamma_a/b$. In this case we have $\mathcal{L}_r \gamma_a \stackrel{d}{=} \gamma_{a+r} \stackrel{d}{=} \gamma_a + \gamma_r$. If $c > 0$ it does not seem possible to identify the law $L(Z)$ in (3), even in the case $r = 1$.

Suppose that μ is a real constant, $\sigma > 0$, and $\mathcal{N}$ has the standard normal law. Then $X = \exp(\mu + \sigma \mathcal{N})$ has the lognormal law whose density function $\propto x^{(\mu/\sigma^2)-1} \exp(-(\log x)^2/2\sigma^2)$. This density function is HCM (page 59 in [9]), and clearly $\widehat{X}_r$ has a lognormal law for all real r .

More examples of HCM densities may be found in [9] and the fact that if $L(X)$ has a HCM density, then so does $L(X^c)$ if $|c| > 1$.

The next example shows the existence of laws $L(\widehat{X}_r)$ which are not infdiv for any $r \geq 0$.

Example 4. It is known that infinite divisibility of a positive law $L(X)$ imposes constraints on how fast the survivor function $\bar{F}(x) = P(X > x)$ can go to zero as $x \rightarrow \infty$. In fact, if $L(X)$ is infdiv, then it follows from (9.6) on page 114 in [1] that $\bar{F}(x) \geq \exp(-kx \log x)$ for any $k > 0$. Laws with a bounded support comprise the extreme contradiction of this condition; they are not infdiv and neither are their length-biased versions. Laws with unbounded support where $\bar{F}(x) = O(\exp(-kx^\delta))$ where $k > 0$ and $\delta > 1$ likewise are not infdiv; their right-hand tail is too thin. Clearly $m_r < \infty$ if $r > 0$, and the length-biased laws of all positive orders satisfy a similar tail constraint and hence are not infdiv.

Our next example shows that length-biasing laws which are not infdiv can yield infdiv laws.

Example 5. Let $0 < p < 1$, $a > 0$, and $L(X)$ have the LST

$$\phi(s) = p + \frac{1-p}{(1+s)^a}, \quad (11)$$

representing the scale mixture $\gamma_a B$, where B has the Bernoulli law with $P(B = 1) = 1 - p$. This LST has the form (7) and hence it is infdiv if $a \leq 2$, but it is not SD (page 346 in [1]). On the other hand, if $a > 2$ then $L(X)$ is not infdiv (page 409 in [1]). For any $a > 0$, the order $r > 0$ length-biased version of $L(X)$ is the gamma($a + r$) law, and hence it is infdiv.

We will look at the case $a \leq 2$ in more detail. The LST of $L(Z)$ in (3) is

$$\zeta(s; a) = \frac{\psi(s; a)}{1+s}, \quad \text{where } \psi(s; a) = \frac{1}{1-p+p(1+s)^a}. \quad (12)$$

Suppose first that $0 < a \leq 1$ and let $L(\sigma_a)$ denote the stable law of index a , denoted by $\text{stable}(a)$ and having the LST

$$E(e^{-s\sigma_a}) = \exp(-s^a). \quad (13)$$

Note that $P(\sigma_1 = 1) = 1$. If $\varepsilon \sim \text{Exp}(1)$ is independent of σ_a and if $c > 0$ is a constant, then

$$E\left(\exp\left(-s(c\varepsilon)^{1/a}\sigma_a\right)\right) = E\left(\exp\left(-cs^a\varepsilon\right)\right) = (1 + cs^a)^{-1}, \quad (14)$$

the LST of a scaled positive Linnik law (also called the Mittag-Leffler law); see page 219 in [12] (and its references) and page 38 in [9] for a proof that Linnik laws are GGCs.

Next, let $\tau \geq 0$ and let $\mathcal{E}_\tau$ denote the exponential tilt operator; $\tilde{F}_\tau(x) := \mathcal{E}_\tau F(x) \propto \int_0^x e^{-\tau y} dF(y)$. The corresponding LST is $\tilde{\phi}_\tau(s) = \phi(s + \tau)/\phi(\tau)$, and hence $\tilde{F}_\tau$ is infdiv if F is infdiv. Each of the classes SD, BO, and GGC is closed under exponential tilting. If $L(X)$ has the DF F , then $\tilde{X}_\tau := \mathcal{E}_\tau X$ denotes a random variable whose DF is $\tilde{F}_\tau$.

Still assuming that $0 < a \leq 1$ and referring to (12), it follows that $\psi(s; a)$ is the LST of $\mathcal{E}_1((c\varepsilon)^{1/a}\sigma_a)$, where $c = p/(1 - p)$. We thus can express the defect random variable in the representation (3) as

$$Z = \varepsilon' + \mathcal{E}_1\left((c\varepsilon)^{1/a}\sigma_a\right), \quad (15)$$

where $\varepsilon' \stackrel{d}{=} \varepsilon$. Thus $L(Z) \in \text{GGC}$, and if $a = 1$, then we have the simpler representation $Z = \varepsilon' + p\varepsilon$.

