Research Article

When Is the Complement of the Comaximal Graph of a Commutative Ring Planar?

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Let R be a commutative ring with identity. In this paper we classify rings R such that the complement of comaximal graph of R is planar. We also consider the subgraph of the complement of comaximal graph of R induced on the set S of all nonunits of R with the property that each element of S is not in the Jacobson radical of R and classify rings R such that this subgraph is planar.

1. Introduction

All rings considered in this paper are commutative with identity $1 \neq 0$. The graphs considered here are undirected and simple. Sharma and Bhatwadekar in [1] introduced a graph $\Omega(R)$ on a commutative ring R , whose vertices are the elements of R and two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. In [1, Theorem 2.3], the authors showed that $\chi(\Omega(R)) < \infty$ if and only if the ring R is finite and in this case $\chi(\Omega(R)) = \omega(\Omega(R)) = t + l$, where t and l denote, respectively, the number of maximal ideals of R and number of units of R . In [2], Maimani et al. named the graph studied by Sharma and Bhatwadekar as the comaximal graph $\Gamma(R)$ of R . Also in [2], the authors studied the subgraphs $\Gamma_1(R)$, $\Gamma_2(R)$, and $\Gamma_2(R) \setminus J(R)$, where $\Gamma_1(R)$ is the subgraph of $\Gamma(R)$ induced on the set of units of R , $\Gamma_2(R)$ is the subgraph of $\Gamma(R)$ induced on the set of nonunits of R , and $\Gamma_2(R) \setminus J(R)$ is the subgraph of $\Gamma(R)$ induced on the set of nonunits of R which are not in $J(R)$ = the Jacobson radical of R . Several other researchers also investigated the comaximal graphs of rings [3–5]. In [6], Gaur and Sharma studied the graph whose vertices are the elements of a ring R and two distinct vertices x and y are adjacent if and only if there exists a maximal ideal of R containing both x and y . They called this graph the maximal graph of the ring R .

Let $G = (V, E)$ be a simple graph. Recall from [7, Definition 1.1.13] that the complement G^c of G is defined by taking $V(G^c) = V(G) = V$ and two distinct vertices u

and v are adjacent in G^c if and only if they are not adjacent in G . Thus the maximal graph of a ring studied in [6] is the complement of the comaximal graph of a ring studied by Sharma and Bhatwadekar in [1].

For any set A , $|A|$ denotes the cardinality of A . For any ring R , $U(R)$ and $NU(R)$ denote, respectively, the set of units of R and the set of nonunits of R . We denote $R \setminus \{0\}$ by R^* and the set of all maximal ideals of R by $\text{Max}(R)$. For the sake of convenience, for any ring R with at least two maximal ideals we denote $\Gamma_2(R) \setminus J(R)$ by $G(R)$. For any $n \geq 2$, $\mathbb{Z}_n$ denotes the ring of integers modulo n . For any prime number p and $n \geq 1$, $\mathbb{F}_{p^n}$ denotes the finite field with exactly p^n elements.

We first recall the following definitions and results from graph theory. A graph $G = (V, E)$ is said to be complete if every pair of distinct vertices of G are adjacent in G . A complete graph on n vertices is denoted by K_n [7, Definition 1.1.11]. A graph $G = (V, E)$ is said to be bipartite if the vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and another one in Y . The pair (X, Y) is called a bipartition of G . A bipartite graph G with bipartition (X, Y) is denoted by $G(X, Y)$. A bipartite graph $G(X, Y)$ is said to be complete if each vertex of X is adjacent to all the vertices of Y . If $G(X, Y)$ is a complete bipartite graph with $|X| = m$ and $|Y| = n$, then it is denoted by $K_{m,n}$ [7, Definition 1.1.12]. Let $G = (V, E)$ be a graph. By a clique of G , we mean a complete subgraph of G [7, Definition 1.2.2]. We say that the clique number of G equals n if n is

the largest positive integer such that K_n is a subgraph of G [7, page 185]. The clique number of a graph G is denoted by the notation $\omega(G)$. If G contains K_n as a subgraph for all $n \geq 1$, then we set $\omega(G) = \infty$.

