Research Article

A Few Complex Equations Constituted by an Operator Consisting of Fractional Calculus and Their Consequences

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A few complex (differential) equations constituted by certain operators consisting of fractional calculus are first presented and some of their comprehensive consequences relating to (analytic and) geometric function theory are then pointed out.

1. Introduction, Definitions, Notations, and Motivation

From the literature, as we know, fractional calculus (FC) is a generalization of ordinary differentiation and integration to arbitrary noninteger order. This subject is as old as the differential calculus and also goes back to time when Leibniz and Newton invented differential calculus. The efficient usage of FC has been a subject of interest not only among mathematicians but also among physicists and engineers, appearing in rheology, viscoelasticity, electrochemistry, electromagnetism, and so forth. For example, see the works in the references in [1–11].

Fractional differential equations (FDE), that is, differential equations determined by FC, have also many applications in modeling of physical and chemical processes and in engineering. In their turn, mathematical aspects of studies on FDC were discussed by several authors. For those, can be also seen the works in [3, 12–29].

As we have emphasized just above, the main purpose of this investigation is both to present a novel work relating to analytic and/or geometric function theory (AGFT) and FDC and to reveal some (comprehensive) results between certain complex valued functions and complex (differential) equations constituted by certain operators dealing with FC. In

particular, special consequences of the main results are also pointed out in the concluding section of this paper.

Now, there is a need to introduce some notations and definitions which will be used in this work.

First, let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ be the set of real numbers, the set of complex numbers and the set of positive integers, respectively. Also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{C}^* := \mathbb{C} - \{0\}$, and $\mathbb{R}^* := \mathbb{R} - \{0\}$.

For $0 \leq \mu < 1$ and an analytic function $\kappa := \kappa(z)$, the symbol $\mathcal{D}_z^\mu[\kappa]$ denotes an operator of FC, which is defined as follows (cf., e.g., [9, 30–33]).

Let $\kappa(z)$ be an analytic function in a simply-connected region of the z -plane containing the origin. Then, the fractional derivative of order μ is defined by

$$\mathcal{D}_z^\mu[\kappa] = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{\kappa(\xi)}{(z-\xi)^\mu} d\xi \quad (0 \leq \mu < 1), \quad (1)$$

where the multiplicity of $(z-\xi)^{-\mu}$ above is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$. Note that, here and throughout this investigation, the function Γ is the well-known gamma function.

Under the hypotheses of the definition above, for an analytic function $\kappa(z)$, the fractional derivative of order $m + \mu$ is also defined by

$$\mathcal{D}_z^{m+\mu}[\kappa] = \frac{d^m}{dz^m} (\mathcal{D}_z^\mu[\kappa]) \quad (0 \leq \mu < 1; m \in \mathbb{N}_0). \quad (2)$$

By means of (1) and (2), for a function $\kappa(z) = z^\nu$, it can be easily determined that

$$\mathcal{D}_z^{m+\mu} [z^\nu] = \frac{\Gamma(\nu+1)}{\Gamma(\nu-m-\mu+1)} z^{\nu-m-\mu} \quad (3)$$

for some $0 \leq \mu < 1$ and for all $m \in \mathbb{N}_0$ with $m < \nu - \mu + 1$.

In the usual notation, let $\mathcal{A}$ denote the family of the functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \quad (4)$$

$$(a_{n+1} \in \mathbb{C}; n \in \mathbb{N}),$$

which are analytic in the unit open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the family of functions belonging to $\mathcal{A}$ which are univalent in the open disk $\mathbb{U}$. As is known, the functions family $\mathcal{A}$ has important roles for AGFT (see [34, 35]). In particular, some of the important and well-investigated families of the univalent function family $\mathcal{S}$ include the family $\mathcal{S}^*(\tau)$ of starlike functions of order τ and the family $\mathcal{K}(\tau)$ of convex functions of order τ ($0 \leq \tau < 1$) in the domain $\mathbb{U}$. As their definitions, we also recall

$$\mathcal{S}^*(\tau) := \left\{ f \in \mathcal{S} : \Re \left(\frac{zf'(z)}{f(z)} \right) > \tau \quad (0 \leq \tau < 1; z \in \mathbb{U}) \right\},$$

$$\mathcal{K}(\tau) := \left\{ f \in \mathcal{S} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \tau \quad (0 \leq \tau < 1; z \in \mathbb{U}) \right\}. \quad (5)$$

It readily follows from definitions (5) that $f(z) \in \mathcal{K}(\tau)$ if and only if $zf'(z) \in \mathcal{S}^*(\tau)$. For their details, one may refer to [34, 35].

Now, by using the operator $\mathcal{D}_z^\mu[\cdot]$, for a function $f(z)$ in $\mathcal{A}$ (or, in $\mathcal{S}$), given by (1), we can again define a linear operator $J_z^\mu[f]$ as in the following form:

$$J_z^\mu[f] = \Gamma(2-\mu) z^\mu \mathcal{D}_z^\mu[f]$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k-\mu+1)} a_k z^k, \quad (6)$$

where $\mu \in \mathbb{R} := \mathbb{R} - \{2, 3, 4, \dots\}$.

