Research Article

A Priori Estimation of the Solution for Mixed Problems with Integral Condition for Singular Parabolic Equations

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Received 31 October 2013; Accepted 15 December 2013; Published 4 February 2014

Academic Editors: M. Escobedo and C. Zhu

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We prove a theorem in which we get a priori estimation of the solution for mixed problems with integral condition for singular parabolic equations. Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in many works lately. Our result plays an important role in the theory of heat transmission, thermoelasticity, chemical engineering, underground water flow, and plasma physics.

1. Introduction

The importance of problems with integral condition has been pointed out by Samarskii [1]. Mathematical modelling by evolution problems with a nonlocal constraint of the form $(1/(1 - \alpha)) \int_{\alpha}^1 u(x, t) dx = E(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics. For background information, we refer the reader to Benouar and Yurchuk [2], Bouziani and Benouar [3], Bouziani [4], Cannon and van der Hoek [5–7], Ionkin and Moiceev [8, 9], Kamynin [10], and Yurchuk [11, 12]. Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [13], Byszewski [14–16], Gasymov [17], Ionkin [8, 9], Lazhar [18], and Said and Nadia [19]. The results and the method used here are a further elaboration of those in [2]. We should mention here that the presence of integral term in the boundary condition can greatly complicate the application of standard functional and numerical techniques. This work can be considered as a continuation of the results in [11, 20].

We consider the following mixed problem in the rectangle $Q = (0, l) \times (0, T)$:

$$Lu = \frac{\partial u}{\partial t} - \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) = f(x, t), \quad m > 0, \quad (1)$$

$$\begin{aligned} lu &= u(x, 0) = \varphi(x), \quad |u(0, t)| < \infty, \\ \int_{\alpha}^l x^m u(x, t) dx &= 0, \quad \alpha > 0. \end{aligned} \quad (2)$$

2. A Priori Estimate

Let E be the Hilbert space of all sufficiently smooth functions u satisfying the second and third conditions in (2) and equipped with the norm

$$\begin{aligned} \|u\|_E^2 &= \int_Q (l - x) \left[x^m \left| \frac{\partial u}{\partial t} \right|^2 + \frac{1}{x^m} \left| \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) \right|^2 \right] dx dt \\ &+ \sup_{0 \leq t \leq T} \left\{ \int_0^l (l - x) x^m \left(\frac{\partial x}{\partial u} \right)^2 dx + \frac{m}{l - \alpha} \int_{\alpha}^l x^{m-1} u^2 dx \right\}. \end{aligned} \quad (3)$$

The equality

$$x^{m+1} \frac{\partial u}{\partial x} = \int_0^x d\eta \int_{\eta}^x \frac{\partial}{\partial \xi} \left(\xi^m \frac{\partial u}{\partial \xi} \right) d\xi + \int_0^x \xi^m \frac{\partial u}{\partial \xi} d\xi \quad (4)$$

implies the following inequality:

$$x^m \left| \frac{\partial u}{\partial x} \right| \leq x^{1/2} \sqrt{\int_0^x \left(\frac{\partial}{\partial \xi} \left(\xi^m \frac{\partial u}{\partial \xi} \right) \right)^2 d\xi} + \sqrt{\int_0^x \left(\xi^m \frac{\partial u}{\partial \xi} \right)^2 d\xi}. \quad (5)$$

By (5) it follows that

$$\lim_{x \rightarrow 0} x^m \frac{\partial u}{\partial x} = 0, \quad (6)$$

for any $u \in E$.

We will use (5) for the solutions u of the problem (1)-(2). For the right hand side f of (1) and initial condition of u from (2) we introduce the space F which is consisted of the vector function $\mathcal{F} = (f, \varphi)$ with the norm

$$\begin{aligned} \|\mathcal{F}\|_F^2 &= \int_Q x^m |f(x, t)|^2 dx + \int_0^l x^m \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\ &+ \frac{m}{l - \alpha} \int_\alpha^l x^{m-1} \varphi^2 dx. \end{aligned} \quad (7)$$

Here it is assumed that

$$|\varphi(0)| < \infty, \quad \int_\alpha^l x^m \varphi(x) dx = 0. \quad (8)$$

Theorem 1. For any function $u \in E$ such that $x^{m/2}(\partial u / \partial t) \in L_2(Q)$ and $x^{-m/2}(\partial / \partial x)x^m(\partial u / \partial x) \in L_2(Q)$, the following inequality holds:

$$\|u\|_E^2 \leq c \|\mathcal{F}\|_F^2, \quad (9)$$

where $c = 2(l + \exp(T/2\alpha^2))$.