A graph G is said to be planar if it can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G [7, Definition 8.1.1]. Two adjacent edges of a graph G are said to be in series if their common vertex is of degree two [8, page 9]. Two graphs are said to be homeomorphic if one graph can be obtained from the other graph by the creation of edges in series (i.e., by insertion of vertices of degree two) or by the merger of edges in series [8, page 100]. Recall from [8, page 93] that K_5 is referred to as Kuratowski's first graph and $K_{3,3}$ is referred to as Kuratowski's second graph. A celebrated theorem of Kuratowski says that a necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [8, Theorem 5.9].

Motivated by the interesting theorems proved on comaximal graphs of rings in [1–5] and also the theorems proved on its complement in [6], in this paper we attempt to characterize commutative rings R such that $(\Omega(R))^c$ is planar, where $\Omega(R)$ is the comaximal graph of a ring R . Since any subgraph of a planar graph is planar, it follows that $(G(R))^c (= (\Gamma_2(R) \setminus J(R))^c)$ is planar if $(\Omega(R))^c$ is planar. Hence we first focus our study on investigating when $(G(R))^c$ is planar and at the end of this paper we classify the rings R for which $(\Omega(R))^c$ is planar. While investigating when $(G(R))^c$ is planar, we assume that R has at least two maximal ideals.

In view of Kuratowski's Theorem, [8, Theorem 5.9] we introduce the following definitions.

Let $G = (V, E)$ be a graph.

- (i) We say that G satisfies Ku_1 if G does not contain K_5 as a subgraph.
- (ii) We say that G satisfies Ku_2 if G does not contain $K_{3,3}$ as a subgraph.
- (iii) We say that G satisfies Ku_1^* if G satisfies Ku_1 and, moreover, G does not contain any subgraph homeomorphic to K_5 .
- (iv) We say that G satisfies Ku_2^* if G satisfies Ku_2 and, moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

If a graph G is planar, then it follows from Kuratowski's theorem [8, Theorem 5.9] that G satisfies both Ku_1^* and Ku_2^* . Hence G satisfies both Ku_1 and Ku_2 . It is interesting to note that a graph G may be nonplanar even if it satisfies both Ku_1 and Ku_2 . For example of this type refer to [8, Figure 5.9(a), page 101] and the graph G in this example does not satisfy Ku_1^* . It is not difficult to construct an example of a graph G such that G satisfies Ku_1 but G does not satisfy Ku_1^* .

Let R be a ring with at least two maximal ideals. We now give a brief summary of the results proved in this paper. It is shown in Lemma 8 that if $(G(R))^c$ satisfies Ku_1 or Ku_2 , then $|\text{Max}(R)| \leq 3$. For a ring R with exactly two maximal ideals, it is proved in Theorem 19 that $(G(R))^c$ satisfies Ku_1 if and only if $(G(R))^c$ satisfies Ku_2 if and only if $(G(R))^c$ is

planar. Moreover, the statement (2) of Theorem 19 describes such rings up to isomorphism. If R is a ring with exactly three maximal ideals, then it is shown in Theorem 24 that $(G(R))^c$ satisfies Ku_1 if and only if $(G(R))^c$ satisfies Ku_2 if and only if $(G(R))^c$ is planar if and only if $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

Let R be a ring with at least two maximal ideals. As $(G(R))^c$ is a subgraph of $(\Omega(R))^c$, we obtain from Lemma 8 that if $(\Omega(R))^c$ satisfies Ku_1 or Ku_2 , then $|\text{Max}(R)| \leq 3$. It is proved in Theorem 32 that, for a ring R with exactly two maximal ideals, $(\Omega(R))^c$ satisfies Ku_1 and Ku_2 if and only if $(\Omega(R))^c$ satisfies Ku_1 if and only if $(\Omega(R))^c$ is planar. Also, in statement (3) of Theorem 32, rings R with $|\text{Max}(R)| = 2$ such that $(\Omega(R))^c$ is planar are listed up to isomorphism of rings. In Remark 34, an example of a ring R with $|\text{Max}(R)| = 2$ is provided to illustrate that $(\Omega(R))^c$ which satisfies Ku_2 need not imply that it satisfies Ku_1 . Moreover, in Theorem 35, rings R with $|\text{Max}(R)| = 2$ such that $(\Omega(R))^c$ satisfies Ku_2 are listed up to isomorphism of rings. For a ring R with $|\text{Max}(R)| = 3$, it is proved in Theorem 36 that $(\Omega(R))^c$ satisfies Ku_1 if only if $(\Omega(R))^c$ satisfies Ku_2 if and only if $(\Omega(R))^c$ is planar if and only if $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