The case $1 < a \leq 2$ is more opaque. In particular, the factor $\psi(s; a)$ need not be a Laplace transform; it is not convex if $p < (a - 1)/2a$. If $a = 2$, some algebra yields

$$\zeta(s; 2) = (1 - p)^{-1} \left(\frac{1}{1 + s} - \frac{1 + s}{(1 + s)^2 + \rho^2} \right), \quad (16)$$

where $\rho^2 = (1 - p)/p$. It follows that $L(Z)$ has the density

$$g(x; 2) = (1 - p)^{-1} e^{-x} (1 - \cos(\rho x)). \quad (17)$$

It clearly has zeros in $(0, \infty)$ and hence $L(Z)$ is *not* infdiv [1], although, as remarked above, $L(\tilde{X})$ is the gamma(3) law which is a GGC.

The next example shows that length biasing an infdiv law can result in a non-infdiv law. This possibility is noted in Example 12.6 of [1] (page 413) for two cases where $L(\tilde{X})$ is not infdiv, although it is not clear whether or not $L(X)$ is infdiv. The following example rests on the fact that if $L(X)$ is infdiv then $\phi(s) = \exp(-c(s)) \neq 0$ if $\Re s > s_a$, where $s_a \leq 0$ is the abscissa of convergence of the integral defining ϕ . Note that if ϕ can be analytically continued into $\Re s < s_a$, then it can have zeros there.

Example 6. If $c > 1$ then $L(X)$ having the LST

$$\phi(s) = \frac{1}{2} \left[\frac{1}{(1 + s)^2} + \left(\frac{c}{c + s} \right)^2 \right] \quad (18)$$

is a scale mixture of gamma(2) laws, and hence it is infdiv. This LST is holomorphic in $\Re s > -1$.

Since $\phi_1(s) = -\phi'(s)/m$, it is clear that $L(\tilde{X})$ is a mixture of gamma(3) laws which may, or may not, be infdiv. We show now that the latter can occur. Calculation shows that $\phi_1(s) = 0$

if $c^2(1 + s)^3 = -(c + s)^3 = e^{i\pi}(c + s)^3$, that is, if $s = s_0$ where, since $e^{i\pi/3} = (1/2)(1 + i\sqrt{3})$,

$$s_0 = \frac{(1/2)c - c^{2/3} + (1/2)ic\sqrt{3}}{c^{2/3} - (1/2) - (1/2)i\sqrt{3}}. \quad (19)$$

The real part of s_0 has the same sign as

$$\begin{aligned} & \left(\frac{1}{2}c - c^{2/3} \right) \left(c^{2/3} - \frac{1}{2} \right) - \frac{3c}{4} \\ &= \frac{1}{2}c^{2/3} \left(c - 2c^{1/3} - 2c^{2/3} + 1 \right). \end{aligned} \quad (20)$$

With $c = u^3$, the right-hand side bracketed term is the cubic polynomial $p(u) = u^3 - 2u^2 - 2u + 1$. We have $p(u) = -2$ if $u = u' := (1/2)(1 + \sqrt{13}) = 2.3025$ and $p(u)$ is increasing in (u', ∞) . It follows that $s_0 > -1$ if $c > c' := (u')^3 = 12.2111$. Hence $\phi_1(s)$ has a conjugate pair of zeros in the half-plane $\Re s > -1$ if $c > c'$, and hence $L(\tilde{X})$ is *not* infdiv even though $L(X)$ is infdiv.

Restricting the length-bias order to positive integer values $n = 1, 2, \dots$, it is clear that the LST of $L(\tilde{X}_n)$ is $\phi_n(s) = (-1)^n \phi^{(n)}(s)/m_n$. It follows from (3) that $L(\tilde{X}_n)$ is infdiv if and only if there is a defect law $L(Z_n)$ such that

$$\widehat{X}_{n+1} \stackrel{d}{=} \widehat{X}_n + Z_n, \quad (n = 0, 1, \dots), \quad (21)$$

in which case the LST of Z_n is

$$\zeta_n(s) = \frac{\phi_{n+1}(s)}{\phi_n(s)} = -\frac{m_n}{m_{n+1}} \cdot \frac{\phi^{(n+1)}(s)}{\phi^{(n)}(s)}. \quad (22)$$

Thus $Z = Z_0$.

The following result provides a recursive computation scheme for $\zeta_n(s)$ which is simpler to use than (22). Assuming the moments are finite, the first equality in (22) defines a real-analytic function ζ_n irrespective of whether or not $L(\tilde{X}_n)$ is infdiv and, in addition, $\zeta_n \in (0, 1)$. The upper bound is a consequence of the stochastic order relation $\widehat{X}_n \leq_{st} \widehat{X}_{n+1}$. Thus the function $\xi_n(s) = -\log \zeta_n(s)$ is positive-valued and real-analytic. Of course $\zeta_n \in \mathcal{C.M}$ and $\xi_n \in \mathcal{B}$ if and only if $L(Z_n)$ is infdiv.