Proof. We set $Mu = x^m u$ with $0 \leq x \leq \alpha$ and

$$\begin{aligned} Mu &= \frac{l-x}{l-\alpha} x^m u(x, t) + \frac{1}{l-\alpha} J(x^m u), \\ J\vartheta(x) &= \int_\alpha^x \vartheta(\xi) d\xi, \end{aligned} \quad (10)$$

with $\alpha \leq x \leq l$.

Consider the following equality:

$$\begin{aligned} &\int_0^\tau \int_0^l LuM \frac{\partial u}{\partial t} dx dt \\ &= \int_0^\tau \int_0^l \frac{\partial u}{\partial t} M \frac{\partial u}{\partial t} dx dt \\ &\quad - \int_0^\tau \int_0^l \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) M \frac{\partial u}{\partial t} dx dt. \end{aligned} \quad (11)$$

It can be seen that the following equalities hold:

$$\begin{aligned} &\int_0^\tau \int_0^l \frac{\partial u}{\partial t} M \frac{\partial u}{\partial t} dx dt \\ &= \int_0^\tau \int_0^\alpha x^m \left| \frac{\partial u}{\partial t} \right|^2 dx \\ &\quad + \int_0^\tau \int_\alpha^l \left[\left(\frac{l-x}{l-\alpha} \right) x^m \left| \frac{\partial u}{\partial t} \right|^2 \right. \\ &\quad \left. + \frac{1}{l-\alpha} \frac{\partial u}{\partial t} \int_\alpha^x \frac{\xi \partial u(\xi, t)}{\partial t} d\xi \right] dx dt, \end{aligned} \quad (12)$$

$$\begin{aligned} &\int_\alpha^l \frac{\partial u}{\partial t} \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi dx \\ &= \frac{m}{2} \int_\alpha^l \left| \frac{\partial u}{\partial t} \right| \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi^2 dx. \end{aligned}$$

By (12), we obtain the following equality:

$$\begin{aligned} &\int_0^\tau \int_0^l \frac{\partial u}{\partial t} M \frac{\partial u}{\partial t} dx dt \\ &= \int_0^\tau \int_0^\alpha x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\ &\quad + \int_0^\tau \int_\alpha^l \frac{l-x}{l-\alpha} x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\ &\quad + \frac{m}{l-\alpha} \int_0^\tau \int_\alpha^l \frac{1}{x^{m+1}} \left| \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi \right|^2 dx dt. \end{aligned} \quad (13)$$

Integrating by parts (and using (6)), we get

$$\begin{aligned} &-\int_0^\tau \int_0^\alpha \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) x^m \frac{\partial u}{\partial t} dx dt \\ &= -\int_0^\tau x^m \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \Big|_{x=0}^{x=\alpha} + \int_0^\tau \int_0^\alpha x^m \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} dx dt \\ &= -\int_0^\tau \alpha^m \frac{\partial u(\alpha, t)}{\partial \alpha} \frac{\partial u(\alpha, t)}{\partial t} dt \\ &\quad + \frac{1}{2} \int_0^\alpha x^m \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx - \frac{1}{2} \int_0^\alpha x^m \left(\frac{d\varphi(x)}{dx} \right)^2 dx, \end{aligned}$$