In Section 6, quasilocal rings (R, M) which are not fields are considered and necessary and sufficient conditions are determined for $(\Omega(R))^c$ to be planar. It is proved in Theorem 37 that, for a quasilocal ring (R, M) which is not a field, $(\Omega(R))^c$ satisfies Ku_1 and Ku_2 if and only if $(\Omega(R))^c$ satisfies Ku_1 if and only if $|M| \leq 4$, $|R/M| \leq 4$, and $|R| \leq 16$ if and only if $(\Omega(R))^c$ is planar. It is observed in Lemma 38 that for a quasilocal ring (R, M) which is not a field, $(\Omega(R))^c$ satisfies Ku_2 if and only if $|M| \leq 5$, $|R/M| \leq 5$, and $|R| \leq 25$. In Remark 40, up to isomorphism of rings, all local rings which are not fields of order 4, 8, 9, 25 and all local rings (R, M) with $|R| = 16$ and $|M| = 4$ are mentioned with the help of the theorems proved in [9] and from the results proved in [10].

2. Preliminaries

As is mentioned in the introduction, the rings considered in this paper are commutative with identity $1 \neq 0$. In this section we state and prove several lemmas which are needed for proving the main results of this paper.

Lemma 1. *Let R be a ring which admits at least one nonzero nonunit. If the set of nonunits of R is finite, then the ring R is finite.*

Proof. Suppose that R has an infinite number of units. Let “ a ” be any nonzero nonunit of R . Then $\{ua \mid u \in U(R)\} \subseteq NU(R)$ = the set of all nonunits of R . As $NU(R)$ is finite, there exist $1 \leq i_1 < i_2 < i_3 < \dots$ such that $u_{i_1}a = u_{i_j}a$ for $j = 2, 3, \dots$. So $(u_{i_1} - u_{i_j})a = 0$ for $j = 2, 3, \dots$ which implies that $u_{i_1} - u_{i_j}$ are distinct nonunits of R , for $j = 2, 3, \dots$. Hence we obtain that $NU(R)$ is not finite. This is a contradiction. Thus $U(R)$ must be finite. Therefore, R is a finite ring. $\square$

Lemma 2. *Let $n \geq 4$. If the ring R is such that $\omega((G(R))^c) \leq n$, then $|\text{Max}(R)| \leq n - 1$.*

Proof. Suppose that R has more than $n - 1$ maximal ideals. Let $\{M_1, M_2, M_3, \dots, M_n\} \subseteq \text{Max}(R)$. Let $a_i \in (\bigcap_{j \in \{1, 2, \dots, n\} \setminus \{i\}} M_j) \setminus M_i$ for $i = 1, 2, \dots, n$ and $a_{n+1} \in (M_{n-1} \cap M_n) \setminus (\bigcup_{j \in \{1, 2, \dots, n-2\}} M_j)$. We claim that the subgraph of $(G(R))^c$ induced on $\{a_i \mid i \in \{1, 2, \dots, n+1\}\}$ is a clique. Let $i, j \in \{1, 2, \dots, n\}, i \neq j$. Now $a_i \notin M_i$ but $a_j \in M_i$. So a_i and a_j are distinct. Note that $a_i \in M_1$ for $2 \leq i \leq n$ and $a_1 \in M_2$ but $a_{n+1} \notin M_1 \cup M_2$. Hence $a_{n+1} \notin \{a_1, a_2, \dots, a_n\}$. Let $1 \leq i, j \leq n, i \neq j$. Since $n \geq 4, n \geq 3$. Hence there exists $k \in \{1, 2, \dots, n\} \setminus \{i, j\}$. Note that both a_i and a_j are in M_k . It is clear that a_{n+1} and a_i are in M_n for $1 \leq i \leq n-1$. Observe that both a_{n+1} and a_n belong to M_{n-1} . Hence the subgraph of $(G(R))^c$ induced on $\{a_1, a_2, \dots, a_{n+1}\}$ is a clique. This is in contradiction to the assumption that $\omega((G(R))^c) \leq n$. Therefore, $|\text{Max}(R)| \leq n - 1$. $\square$

Lemma 3. Let R be a semiquasilocal ring which is not quasilocal. If there exists $k \in \mathbb{N}$ such that $|M \setminus J(R)| \leq k$ for all $M \in \text{Max}(R)$, then R is finite.