2. Main Result and Certain Consequences

In this section, firstly, in order to prove main result, we need to recall the following assertion given by [36].

Lemma 1. Let $p(z)$ be an analytic function in the disk $\mathbb{U}$ with $p(0) = 1$. If there exists a point z_0 in $\mathbb{U}$ such that

$$\Re(p(z)) > 0 \quad (|z| < |z_0|), \quad (7)$$

$$\Re(p(z_0)) = 0, \quad p(z_0) \neq 0,$$

then

$$p(z_0) = ia,$$

$$\left. \frac{zp'(z)}{p(z)} \right|_{z=z_0} = ik \left(a + \frac{1}{a} \right) \quad (k \geq 1; a \in \mathbb{R}^*). \quad (8)$$

The comprehensive main result involving certain analytic functions and complex equations constituted by FC, which is fractional derivatives, is contained in Theorem 2 below.

Theorem 2. Let $\psi(z)$ be an analytic function and satisfy any of the following inequalities:

$$\Re(\psi(z)) \neq 0, \quad |\Im(\psi(z))| < 2 \quad (9)$$

and also let the function $\mathcal{F}(z)$ be defined in the following form:

$$\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z) \quad (0 \leq \lambda \leq 1; f(z) \in \mathcal{S}). \quad (10)$$

If $\mathcal{F} := \mathcal{F}(z)$ satisfies the following complex equation:

$$z \cdot \left(\frac{J_z^\alpha[\mathcal{F}]}{J_z^\beta[\mathcal{F}]} \right)' - \psi(z) \cdot \frac{J_z^\alpha[\mathcal{F}]}{J_z^\beta[\mathcal{F}]} = -\gamma \cdot \psi(z), \quad (11)$$

then

$$\Re \left(\frac{J_z^\alpha[\mathcal{F}]}{J_z^\beta[\mathcal{F}]} \right) > \gamma \quad (0 \leq \gamma < 1; \alpha \in \mathbb{R}; \beta \in \mathbb{R}; z \in \mathbb{U}). \quad (12)$$

Proof. For a function $f(z) \in \mathcal{S}$, we assume that $\mathcal{F}(z)$ has got the form as in (10). Then, in view of $J_z^\alpha[\mathcal{F}]$ together with (6), it can be easily derived that

$$\frac{J_z^\alpha[\mathcal{F}]}{J_z^\beta[\mathcal{F}]} = \frac{\Gamma(2-\alpha) z^\alpha \mathcal{D}_z^\alpha[\mathcal{F}]}{\Gamma(2-\beta) z^\beta \mathcal{D}_z^\beta[\mathcal{F}]}$$

$$= \left(\left(z + \sum_{k=2}^{\infty} \frac{(k\lambda - \lambda + 1)\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k-\alpha+1)} a_k z^k \right) \right.$$

$$\times \left. \left(z + \sum_{k=2}^{\infty} \frac{(k\lambda - \lambda + 1)\Gamma(k+1)\Gamma(2-\beta)}{\Gamma(k-\beta+1)} a_k z^k \right)^{-1} \right)$$

$$=: h(z), \quad (13)$$

where $0 \leq \gamma < 1$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and $z \in \mathbb{U}$. Clearly, the function $h(z)$ has a removable singular point at $z = 0$ and also its series expansion is in the following form:

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots \quad (z \in \mathbb{U}). \quad (14)$$

It follows from equality (13) along with (14) that one can define a function $p(z)$ by

$$\frac{1}{1-\gamma} \cdot \left(\frac{J_z^\alpha[\mathcal{F}]}{J_z^\beta[\mathcal{F}]} - \gamma \right) = p(z) \quad (0 \leq \gamma < 1; z \in \mathbb{U}). \quad (15)$$

Obviously, $p(z)$ has a similar series expansion to the function $h(z)$ in (14) and is an analytic function in $\mathbb{U}$ with $p(0) = 1$. By the help of equality (15), we obtain

$$\frac{z \cdot \left[(1/(1-\gamma)) \cdot \left(J_z^\alpha[\mathcal{F}]/J_z^\beta[\mathcal{F}] - \gamma \right) \right]'}{(1/(1-\gamma)) \cdot \left(J_z^\alpha[\mathcal{F}]/J_z^\beta[\mathcal{F}] - \gamma \right)} = \frac{zp'(z)}{p(z)} \quad (16)$$

or, equivalently,

$$\frac{z \cdot (J_z^\alpha [\mathcal{F}] / J_z^\beta [\mathcal{F}])'}{J_z^\alpha [\mathcal{F}] / J_z^\beta [\mathcal{F}] - \gamma} \left(= \frac{zp'(z)}{p(z)} \right) = \psi(z). \quad (17)$$