Theorem 7. (i) If $m_{n+2} < \infty$, then

$$\frac{m_{n+2}}{m_{n+1}} \zeta_{n+1}(s) = \frac{m_{n+1}}{m_n} \zeta_n(s) + \xi'_n(s). \quad (23)$$

(ii) If also $L(\tilde{X}_n)$ and $L(Z_n)$ are infdiv and the latter has a Lévy density $l_n(z)$ and hence a density $g_n(z)$, then $L(Z_{n+1})$ has a density given by

$$\frac{m_{n+2}}{m_{n+1}} g_{n+1}(z) = \frac{m_{n+1}}{m_n} g_n(z) + z l_n(z). \quad (24)$$

Proof. Since $\phi_{n+2}(s) = (m_{n+1}/m_{n+2})(-\phi'_{n+1}(s))$, we have

$$\begin{aligned}\phi_{n+2}(s) &= \phi_{n+1}(s) \zeta_{n+1}(s) \\ &= -\frac{m_{n+1}}{m_{n+2}} (\phi'_n(s) \zeta_n(s) + \phi_n(s) \zeta'_n(s)) \\ &= \frac{m_{n+1}}{m_{n+2}} \left(\frac{m_{n+1}}{m_n} \phi_{n+1}(s) \zeta_n(s) - \phi_n(s) \zeta'_n(s) \right).\end{aligned}\quad (25)$$

But, $-\zeta'_n(s) = \zeta_n(s)\xi'_n(s) = (\phi_{n+1}(s)/\phi_n(s))\xi'_n(s)$. Substituting into the last display and dividing through by $\phi_{n+1}(s)$ yield (23) and Assertion (ii) follows from (i). $\square$

The recursion (23) can be used as follows. If $L(\widehat{X}_n)$ is known to be infdiv, then $\zeta_n \in \mathcal{CM}$ and in principle it can be inverted to yield the density $g_n(z)$ (or DF) of $L(Z_n)$. If also $\xi'_n \in \mathcal{CM}$, then $L(Z_n)$ is infdiv and hence so is $L(\widehat{X}_{n+1})$ and the process can be continued. However, as shown in Example 5, it is possible that $L(\widehat{X}_{n+1})$ is infdiv even though $L(Z_n)$ is not. Although (24) provides in principle a direct evaluation of $g_n(z)$, deciding whether $L(Z_n)$ is infdiv usually requires a consideration of its LST.

We end this section with two simple deductions from (3). First, assuming that $L(\widehat{X}_j)$ is infdiv for $j = 1, \dots, n-1$, then iterating (21) yields the additive representation

$$\widehat{X}_n \stackrel{d}{=} X + \sum_{j=1}^{n-1} Z_j. \quad (26)$$

Weakening the assumption here to requiring only that $L(X)$ is infdiv a second, more complicated representation is implied by the following lemma about the application of $\mathcal{L}_n$ to the sum of random variables. We use the following compact notation. Suppose that $L(Y)$ satisfies $E(Y^n) < \infty$ for a positive integer n and N is a random variable taking values in $\{1, \dots, n\}$. Then $\widehat{Y}_N$ denotes a random variable whose value is $\widehat{Y}_n$ with probability $P(N = n)$.

Lemma 8. Suppose that $Y(i)$ ($i = 1, 2$) are independent random variables having finite moments $m_{i,j} = E(Y^j(i))$ ($j = 1, \dots, n$), let $\mu_n = E[(Y(1) + Y(2))^n]$, and

$$p_{n,j} = \binom{n}{j} \frac{m_{1,j} m_{2,n-j}}{\mu_n}. \quad (27)$$

Then $\{p_{nj} : j = 0, 1, \dots, n\}$ is the probability mass function of a random variable, N say, and

$$\mathcal{L}_n(Y(1) + Y(2)) \stackrel{d}{=} \widehat{Y}_N(1) + \widehat{Y}_{n-N}(2), \quad (28)$$

where the random variables on the right-hand side are independent.

Proof. Let $\chi_i(s)$ denote the LST of $Y(i)$ and $\chi(s) = \chi_1(s)\chi_2(s)$. Applying the Leibniz differentiation rule to this product shows that the LST of $\mathcal{L}_n(Y(1) + Y(2))$ is

$$\begin{aligned}(-1)^n \mu_n^{-1} \chi^{(n)}(s) \\ = \sum_{j=0}^n p_{nj} \frac{(-1)^j \chi_1^{(j)}(s)}{m_{1,j}} \cdot \frac{(-1)^{n-j} \chi_2^{(n-j)}(s)}{m_{2,n-j}}.\end{aligned}\quad (29)$$

The mixture LST on the right-hand side is that of the right-hand side of (28). $\square$

4. A Compound Poisson Law

We illustrate Theorem 7 with an example which also illustrates the mounting algebraic difficulties as n is increased. It has a relation to the Lambert example discussed in Section 5. Let F be the DF of the unit rate compound Poisson law $L(X)$ whose jump law is the Exp(35) law. Thus $P(X = 0) = e^{-1}$ and the absolutely continuous component of F has the density

$$f(x) = e^{-1} \sum_{j=1}^{\infty} e^{-x} \frac{x^{j-1}}{(j-1)!j!}. \quad (30)$$