Proof. Let $\{M_1, M_2, \dots, M_n\}$ be the set of all maximal ideals of R . Note that $n \geq 2$. In view of Lemma 1, to prove R is finite, it is enough to prove that $NU(R)$ is finite. Observe that $NU(R) = J(R) \cup (\bigcup_{i \in \{1, 2, \dots, n\}} (M_i \setminus J(R)))$. By hypothesis, $|M_i \setminus J(R)| \leq k$ for $i = 1, 2, \dots, n$. Thus that $NU(R)$ is finite will follow if we show that $J(R)$ is finite. We claim that $|J(R)| \leq k$. Let $m \in M_1 \setminus J(R)$. Note that $\{m + j \mid j \in J(R)\} \subseteq M_1 \setminus J(R)$ and so $|\{m + j \mid j \in J(R)\}| \leq |M_1 \setminus J(R)| \leq k$. Hence $|J(R)| = |\{m + j \mid j \in J(R)\}| \leq k$. This proves that $NU(R)$ is finite and so R is a finite ring. $\square$

Corollary 4. Let $n \geq 4$. Let R be a ring with at least two maximal ideals. If $\omega((G(R))^c) \leq n$, then R is finite.

Proof. By assumption, $\omega((G(R))^c) \leq n$. Hence by Lemma 2, $|\text{Max}(R)| \leq n - 1$. Moreover, since for any maximal ideal M of R , the subgraph of $(G(R))^c$ induced on $M \setminus J(R)$ is a clique, it follows that $|M \setminus J(R)| \leq n$ for any $M \in \text{Max}(R)$. Therefore, we obtain from Lemma 3 that R is finite. $\square$

Lemma 5. Let $\{M_1, M_2\}$ be the set of all maximal ideals of a ring R . Then $(G(R))^c$ has exactly two components and if $|M_i \setminus J(R)| \leq 4$ for $i = 1, 2$, then $(G(R))^c$ is planar.

Proof. Let g_1 (resp., g_2) be the subgraph of $(G(R))^c$ induced on $M_1 \setminus M_2$ (resp., $M_2 \setminus M_1$). Let $V_1 = M_1 \setminus M_2$ and $V_2 = M_2 \setminus M_1$. Observe that $V_1 \neq \emptyset, V_2 \neq \emptyset$ and $V_1 \cap V_2 = \emptyset$. For any $a \in V_1$ and $b \in V_2, Ra + Rb = R$. Hence there is no edge of $(G(R))^c$ whose one end vertex is in V_1 and the other is in V_2 . Therefore, there is no path in $(G(R))^c$ between any $a \in V_1$ and $b \in V_2$. Moreover, it is clear that g_1 and g_2 are complete. This proves that g_1 and g_2 are the only components of $(G(R))^c$. It is well known that any complete graph on at most four vertices is planar. Hence if $|M_i \setminus J(R)| \leq 4$ for $i = 1, 2$, it follows that $(G(R))^c$ is planar. $\square$

Lemma 6. Let (R, M) be a quasilocal ring with $M \neq (0)$. If $n \in \mathbb{N}$ with $n \geq 2$ is such that $|M| \leq n$, then $|R/M| \leq n$. In particular, if $|R| = 8$, then $|M| = 4$.

Proof. Now $|M| \leq n$. We assert that $|R/M| \leq n$. Suppose that $|R/M| \geq n + 1$. Let $\{0 + M, 1 + M, r_1 + M, r_2 + M, \dots, r_{n-1} + M\} \subseteq R/M$. Let $m \in M \setminus (0)$. Note that for any quasilocal ring (R, M) , if $a, b \in R$ with $a - b \notin M$, then for any $m \in M \setminus (0), am \neq bm$. Therefore, it follows that $\{0, m, r_1m, r_2m, \dots, r_{n-1}m\} \subseteq M$. Hence $|M| \geq n + 1$. This is a contradiction to the hypothesis that $|M| \leq n$. Therefore, $|R/M| \leq n$.

