Research Article

Translation, Creation and Annihilation of Poles and Zeros with the Biernacki and Ruscheweyh Operators, Acting on Meijer's G -Functions

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Meijer's G -functions are studied by the Biernacki and Ruscheweyh operators. These operators are special cases of the Erdélyi-Kober operators (for $m = 1$). The effect of operators on Meijer's G -functions can be shown as the change in the distribution of poles and zeros on the complex plane. These poles and zeros belong to the integrand, a ratio of gamma functions, defining the Meijer's G -function. Displacement in position and increasing or decreasing in number of poles and zeroes are expressed by the transporter, creator, and annihilator operators. With special glance, three basic univalent Meijer's G -functions, Koebe, and convex functions are considered.

1. Introduction

In studying of analytic functions, consideration of existing zeros and poles on the complex plane is the first basic step. Through all analytic functions, Gamma function $\Gamma(z)$ has infinity poles at $n = 0, -1, -2$, and so forth, whereas it does not have any zeros [1].

Consider the following:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}. \quad (1)$$

It is meromorphic function, meaning that it is analytic except for isolated singularities which are poles. However, the function $1/\Gamma(z)$ does not have any poles, instead it has infinity zeroes. This function is an entire function, and its Weierstrass product is [1]

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad (2)$$

where γ is known as the Euler constant γ . Gamma function with different argument has infinity poles in other places. In

integral definition of Meijer's G -functions, we face the products of Gamma functions in numerator and denominator.

Meijer's G -function has been useful in mathematical physics because of its analytical properties and because it can be expressed as a finite number of generalized hypergeometric functions which have well-known series expansions. These functions are defined as follows.

Definition 1. A definition of the Meijer's G -function is given by the following path integral in the complex plane [2–6]:

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \times \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \times \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds. \quad (3)$$

This integral is included in the so-called Mellin-Barnes type and may be viewed as an inverse Mellin transform. Here, an empty product means unity, and the integers m, n, p , and q are

called the “orders” of the G -function or the components of the order $(m; n; p; q)$. Here, a_p and b_q are called “parameters” and, generally, they are complex numbers. The definition holds under the following assumptions: $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p , and q are integer numbers. Further, $a_j - b_k \neq 1, 2, 3, \dots$ for $k = 1, \dots, n$ and $j = 1, 2, \dots, m$ imply that no pole of any $\Gamma(b_j - s)$, $j = 1, \dots, m$ coincides with any pole of any $\Gamma(1 - a_k + s)$, $k = 1, \dots, n$.

Based on the definition, the following basic property is easily derived:

$$z^\alpha G_{p,q}^{m,n} \left[\begin{matrix} \mathbf{a}_p \\ \mathbf{b}_q \end{matrix} \middle| z G_{p,q}^{m,n} \left[\begin{matrix} \mathbf{a}_p + \alpha \\ \mathbf{b}_q + \alpha \end{matrix} \middle| z \right] \right. \quad (4)$$

where the multiplying term z^α changes the parameters of the G -function.

Definition 2 (see [1, 7]). The Mellin transform of a function is given by

$$F(s) = \int_0^\infty x^{s-1} f(x) dx. \quad (5)$$

Definition 3 (see [1, 7]). The inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) ds, \quad c > 0. \quad (6)$$

In [8] Pishkoo and Darus assign a more substantive role to Meijer's G -functions in the univalent functions theory. They use some selected Meijer's G -functions as *univalent Meijer's G -functions* and path integral representation instead of their representative series form. The results are shown in tables wherein each univalent Meijer's G -function of an upper rank is obtained from another univalent Meijer's G -function of lower rank. Herein, and overall, it so suffices that we consider the three basic univalent Meijer's G -functions: $G_{0,2}^{1,0}$; $G_{1,2}^{1,1}$; $G_{1,1}^{1,1}$ and from which a number of univalent Meijer's G -functions can be obtained. One of the most important results in the work mentioned above is that the Erdélyi-Kober (E-K) operators (for $m = 1, 2$) relate Meijer's G -functions together.

In [9], Kiryakova et al. applied generalized fractional calculus in univalent function theory and derived equivalent forms of some well-known operators in terms of E-K fractional integral and derivative. The Biernacki operator was obtained in terms of the E-K fractional integral.