Note that $P(X > 0) = \int_0^{\infty} f(x)dx = 1 - e^{-1}$. If $r > 0$ then $x^r f(x)$ can be normalised to be an honest density. Now $\phi(s) = \exp(-(1 + (1 + s)^{-1}))$, and some algebra shows that

$$m_1 = 1, \quad m_2 = 3, \quad m_3 = 13, \quad m_4 = 73. \quad (31)$$

Case $n = 0$. Since $L(X)$ is infdiv, (3) holds and differentiation shows that $\zeta_0(s) = \zeta(s) = (1 + s)^{-2}$. Thus $L(Z)$ is the gamma(2) law, and hence $L(\widehat{X})$ itself is infdiv. The Lévy density of the gamma(a) law is $(a/x)e^{-x}$, whence the Lévy density of $L(\widehat{X})$ is $\ell_1(x) = e^{-x}(1 + 2/x)$. This is completely monotone, and hence $L(\widehat{X}) \in \text{B0}$. The second-order derivative of $x\ell_1(x)$ is $x e^{-x}$, which is not completely monotone, so $L(\widehat{X}) \notin \text{GGC}$.

Case $n = 1$. We have from the $n = 0$ case that $L(\widehat{X}_1) \in \text{B0}$. Since $\xi'_1(s) = 2/(1 + s)$, it follows from Theorem 7 and (23) with $n = 0$ that

$$\begin{aligned}\zeta_1(s) &= \frac{m_1}{m_2} \left(\frac{m_1}{m_0} \zeta(s) + \xi'_1(s) \right) \\ &= \frac{1}{3} \left(\frac{2}{1 + s} + \frac{1}{(1 + s)^2} \right) = \frac{3 + 2s}{3(1 + s)^2}.\end{aligned}\quad (32)$$

This has the form (7) so Lemma 2 implies that $L(Z_1)$ is infdiv. It follows that $L(\widehat{X}_2)$ too is infdiv and also that $Z_1 \stackrel{d}{=} \gamma_{1+N}$, where $N \sim \text{Bern}(1/3)$ is independent of the gamma random variable. The rational form of ζ_1 yields

$$\xi'_1(s) = \frac{2}{1 + s} - \frac{2}{3 + 2s} \quad (33)$$

and hence the Lévy density of $L(Z_1)$ is

$$l_1(z) = z^{-1} (2e^{-z} - e^{-3z/2}). \quad (34)$$

Denoting the unit (Heaviside) step function with jump at a by $U(y - a)$, it is clear that $l_1(z)$ is the Laplace transform of

$2U(y-1)-U(y-3/2) \geq 0$. Hence $L(Z_1) \in \text{B0}$, but the second-order derivative of $z l_1(z)$ changes sign, so $L(Z_1) \notin \text{GGC}$.

Case $n = 2$. It follows from the $n = 1$ case that $L(\widehat{X}_2) \in \text{B0}$. Using Theorem 7 and (23) with $n = 1$ we obtain

$$\begin{aligned} \zeta_2(s) &= \frac{3}{13} \left(\frac{1}{(1+s)^2} + \frac{4}{1+s} - \frac{1}{(3/2)+s} \right) \\ &= \frac{3}{13} \cdot \frac{13 + 18s + 6s^2}{(1+s)^2(3+2s)}. \end{aligned} \quad (35)$$

The partial fraction form inverts to the density

$$g_2(z) = \frac{3}{13} e^{-z} (z + 4 - e^{-z/2}). \quad (36)$$

It follows directly from the rational form in (35) that

$$\begin{aligned} \xi'_2(s) &= \frac{2}{1+s} + \frac{2}{3+2s} - \frac{18+12s}{13+18s+6s^2} \\ &= \frac{2}{1+s} + \frac{2}{3+2s} - \frac{2(3/2+s)}{(3/2+s)^2 - 1/12}. \end{aligned} \quad (37)$$

Since the Laplace transform of $\cosh \rho z$ is $s/(s^2 - \rho^2)$, the inverse transform of $\xi'_2(s)$ is

$$\begin{aligned} z l_2(z) &= 2e^{-z} + e^{-3z/2} - 2e^{-3z/2} \cosh\left(\frac{z}{\sqrt{12}}\right) \\ &= e^{-z} \left[2 - e^{-(1/2)(1-3^{-1/2})z} + e^{-(1/2)z} \left(1 - e^{-z/2\sqrt{3}} \right) \right] > 0. \end{aligned} \quad (38)$$

Hence $\xi'_2(s) \in \mathcal{CM}$, so $l_2(z)$ is a Lévy density and $L(Z_2)$ is infdiv. In addition, similar to the above treatment of $l_1(z)$, $l_2(z)$ has an inverse Laplace transform comprising a linear combination of unit step functions whose right-continuous version is identically zero in $[0, 1)$ and positive-valued in $[1, \infty)$. Hence $l_2 \in \mathcal{CM}$, so $L(Z_2) \in \text{B0}$. The inverse transform of $z l_2(z)$ is a linear combination of two positive and two negative point masses, so $L(Z_2) \notin \text{GGC}$.