Let $|R| = 8$. Now $8 = |R| = |M||R/M|$. Hence $|M| = 2^i$ for some $i, 1 \leq i < 3$. If $i = 1$, then we obtain $|M| = 2$ and so it follows from the previous paragraph that $|R/M| = 2$. Hence $|R| = |M||R/M| = 2 \times 2 = 4$. This is a contradiction. Therefore, $|M| = 4$. $\square$

Remark 7. Let (R, M) be a finite local ring which is not a field. If $|R| = 9$, then $|M| = 3$.

Proof. Now $|R| = 9 = |M||R/M|$. As $|M| \geq 2$ and $|R/M| \geq 2$, it follows that $|M| = 3 = |R/M|$. $\square$

Observation 1. Let $G = (V, E)$ be a graph. If G contains K_6 as a subgraph, then G neither satisfies Ku_1 nor satisfies Ku_2 .

Proof. It is obvious. $\square$

Lemma 8. Let R be a ring with at least four maximal ideals. Then $(G(R))^c$ neither satisfies Ku_1 nor satisfies Ku_2 .

Proof. By hypothesis, R has at least four maximal ideals. Let $\{M_1, M_2, M_3, M_4\} \subseteq \text{Max}(R)$. Consider the elements a, b, c, d, e, f defined by

$$\begin{aligned} a &\in (M_1 \cap M_2) \setminus (M_3 \cup M_4) \\ b &\in (M_1 \cap M_3) \setminus (M_2 \cup M_4) \\ c &\in (M_1 \cap M_4) \setminus (M_2 \cup M_3) \\ d &\in (M_1 \cap M_2 \cap M_3) \setminus M_4 \\ e &\in (M_1 \cap M_2 \cap M_4) \setminus M_3 \\ f &\in (M_1 \cap M_3 \cap M_4) \setminus M_2. \end{aligned}$$

It is clear that the above elements a, b, c, d, e, f are distinct. As they all belong to $M_1 \setminus J(R)$, it follows that the subgraph of $(G(R))^c$ induced on $\{a, b, c, d, e, f\}$ is a clique. Hence $(G(R))^c$ contains K_6 as a subgraph. Therefore, by Observation 1, we obtain that $(G(R))^c$ neither satisfies Ku_1 nor satisfies Ku_2 . $\square$

Lemma 9. Let R be a ring with at least two maximal ideals. If $(G(R))^c$ satisfies Ku_1 or Ku_2 , then R is finite.

Proof. Assume that $(G(R))^c$ satisfies either Ku_1 or Ku_2 . Then it is clear that $\omega((G(R))^c) \leq 5$. Therefore, we obtain from Corollary 4 that R is a finite ring. $\square$

Lemma 10. Let R be a ring with at least two maximal ideals. If $(G(R))^c$ satisfies Ku_1 or Ku_2 , then R is isomorphic to the direct product of finite local rings (R_i, M_i) ($i = 1, 2, \dots, n$) with $2 \leq n \leq 3$.

Proof. We know from Lemma 8 that $|\text{Max}(R)| \leq 3$ and from Lemma 9 that R is finite. Since any finite ring is isomorphic to

the direct product of finite local rings and as $2 \leq |\text{Max}(R)| \leq 3$, it follows that either $R \approx R_1 \times R_2$ or $R \approx R_1 \times R_2 \times R_3$ as rings, where (R_i, M_i) is a finite local ring for each $i \in \{1, 2, 3\}$. $\square$

3. Characterization of Rings R with Exactly Two Maximal Ideals Such That $(G(R))^c$ Is Planar

The aim of this section is to characterize rings R with $|\text{Max}(R)| = 2$ such that $(G(R))^c$ is planar.

Lemma 11. *Let $R = F_1 \times F_2$, where F_1 and F_2 are fields. If $(G(R))^c$ satisfies Ku_1 or Ku_2 , then $|F_i| \leq 5$ for $i = 1, 2$.*

Proof. Now $R = F_1 \times F_2$. Note that R has exactly two maximal ideals M_1 and M_2 , where $M_1 = (0) \times F_2$ and $M_2 = F_1 \times (0)$. Observe that $J(R) = \{(0, 0)\}$. As $(G(R))^c$ satisfies either Ku_1 or Ku_2 , $|M_i \setminus J(R)| \leq 5$ for $i = 1, 2$. Hence $|M_i| \leq 6$ for $i = 1, 2$. Therefore, $|F_i| \leq 6$ for $i = 1, 2$. As there is no field with exactly 6 elements, it follows that $|F_i| \leq 5$ for $i = 1, 2$. $\square$