$$\begin{aligned}
& - \int_0^\tau \int_\alpha^l \frac{l-x}{l-\alpha} \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) x^m \frac{\partial u}{\partial t} dx dt \\
& = - \int_0^\tau \frac{l-x}{l-\alpha} x^m \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dt \Big|_{x=\alpha}^{x=l} \\
& \quad + \int_0^\tau \int_\alpha^l \frac{l-x}{l-\alpha} x^m \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} dx dt \\
& \quad - \frac{1}{l-\alpha} \int_0^\tau \int_\alpha^l x^m \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\
& = \int_0^\tau \alpha \frac{\partial u(\alpha, t)}{\partial \alpha} \frac{\partial u(\alpha, t)}{\partial t} dt \\
& \quad + \frac{1}{2} \int_\alpha^l \frac{l-x}{l-\alpha} x^m \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \\
& \quad - \frac{1}{2} \int_\alpha^l \frac{l-x}{l-\alpha} x^m \left| \frac{\partial \varphi(x)}{\partial x} \right|^2 dx \\
& \quad - \frac{1}{l-\alpha} \int_0^\tau \int_\alpha^l x^m \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt, \\
& - \int_0^\tau \int_\alpha^l \frac{1}{l-\alpha} \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi dx dt \\
& = \frac{1}{l-\alpha} \int_0^\tau \int_\alpha^l x^m \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\
& \quad - \frac{m}{l-\alpha} \int_0^\tau \int_\alpha^l \frac{1}{x} \frac{\partial u}{\partial x} \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi dx dt \\
& = \frac{1}{l-\alpha} \int_0^\tau \int_\alpha^l x^m \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\
& \quad - \frac{m}{l-\alpha} \int_0^\tau \int_\alpha^l \frac{u}{x^2} \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi dx dt \\
& \quad + \frac{m}{2(l-x)} \int_\alpha^l u^2(x, \tau) x^{m-1} dx \\
& \quad - \frac{m}{2(l-x)} \int_\alpha^l x^{m-1} \varphi^2(x) dx.
\end{aligned} \tag{14}$$

$$\begin{aligned}
& - \int_0^\alpha x^m \left| \frac{\partial \varphi(x)}{\partial x} \right|^2 dx \\
& - \int_\alpha^l \left[\frac{l-x}{l-\alpha} x^m \left| \frac{\partial \varphi(x)}{\partial x} \right|^2 \right. \\
& \quad \left. - \frac{m}{l-\alpha} x^{m-1} \varphi^2(x) \right] dx \Big\} \\
& - \frac{m}{l-\alpha} \int_0^\tau \int_0^\alpha \frac{u(x, t)}{x^2} \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} d\xi dt dx.
\end{aligned} \tag{15}$$

Adding (13) and (15), we get

$$\begin{aligned}
& \int_0^\tau \int_0^l LuM \frac{\partial u}{\partial t} dx dt \\
& = \int_0^\tau \int_0^l \psi(x) x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
& \quad + \frac{1}{2} \left\{ \int_0^l \psi(x) x^m \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \right. \\
& \quad + \frac{m}{(l-\alpha)} \int_\alpha^l x^{m-1} u^2(x, \tau) dx \\
& \quad - \int_0^l \psi(x) x^m \left| \frac{d\varphi}{dx} \right|^2 dx \\
& \quad \left. - \frac{m}{(l-\alpha)} \int_\alpha^l x^{m-1} \varphi^2(x) dx \right\} \\
& \quad + \frac{m}{(l-\alpha)} \int_0^\tau \int_\alpha^l \frac{1}{x^{m+1}} \left| \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} \right|^2 dx dt \\
& \quad - \frac{m}{(l-\alpha)} \int_0^\tau \int_\alpha^l \frac{u}{x^2} \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} dx dt,
\end{aligned} \tag{16}$$

where the function

$$\psi(x) = \begin{cases} 1, & 0 \leq x \leq \alpha \\ \frac{l-x}{l-\alpha}, & \alpha \leq x \leq l. \end{cases} \tag{17}$$

The formulas (14) imply the following:

$$\begin{aligned}
& - \int_0^\tau \int_0^\alpha \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) M \frac{\partial u}{\partial t} dx dt \\
& = \frac{1}{2} \left\{ \frac{1}{2} \int_0^\alpha x^m \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \right. \\
& \quad \left. + \int_\alpha^l \left[\frac{l-x}{l-\alpha} x^m \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 + \frac{m \cdot x^m}{(l-\alpha)} u^2(x, \tau) \right] dx \right.
\end{aligned}$$