Case $n = 3$. This case illustrates the growing algebraic complexity of our procedures. It follows from the previous case that $L(\widehat{X}_3) \in \text{B0}$. Using (23), (35), and (37), we have

$$\begin{aligned} \zeta_3(s) &= \frac{13}{73} \left(\frac{1}{(1+s)^2} + \frac{6}{1+s} - \frac{18+12s}{13+18s+6s^2} \right) \\ &= \frac{13}{73} \cdot \frac{73 + 156s + 108s^2 + 24s^3}{(1+s)^2(13+18s+6s^2)}. \end{aligned} \quad (39)$$

The partial fraction form inverts to

$$g_3(z) = \frac{13}{73} \left(6e^{-z} + ze^{-z} - 2e^{-3z/2} \cosh\left(\frac{z}{\sqrt{12}}\right) \right), \quad (40)$$

which we know is a density function because $L(\widehat{X}_3)$ is infdiv.

The rational form of ζ_3 yields

$$\begin{aligned} \xi'_3(s) &= \frac{2}{1+s} + \frac{18+12s}{13+18s+6s^2} \\ &\quad - \frac{156+216s+72s^2}{73+156s+108s^2+24s^3}. \end{aligned} \quad (41)$$

Replacing 73 with 72 in the denominator $D(s)$ of the subtracted rational term $r_3(s)$ yields a polynomial which factorizes as $24(1+s)(3/2+s)(2+s)$. It follows that the zeros $-s_i$ of $D(s)$ are real and that $s_1 - 1$, $3/2 - s_2$, and $s_3 - 2$ are small positive numbers. In fact $s_1 \approx 1.1289$, $s_2 \approx 1.3025$, and $s_3 \approx 2.0686$. Calculation shows that

$$r_3(s) \approx \frac{0.9994}{s_1+s} + \frac{1.00047}{s_2+s} + \frac{1.00013}{s_3+s} \quad (42)$$

and hence the inverse Laplace transform of $\xi'_3(s)$ is

$$z l_3(z) \approx e^{-z} b_3(z), \quad (43)$$

where

$$\begin{aligned} b_3(z) &= 2 + e^{-0.2113z} + e^{-0.7887z} - 0.9994e^{-0.2889z} \\ &\quad - 1.00047e^{-0.3025z} - 1.00013e^{-1.0686z}. \end{aligned} \quad (44)$$

Note first that $b_3(0) = 1$, $b'_3(0) = 0.500093$, and $b_3(z) \rightarrow 2$ as $z \rightarrow \infty$. Denote the successive exponential terms (neglecting their signs) in b_3 by $e_i(z)$ ($i = 1, \dots, 5$). Clearly $1 - e_3(z) > 0$ and a little arithmetic shows that $e_1(z) - e_4(z) \leq 0$ if and only if $0 \leq z \leq 0.00515$ and its least value is -0.00047 and that $e_2(z) - e_5(z) \leq 0$ if and only if $0 \leq z \leq 0.00168$ and its least value is -0.00013 . Thus we conclude that $b_3(z) > 0$ in $(0, \infty)$ and hence that $L(Z_3)$ and $L(\widehat{X}_4)$ are infdiv laws.

Finally, the Lévy density $l_3(z)$ has an inverse transform which again is a linear combination of unit step functions which is positive-valued in $[1, \infty)$. Hence $L(Z_3) \in \text{B0}$, but similar to the above, $L(Z_3) \notin \text{GGC}$. It follows from our calculations that $L(\widehat{X}_n) \in \text{B0}$ for $n = 1, \dots, 4$ and we conjecture that this holds for all n .

5. The Lambert Law

The principal solution $W(s)$ of the functional equation

$$We^W = s \quad (45)$$

is called the Lambert W -function (named after J. H. Lambert (1728–1777) and not the author of [6]). It is a concave-increasing function mapping $[-e^{-1}, \infty)$ into $[-1, \infty)$ with $W(-e^{-1}) = -1$, $W(0) = 0$, and $W(\infty) = \infty$. Its first-order derivative is (from (45))

$$W'(s) = \frac{\exp(-W(s))}{1+W(s)} = \frac{W(s)}{s(1+W(s))}. \quad (46)$$

Many properties and applications are given in [13] and it is classified as an elementary function in [14].

The key property for our purposes is that $W \in \mathcal{B}$. See [4, 15, 16] for various proofs of this fact. It is shown in [4] (Theorem 3.1) that

$$W(s) = \int_0^\infty (1 - e^{-sx}) \omega(x) dx, \quad (47)$$

where the Lévy density is

$$\omega(x) = x^{-1} \int_0^\infty e^{-xy} \nu(dy), \quad (48)$$

and ν is a probability measure with $\text{supp}(\nu) = [e^{-1}, \infty)$. Consequently, as observed in the remarks following the proof of Theorem 3.1 in [4], $W(s) \in \mathcal{TB}$.