Consider the following:

$$Bf(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi = \int_0^1 \frac{f(z\sigma)}{\sigma} d\sigma = I_1^{-1,1} f(z). \quad (7)$$

Meanwhile, the so-called Ruscheweyh derivative defined by

$$D^\alpha f(z) = \left(\frac{z}{(1-z)^{1+\alpha}} * f \right)(z) \quad (\alpha \geq 0) \quad (8)$$

was also obtained in terms of the E-K fractional derivative of order α , as follows:

$$\begin{aligned} D^\alpha f(z) &= \frac{1}{\Gamma(\alpha+1)} \left[z \left(\frac{d}{dz} \right)^\alpha z^{\alpha-1} \right] f(z) \\ &= \frac{1}{\Gamma(\alpha+1)} D_1^{-1,\alpha} f(z). \end{aligned} \quad (9)$$

If $\alpha = 1$, then we have

$$D^1 f(z) = z \frac{d}{dz} f(z) = D_1^{-1,1} f(z). \quad (10)$$

referring to (3), path integral definition of Meijer's G -function, for each G -function there exists special integrand that has poles and zeroes which can be shown by their positions. In [10, 11] we show that Meijer's G -functions are the solution of physical models and in micro- and nanostructures. In quantum mechanics, operators are measurable physical quantities and wave functions as physical systems. In this paper we study Meijer's G -function as appropriate candidate for describing physical systems and operators prepared in the language of “Generalized fractional calculus operators” which are observables. Here, studying distribution of poles and zeroes before and after action of the operator on Meijer's G -function leads to three definitions for operators; namely, transporter, creator, and annihilator operators.

2. Preliminaries

Using (3) for three basic univalent G -functions $G_{0,2}^{1,0}$; $G_{1,2}^{1,1}$; $G_{1,1}^{1,1}$, the following are obtained.

2.1. The First Basic Univalent G -Function. Consider the following:

$$G_{0,2}^{1,0} \left[\begin{matrix} - \\ b_1, b_2 \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_1 - s) z^s}{\Gamma(1 - b_2 + s)} ds. \quad (11)$$

Position of poles: $s = b_1 + n$; $n = 0, 1, 2, \dots$

Position of zeroes: $s = b_2 - 1 - n$; $n = 0, 1, 2, \dots$

If $b_1 = 0$, $b_2 = 1/2$, and $z \rightarrow z^2/4$, then then (see Figure 1) we get

$$\cos z = \sqrt{\pi} \left(\frac{1}{2\pi i} \right) \int_L \frac{\Gamma(-s) z^{2s}}{4^s \Gamma(1/2 + s)} ds. \quad (12)$$

If $b_1 = 1/2$, $b_2 = 0$, and $z \rightarrow z^2/4$, then (see Figure 2) we get

$$\sin z = \sqrt{\pi} \left(\frac{1}{2\pi i} \right) \int_L \frac{\Gamma(1/2 - s) z^{2s}}{4^s \Gamma(1 + s)} ds. \quad (13)$$

2.2. The Second Basic Univalent G -Function. Consider the following:

$$G_{1,2}^{1,1} \left[\begin{matrix} a_1 \\ b_1, b_2 \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_1 - s) \Gamma(1 - a_1 + s) z^s}{\Gamma(1 - b_2 + s)} ds. \quad (14)$$

FIGURE 1: Poles and zeroes related to $\cos z$. FIGURE 2: Poles and zeroes related to $\sin z$.

Position of poles: $s = b_1 + n; n = 0, 1, 2, \dots$ and $s = a_1 - 1 - n; n = 0, 1, 2, \dots$

Position of zeroes: $s = b_2 - 1 - n; n = 0, 1, 2, \dots$

If $a_1 = 1 + i, b_1 = 0$, and $b_2 = i + 1/2$ then (see Figure 3) we get

$$G_{1,2}^{1,1} \left[\begin{matrix} 1+i \\ 0, i+\frac{1}{2} \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int_L \frac{\Gamma(-s) \Gamma(-i+s)}{\Gamma(1/2+i+s)} z^s ds. \quad (15)$$

If $a_1 = b_2$ then $G_{1,2}^{1,1} \left[\begin{matrix} a_1 \\ b_1, b_2 \end{matrix} \middle| z \right] = G_{0,1}^{1,0} \left[\begin{matrix} - \\ b_1 \end{matrix} \middle| z \right] = (1/2\pi i) \int_L \Gamma(b_1 - s) z^s ds.$

Position of poles: $s = b_1 + n; n = 0, 1, 2, \dots$

Position of zeroes: there are no zeroes.

If we put $b_1 = 0$, then we get exponential function

$$e^{-z} = G_{0,1}^{1,0} [0 \mid z] = \frac{1}{2\pi i} \int_L \Gamma(-s) z^s ds. \quad (16)$$

2.3. The Third Basic Univalent G-Function. Consider the following:

$$G_{1,1}^{1,1} \left[\begin{matrix} a_1 \\ b_1 \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int_L \Gamma(b_1 - s) \Gamma(1 - a_1 + s) z^s ds. \quad (17)$$

FIGURE 3: Poles and zeroes related to $G_{1,2}^{1,1} \left[\begin{matrix} 1+i \\ 0, i+1/2 \end{matrix} \middle| z \right]$.