Now, it can be shown that the following inequalities hold:

$$\begin{aligned}
& \int_0^\tau \int_\alpha^l \frac{u}{x^2} \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} dx dt \\
& \leq \int_0^\tau \int_\alpha^l \frac{1}{x^{m+1}} \left| \int_\alpha^x \xi^m \frac{\partial u(\xi, t)}{\partial t} \right|^2 dx dt \\
& \quad + \frac{1}{4\alpha^2} \int_0^\tau \int_\alpha^l x^{m-1} u^2(x, t) dx dt,
\end{aligned}$$

$$\begin{aligned}
& \int_0^\tau \int_0^l LuM \frac{\partial u}{\partial t} dx dt \\
& \leq \frac{1}{2} \int_0^\tau \int_0^l x^m |Lu|^2 dx dt \\
& \quad + \frac{1}{2} \int_0^\tau \int_0^l \psi(x) x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt, \\
& \quad l - x \leq \psi(x) \leq 1.
\end{aligned} \tag{18}$$

The equality (16) and the inequalities (18) imply the following inequality:

$$\begin{aligned}
& \int_0^\tau \int_0^l (l-x) x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
& \quad + \int_0^l (l-x) x^m \left| \frac{\partial u(x, \tau)}{\partial x} \right|^2 dx \\
& \quad + \frac{m}{l-\alpha} \int_\alpha^l x^{m-1} u^2(x, \tau) dx \\
& \leq \frac{1}{2\alpha^2} \frac{m}{l-\alpha} \int_0^\tau \int_\alpha^l x^{m-1} u^2 dx dt \\
& \quad + \int_0^l x^m \left| \frac{\partial \varphi}{\partial x} \right|^2 dx \\
& \quad + \frac{m}{l-\alpha} \int_\alpha^l x^{m-1} \varphi^2(x) dx \\
& \quad + \int_0^\tau \int_0^l x^m (Lu)^2 dx dt.
\end{aligned} \tag{19}$$

Lemma 2. Let on $[0, T]$ continuous nonnegative functions g_1, g , and h be given, where h is nondecreasing. Then from the inequality

$$g_1(\tau) + g(\tau) \leq c \int_0^\tau g(t) dt + h(\tau) \tag{20}$$

implies the inequality

$$g_1(\tau) + g(\tau) \leq e^{c\tau} h(\tau). \tag{21}$$

The above lemma can be proved by iteration method. We omit the details.

In order to apply the lemma, we set

$$g(\tau) = \frac{m}{l-\alpha} \int_\alpha^l x^{m-1} u^2(x, \tau) dx. \tag{22}$$

Let $g_1(\tau)$ be the first two terms in the left hand side of (19), $h(\tau)$ the last three terms in the right hand side of (19), and $c = 1/2\alpha^2$.

As a consequence of (19), we obtain the following inequality:

$$\begin{aligned}
& \int_0^\tau \int_0^l (l-x) x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
& \quad + \int_0^l (l-x) x^m \left| \frac{\partial u(x, \tau)}{\partial x} \right|^2 dx \\
& \quad + \frac{m}{l-\alpha} \int_\alpha^l x^{m-1} u^2(x, \tau) dx \leq e^{T/2\alpha^2} \|\mathcal{F}\|_F^2.
\end{aligned} \tag{23}$$

Here we have used the notation (7). The right hand side of inequality (23) does not depend on $\mathcal{F}$. Consequently, in the right hand side of (23) the supremum can be taken. Thus, we get the following inequality:

$$\begin{aligned}
& \int_0^T \int_0^l (l-x) x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
& \quad + \sup_{0 \leq t \leq T} \left[\int_0^l (l-x) x^m \left(\frac{\partial u}{\partial x} \right)^2 dx \right. \\
& \quad \left. + \frac{m}{l-\alpha} \int_\alpha^l x^{m-1} u^2(x, t) dx \right] \\
& \leq e^{T/2\alpha^2} \|\mathcal{F}\|_F^2.
\end{aligned} \tag{24}$$

By (1), we get the following inequality:

$$\begin{aligned}
& \int_0^T \int_0^l (l-x) \frac{1}{x^m} \left(\frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x} \right) \right)^2 dx dt \\
& \leq 2 \int_0^T \int_0^l (l-x) x^m \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
& \quad + 2l \int_0^T \int_0^l x^m |f|^2 dx dt.
\end{aligned} \tag{25}$$

The inequalities (24) and (25) imply inequality (9). $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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