It follows that there is a probability law $L(X) \in \text{GGC}$ (denoted by $L(\Lambda)$ in [4]) whose LST is

$$\phi(s) = e^{-W(s)} = \frac{W(s)}{s} \quad (49)$$

and is named the (standard) Lambert law. In addition, Theorem 3.2 in [4] asserts that $L(X) \in \mathcal{ME}$. There is a corresponding Lambert subordinator $(S(t) : t \geq 0)$, that is, a Lévy process having nondecreasing sample paths, such that $S(0) = 0$ and $S(1) \stackrel{d}{=} X$. Since $W(\infty) = \infty$, the Lambert subordinator is not compound Poisson; its sample paths are nowhere constant step functions.

Lagrange's reversion of series yields the known expansion

$$W(s) = \sum_{j=1}^{\infty} \frac{(-j)^{j-1}}{j!} s^j, \quad (50)$$

converging if $|s| < e^{-1}$ and implying that $L(X)$ has finite moments of all positive orders, $m_j = (j+1)^{j-1}$. In particular

$$m_1 = 1, \quad m_2 = 3, \quad m_3 = 16, \quad m_4 = 125. \quad (51)$$

Differentiating (49) and using (46) and then (49) we find that the LST of $L(\widehat{X})$ is

$$\phi_1(s) = W'(s) e^{-W(s)} = \frac{\phi^2(s)}{1 + W(s)}. \quad (52)$$

Knowing that $L(X)$ is infdiv, it follows that (3) holds and that the defect LST is

$$\zeta(s) = \frac{\phi(s)}{1 + W(s)}. \quad (53)$$

The factor $(1 + W(s))^{-1}$ is the LST of $S(\varepsilon)$, the exponentially stopped Lambert subordinator. Here $\varepsilon \sim \text{Exp}(1)$ is independent of $(S(t))$. It follows that (3) takes the specific form

$$\widehat{X} \stackrel{d}{=} X + X' + S(\varepsilon), \quad (54)$$

where $X' \stackrel{d}{=} X$. The right-hand side is the sum of independent infdiv random variables and hence $L(\widehat{X})$ is infdiv.

This bare conclusion follows from Example 3.2.6 in [9] asserting that the length-biased version of a GGC law is infdiv. The above working yields some extra detail and, in particular, the following refinement.

TABLE 1: Values of the polynomials defined by (56).

n	$Q_n(w)$
1	1
2	$3 + 2w$
3	$16 + 19w + 6w^2$
4	$125 + 204w + 118w^2 + 24w^3$
5	$1296 + 2633w + 2128w^2 + 806w^3 + 120w^4$

Theorem 9. *If $L(X)$ has the Lambert law, then $L(Z) \in \text{BO}$ and hence $L(\widehat{X}) \in \text{BO}$ too.*

Proof. The Laplace exponent of $L(S(\varepsilon))$ is $\log(1 + W(s))$. But $W(s) \in \mathcal{TB} \subset \mathcal{CB}$ and $\log(1 + s) \in \mathcal{CB}$ because the corresponding Lévy density is $x^{-1}e^{-x} \in \mathcal{CM}$. Thus closure of $\mathcal{CB}$ under functional composition implies that $\log(1 + W(s)) \in \mathcal{CB}$, and hence that $L(S(\varepsilon)) \in \text{BO}$. Since $Z = X' + S(\varepsilon)$ is the sum of independent BO random variables and BO is closed under convolution, the first assertion follows, and the second likewise. $\square$

Instead of invoking Theorem 7, computation of $\phi^{(n)}(s)$ for $n = 1, 2, 3$ suggests the following result which follows from an easy induction argument from (22).

Lemma 10. *The higher-order derivatives of the LST (49) have the form*

$$\phi^{(n)}(s) = (-1)^n \frac{(\phi(s))^{n+1}}{(1 + W(s))^{2n-1}} Q_n(W(s)), \quad (n \geq 0), \quad (55)$$

where $Q_0(s) = (1 + s)^{-1}$ and the degree- $(n-1)$ polynomials $Q_n(w)$ are determined from the following recurrence: if $n \geq 0$ then

$$Q_{n+1}(w) = (3n + (n+1)w) Q_n(w) - (1 + w) Q_n'(w). \quad (56)$$

Writing $Q_n(s) = \sum_{j=0}^{n-1} c_{nj} w^j$ and setting $s = 0$ in Lemma 10 yield $c_{n0} = m_n$. Hence (56) yields $c_{n1} = 3nm_n - m_{n+1}$. Dividing (56) by w^n and letting $w \rightarrow \infty$ yield $c_{n+1,n} = (n+1)c_{n,n-1}$; that is, $c_{n,n-1} = n!$. The argument used in part (35) of the proof of Theorem 3.1 in [4] shows that $c_{nj} > 0$ if $n \geq 2$ and $0 \leq j < n$. Table 1 lists exact expressions which we will use below.

The quotients defined at (22) have the form

$$\zeta_n(s) = \phi(s) \cdot \frac{m_n}{m_{n+1}} \cdot \frac{Q_{n+1}(W(s))}{(1 + W(s))^2 Q_n(W(s))}, \quad (n \geq 0), \quad (57)$$

and hence the closure of $\mathcal{CB}$ under functional composition yields the following formal assertion.