Position of poles: $s = b_1 + n; n = 0, 1, 2, \dots$ and $s = a_1 - 1 - n; n = 0, 1, 2, \dots$

Position of zeroes: there are no zeroes.

In [2] all elementary functions can be expressed in terms of G-functions. For instance

$$(1-z)^{-\alpha} = \frac{1}{\Gamma(\alpha)} G_{1,1}^{1,1} \left[\begin{matrix} 1-\alpha \\ 0 \end{matrix} \middle| z \right], \quad (18)$$

wherein, if $\alpha = 2$, then (18) implies that

$$\frac{1}{(1-z)^2} = G_{1,1}^{1,1} \left[\begin{matrix} -1 \\ 0 \end{matrix} \middle| z \right]. \quad (19)$$

It is well known that the Koebe function plays an important role in the theory of univalent functions. Using (17), (19), and (4), if we put $a_1 = 0$ and $b_1 = 1$ then the Koebe function can be obtained (see Figure 4) as follows:

$$\begin{aligned} K(z) &= \frac{z}{(1-z)^2} = G_{1,1}^{1,1} \left[\begin{matrix} 0 \\ 1 \end{matrix} \middle| z \right] \\ &= \frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(1+s) z^s ds. \end{aligned} \quad (20)$$

The path of integration is curved to separate the poles of $\Gamma(1-s)$ from the poles of $\Gamma(1+s)$. Using (17), (18), and (4), if we put $\alpha = 1, a_1 = 1$, and $b_1 = 1$ then the function $z/(1-z)$ can be obtained as follows:

$$\frac{z}{1-z} = G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(s) z^s ds. \quad (21)$$

The path of integration is curved to separate the poles of $\Gamma(1-s)$ from the poles of $\Gamma(s)$.

3. Results and Discussion

The E-K fractional derivative of order 1, namely, $D_1^{-1,1}$, maps $g(z) = z/(1-z)$ onto the Koebe function.

FIGURE 4: Poles related to the Koebe function.

FIGURE 5: Poles related to the function $g(z)$.

Consider (see Figure 5) the following:

$$\begin{aligned}
 g(z) &= \frac{z}{1-z} = G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right] = \frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(s) z^s ds. \\
 &= \frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(1+s) z^s ds \\
 &= D_1^{-1,1} \left(\frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(s) z^s ds \right).
 \end{aligned} \tag{22}$$

So we obtain

$$D_1^{-1,1} G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right] > G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right] \tag{23}$$

or

$$G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right] < G_{1,1}^{1,1} \left[\begin{matrix} 0 \\ 1 \end{matrix} \middle| z \right] \tag{24}$$

Vice versa, the E-K fractional integral, the Biernacki operator $I_1^{-1,1}$, maps the Koebe function onto $g(z) = z/(1-z)$.

Consider the following:

$$\begin{aligned}
 &\frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(s) z^s ds \\
 &= I_1^{-1,1} \left(\frac{1}{2\pi i} \int_L \Gamma(1-s) \Gamma(1+s) z^s ds \right).
 \end{aligned} \tag{25}$$

Similarly we have

$$I_1^{-1,1} G_{1,1}^{1,1} \left[\begin{matrix} 0 \\ 1 \end{matrix} \middle| z \right] < G_{1,1}^{1,1} \left[\begin{matrix} 0 \\ 1 \end{matrix} \middle| z \right]. \tag{26}$$

And more than this we have

$$\begin{aligned}
 I_1^{-1,1} D_1^{-1,1} G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right] &= G_{1,1}^{1,1} \left[\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right], \\
 D_1^{-1,1} I_1^{-1,1} G_{1,1}^{1,1} \left[\begin{matrix} 0 \\ 1 \end{matrix} \middle| z \right] &= G_{1,1}^{1,1} \left[\begin{matrix} 0 \\ 1 \end{matrix} \middle| z \right].
 \end{aligned} \tag{27}$$

Definition 4. Pole (zero) transporter is the Erdélyi-Kober operator that changes the argument of Gamma function(s) which is (are) inside the contour integral definition of Meijer's G-function and shifts position of poles (zeros) of Gamma function(s) on the complex plane.

Definition 5. Pole (zero) creator is the Erdélyi-Kober operator that creates excess Gamma function(s), on the numerator (denominator) of the integrand, inside the contour integral definition of Meijer's G-function and can create a few or infinity excess poles (zeroes) on the complex plane.

Definition 6. Pole (zero) annihilator is the Erdélyi-Kober operator that annihilates Gamma function(s), on the numerator (denominator) of the integrand, inside the contour integral definition of Meijer's G-function and can annihilates a few or infinity poles (zeroes) on the complex plane.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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