Suppose now that there exists a point $z_0 \in \mathbb{U}$ such that

$$\begin{aligned} \Re e(p(z)) &> 0 \quad (|z| < |z_0|), \\ \Re e(p(z_0)) &= 0, \quad p(z_0) \neq 0. \end{aligned} \quad (18)$$

Then, from (8) of Lemma 1, we obtain that

$$\begin{aligned} p(z_0) &= ia, \\ \frac{zp'(z)}{p(z)} \Big|_{z=z_0} &= ik \left(a + \frac{1}{a} \right) \quad (k \geq 1; a \in \mathbb{R}^*). \end{aligned} \quad (19)$$

If we use the related equations in (17), we then arrive at

$$\begin{aligned} \Re e(\psi(z_0)) &= \Re e \left(\frac{zp'(z)}{p(z)} \Big|_{z=z_0} \right) = 0, \\ |\Im m(\psi(z_0))| &= \left| \Im m \left(\frac{zp'(z)}{p(z)} \Big|_{z=z_0} \right) \right| \\ &= k \left| a + \frac{1}{a} \right| \geq 2. \end{aligned} \quad (20)$$

But the results in (20) are contradictions with the assumptions in (9) and (10), respectively. Hence, the equality in (15) yields that $\Re e(p(z)) > 0$ for all $z \in \mathbb{U}$. Therefore, we evidently receive the inequality in (12). This completes the desired proof. $\square$

As we emphasized before, Theorem 2 includes several comprehensive results in relation with AGFT. Specially, some of its consequences containing results between certain complex (differential) equations constituted by certain operators consisting of fractional calculus and the theory of univalent functions are fairly important results. Accordingly, we want to focus on only one of them (which is Proposition 3 below) and also its two useful applications (which are Corollaries 4 and 5 below). The other possible consequences of the main result (which are here omitted) are presented to the attention of the researchers who have been working on the theory of differential equation and/or univalent function.

Proposition 3. *Let $\psi(z)$ be an analytic function and satisfy any of the inequalities given by (9), and also let $\mathcal{F}(z)$ be defined by (10). If $\mathcal{F}(z)$ satisfies the following complex equation:*

$$\begin{aligned} &[\beta J_z^{2+\beta} [\mathcal{F}] + (\psi(z) + 2\beta - 1) J_z^{1+\beta} [\mathcal{F}]] \cdot J_z^\beta [\mathcal{F}] \\ &+ (1 - \beta) [J_z^{1+\beta} [\mathcal{F}]]^2 - \gamma \psi(z) [J_z^\beta [\mathcal{F}]]^2 = 0, \end{aligned} \quad (21)$$

then

$$\Re e \left(\frac{J_z^{1+\beta} [\mathcal{F}]}{J_z^\beta [\mathcal{F}]} \right) > \gamma \quad (0 \leq \gamma < 1; \beta \in \mathbb{R}; z \in \mathbb{U}). \quad (22)$$

Proof. By taking $\alpha := 1 + \beta$ ($0 \leq \beta < 1$) in Theorem 2 and also using the following well-known identity:

$$z \cdot (J_z^\beta [\mathcal{F}])' = (1 - \beta) J_z^{1+\beta} [\mathcal{F}] + \beta J_z^\beta [\mathcal{F}], \quad (23)$$

the proof of Proposition 3 can be easily proven. Its detail is here omitted. $\square$

By letting $\beta := 0$ and $\lambda := 0$ in Proposition 3 (or equivalent choosing in Theorem 2), the result (which is Corollary 4 below) relating to starlikeness of order γ can be next obtained.

Corollary 4. *Let $\psi(z)$ be an analytic function and satisfy any of the inequalities given by (9). If the function $w := f(z) \in \mathcal{S}$ satisfies the following nonlinear complex differential equation:*

$$z [zw'' + (1 - \psi(z)) w'] w - z^2 [w']^2 + \gamma \psi(z) w^2 = 0, \quad (24)$$

then $w \in \mathcal{S}^*(\gamma)$, where $0 \leq \gamma < 1$ and $z \in \mathbb{U}$.

By setting $\beta := 0$ and $\lambda := 1$ in Proposition 3 (or equivalent choosing in Theorem 2), the result (which is Corollary 5 below) dealing with convexity of order γ can be also obtained.

Corollary 5. *Let $\psi(z)$ be an analytic function and satisfy any of the inequalities given by (9). If the function $w := f(z) \in \mathcal{S}$ satisfies the following nonlinear complex differential equation:*

$$\begin{aligned} &z [zw''' + (1 - \psi(z)) w''] w' \\ &- z^2 [w'']^2 - (1 - \gamma) \psi(z) [w']^2 = 0, \end{aligned} \quad (25)$$

then $w \in \mathcal{K}(\gamma)$, where $0 \leq \gamma < 1$ and $z \in \mathbb{U}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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