Theorem 11. *Suppose for each $n = 0, 1, \dots$ that $L(\widehat{X}_n) \in \text{BO}$ and that*

$$\tilde{\zeta}_n(s) = \frac{m_n}{m_{n+1}} \cdot \frac{Q_{n+1}(s)}{(1 + s)^2 Q_n(s)} \quad (58)$$

is the LST of a law $L(\omega_n) \in \text{B0}$. Then $L(\widehat{X}_{n+1}) \in \text{B0}$ and

$$\widehat{X}_{n+1} \stackrel{d}{=} \widehat{X}_n + X' + S(\omega_n), \quad (59)$$

where $X' \stackrel{d}{=} X$ and ω_n is independent of the Lambert subordinator $(S(t))$. A weaker assertion holds where “ $\in \text{B0}$ ” is replaced with “is infdiv.”

Theorems 9 and 11 yield a recursive algorithm for computing the rational form of $\tilde{\zeta}_n(s)$. Checking that in fact it is the LST of a probability law requires resolving it into its partial fraction form. The next theorem gives a first step toward this goal.

Theorem 12. Let $q_n = Q_n(-1)$ and $d_n = Q'_n(-1)$. For $n = 1, 2, \dots$,

$$\frac{m_{n+1}}{m_n} \tilde{\zeta}_n(s) = \frac{A_n}{(1+s)^2} + \frac{B_n}{1+s} + \frac{p_n(s)}{Q_n(s)}, \quad (60)$$

where $A_n = 2n - 1$, $B_n = n + 1 - d_n/q_n$, and

$$p_n(s) = \frac{(d_n/q_n)Q_n(s) - Q'_n(s)}{1+s} \quad (61)$$

is a degree- $(n-2)$ polynomial if $n \geq 2$, and $p_1(s) = 0$. In addition

$$q_{n+1} = (2n-1)q_n = \frac{(2n-1)!}{2^{n-1}(n-1)!}, \quad (62)$$

$d_1 = 0$, and for $n = 1, 2, \dots$,

$$d_{n+1} = 2(n-1)d_n + (n+1)q_n. \quad (63)$$

Proof. It follows from (56) that

$$\begin{aligned} \frac{Q_{n+1}(s)}{(1+s)^2 Q_n(s)} &= \frac{3n + (n+1)s}{(1+s)^2} - \frac{Q'_n(s)}{(1+s)Q_n(s)} \\ &= \frac{2n-1}{(1+s)^2} + \frac{n+1}{1+s} - \frac{Q'_n(s)}{(1+s)Q_n(s)}, \end{aligned} \quad (64)$$

implying (60). The numerator in (61) is a polynomial whose degree is no more than $n-1$. But the definitions of q_n and d_n imply that this numerator vanishes at $s = -1$ and hence $p_n(s)$ is a degree- $(n-2)$ polynomial as asserted. The recursions for q_n and d_n follow from (56). $\square$

Table 2 displays values of the above constants for $n = 1(1)7$. Asymptotic expressions for q_n and d_n can be derived from (62) and (63) and they confirm the rapid increase suggested by the values in Table 2. The algorithm for $p_n(s)$ is straightforward but time consuming to implement. Table 2 lists formulae for $n = 1(1)5$.

We look now at the cases $n = 1-4$, but before starting, we recall that if $L(\widehat{X}_n)$ is infdiv, then $\zeta_n(s) \in \mathcal{CM}$. However, we cannot infer that $\tilde{\zeta}_n(s)$ defined at (58) is completely monotone; this has to be demonstrated.

Case $n = 1$. It follows from Tables 1 and 2 that

$$\tilde{\zeta}_1(s) = \frac{3+2s}{3(1+s)^2} = \frac{1}{3} \left(\frac{1}{(1+s)^2} + \frac{2}{1+s} \right). \quad (65)$$

It follows from Lemma 2 that $L(\omega_1)$ is infdiv. The rational form yields

$$\tilde{\zeta}'_1(s) = \frac{2}{1+s} - \frac{2}{3+2s} \quad (66)$$

which inverts to

$$z\tilde{l}_1(z) = 2e^{-z} - e^{-3z/2}. \quad (67)$$

In turn, $\tilde{l}_1(z)$ has the inverse transform $2U(y-1) - U(y-3/2) \geq 0$. We conclude that $L(\omega_1) \in \text{B0}$ and hence from Theorems 9 and 11 that $L(\widehat{X}_2) \in \text{B0}$.

Case $n = 2$. Here

$$\begin{aligned} \tilde{\zeta}_2(s) &= \frac{3}{16} \cdot \frac{16+19s+6s^2}{(1+s)^2(3+2s)} \\ &= \frac{3}{16} \left(\frac{3}{(1+s)^2} + \frac{1}{1+s} + \frac{4}{3+2s} \right), \end{aligned} \quad (68)$$

again an infdiv mixture of gamma laws. Similar to the case $n = 2$ in Section 4 we have

$$\begin{aligned} \tilde{\zeta}'_2(s) &= \frac{2}{1+s} + \frac{2}{3+2s} - \frac{19+12s}{16+19s+6s^2} \\ &= \frac{2}{1+s} + \frac{2}{3+2s} - 2 \frac{(19/12)+s}{(19/12+s)^2 + 23/144}; \end{aligned} \quad (69)$$

so defining $\alpha = \sqrt{23}/12 = 0.3997$, we obtain the inverse transform

$$z\tilde{l}_2(z) = 2e^{-z} + e^{-3z/2} - 2e^{-19z/12} \cos(\alpha z), \quad (70)$$

which is clearly positive-valued if $z > 0$. Hence $L(\omega_2)$ is infdiv and so too is $L(\widehat{X}_3)$. However it is not clear whether or not $\tilde{l}_2(z) \in \mathcal{CM}$, and hence we cannot conclude that $L(\omega_2) \in \text{B0}$.

Case $n = 3$. Proceeding as above using Table 2 yields the form

$$\begin{aligned} \tilde{\zeta}_3(s) &= \frac{16}{125} \left(\frac{5}{(1+s)^2} + \frac{5/3}{1+s} + \frac{14(s+19/12)-23/6}{6((s+19/12)^2 + 23/144)} \right) \end{aligned} \quad (71)$$

which inverts to

$$\begin{aligned} \tilde{g}_3(z) &= \frac{16}{125} e^{-z} \left[5z + \frac{5}{3} + \frac{1}{3} e^{-7z/12} (7 \cos(\alpha z) - \sqrt{23} \sin(\alpha z)) \right]. \end{aligned} \quad (72)$$

To see that this is a density function, observe first that the trigonometric term $T(z)$ has its first zero where $\tan(\alpha z) = 7/\sqrt{23}$; that is, $z = z_0 = 2.4274$. Thus $T(z) > 0$ in $[0, z_0]$. Critical points of $T(z)$ occur where $\tan(\alpha z) = \sqrt{23}/7$, and the least positive solution is $z_c = 6.3578$ with $T(z_c) = -8.4853$. It

TABLE 2: Values of ingredients required for the partial fraction form (60) of $\tilde{\zeta}_n(s)$.

n	q_n	d_n	A_n	B_n	$p_n(s)$
1	1	0	1	2	0
2	1	2	3	1	4
3	3	7	5	5/3	$(55/3) + 14s$
4	15	40	7	7/3	$(388 + 536s + 192s^2)/3$
5	105	315	9	3	$1255 + 2388s + 1578s^2 + 360s^3$
6	945	3150	11	77/21	—
7	10395	38115	13	191/39	—

follows that the most extreme negative value of the third term inside the square brackets certainly exceeds

$$\frac{1}{3}e^{-7z_0/12}T'(z_c) = -0.6864. \quad (73)$$

Hence $\tilde{g}_3(z)$ is the density of a law $L(\omega_3)$.

The rational form of $\tilde{\zeta}_3(s)$ computed using Table 1 yields

$$\begin{aligned} \tilde{\zeta}_3(s) = & \frac{2}{1+s} + \frac{19+12s}{16+19s+6s^2} \\ & - \frac{204+236s+72s^2}{125+204s+118s^2+24s^3}. \end{aligned} \quad (74)$$

Laplace transform inverses of the first two terms were computed in the case $n = 2$. Denote the cubic polynomial denominator by $D(s)$. It has a zero at $-s_1 \approx -1.8120$ ($D(-s_1) = 0.0005$). It follows that

$$D(s) \approx 24(s + s_1) \left[(s + 1.5523)^2 + 0.6816^2 \right] \quad (75)$$

so the partial fraction form of the subtracted rational term and inverting yields

$$\begin{aligned} z\tilde{l}_3(z) \approx & 2e^{-z} + 2e^{-19z/12} \cos(\alpha z) - e^{-1.8120z} \\ & - 2e^{-1.5523z} \cos(\beta z), \end{aligned} \quad (76)$$

where $\beta = 0.6816$. Write the right-hand side as $e^{-z}(2 - \delta(z))$, where

$$\delta(z) = -2e^{-7z/12} \cos(\alpha z) + e^{-0.8120z} + 2e^{-0.5523z} \cos(\beta z). \quad (77)$$

Clearly $|\delta(z)|$ is bounded above by the sum of the exponential terms, a decreasing function which equals 2.0030 if $z = 3/2$. Hence $\tilde{l}_3(z) > 0$, say, for all $z > 1.6$. Computing actual values of $\delta(z)$ indicates that it is positive and decreasing in $[0, 1.6]$; for example, $\delta(0) = 1$, $\delta(0.5) = 0.6321$, $\delta(1) = 0.3098$, and $\delta(1.5) = 0.0628$.

Thus we can safely conclude that $\tilde{l}_3(z)$ is a Lévy density and hence that $L(\omega_3)$ is infdiv. It follows that $L(\tilde{X}_n)$ is infdiv for $n = 1, \dots, 4$ and we conjecture that this holds for all n .

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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