Lemma 12. *Let $R = F_1 \times F_2$, where F_1 and F_2 are fields. If $|F_i| \leq 5$ for $i = 1, 2$, then $(G(R))^c$ is planar.*

Proof. Note that $M_1 = (0) \times F_2$ and $M_2 = F_1 \times (0)$ are the only maximal ideals of R . Now $|M_1 \setminus M_2| = |(0) \times F_2^*| \leq 4$, and $|M_2 \setminus M_1| = |F_1^* \times (0)| \leq 4$. So we obtain from Lemma 5 that $(G(R))^c$ is planar. $\square$

Lemma 13. *Let $R = F_1 \times R_2$, where F_1 is a field and (R_2, M_2) is a quasilocal ring which is not a field. If $(G(R))^c$ satisfies Ku_1 or Ku_2 , then $|F_1| \leq 3$ and $|R_2| \leq 8$. And in the case $|F_1| = 3$, R_2 must be a local ring with $|R_2| = 4$.*

Proof. Note that $N_1 = (0) \times R_2$ and $N_2 = F_1 \times M_2$ are the only maximal ideals of R . As $(G(R))^c$ satisfies Ku_1 or Ku_2 , $|N_1 \setminus N_2| \leq 5$ and $|N_2 \setminus N_1| \leq 5$. Now $|N_1 \setminus N_2| \leq 5$ implies that $|(0) \times (R_2 \setminus M_2)| \leq 5$. Hence $|R_2 \setminus M_2| \leq 5$. Therefore, $|M_2| \leq 5$ and so $|R_2| = |M_2| + |R_2 \setminus M_2| \leq 10$. Now $|N_2 \setminus N_1| \leq 5$ implies that $|F_1^* \times M_2| \leq 5$. As $|M_2| \geq 2$, we obtain that $2|F_1^*| \leq |F_1^*| |M_2| \leq 5$. Therefore, $|F_1^*| \leq 2$ and this implies that $|F_1| \leq 3$. Now (R_2, M_2) is a local ring which is not a field and $|R_2| \leq 10$. And as the order of any finite local ring is a power of a prime, it follows that $|R_2| \in \{4, 8, 9\}$. Suppose that $|R_2| = 9$ and $|F_1| \in \{2, 3\}$. Then $|R_2 \setminus M_2| = 6$ by Remark 7. Hence $|N_1 \setminus N_2| = |(0) \times (R_2 \setminus M_2)| = 6$. Since the subgraph of $(G(R))^c$ induced on $N_1 \setminus N_2$ is K_6 , it follows that $(G(R))^c$ neither satisfies Ku_1 nor satisfies Ku_2 . Hence $|R_2| = 9$ is not possible. Suppose that $|R_2| = 8$ and $|F_1| = 3$. Then $|M_2| = 4$ by Lemma 6. Note that $|N_2 \setminus N_1| = |F_1^* \times M_2| = |F_1^*| |M_2| = 2 \times 4 = 8$. In this case the subgraph of $(G(R))^c$ induced on $N_2 \setminus N_1$ is K_8 . This is impossible as $(G(R))^c$ satisfies Ku_1 or Ku_2 . This proves that if $(G(R))^c$ satisfies Ku_1 or Ku_2 , then either $|F_1| = 2$ and $|R_2| \in \{4, 8\}$ or $|F_1| = 3$ and $|R_2| = 4$. $\square$

Lemma 14. *Let $R = F_1 \times R_2$, where F_1 is a field with $|F_1| = 2$ and (R_2, M_2) is a quasilocal ring which is not a field with $|R_2| = 4$. Then $(G(R))^c$ is planar.*

Proof. Note that $N_1 = (0) \times R_2$ and $N_2 = F_1 \times M_2$ are the only maximal ideals of R . Now $|N_1 \setminus N_2| = |(0) \times (R_2 \setminus M_2)| = |R_2 \setminus M_2| = 2$ and $|N_2 \setminus N_1| = |F_1^* \times M_2| = |F_1^*| |M_2| = |M_2| = 2$. Hence by Lemma 5, we obtain that $(G(R))^c$ is planar. $\square$

Lemma 15. *Let $R = F_1 \times R_2$, where F_1 is a field with $|F_1| = 2$ and (R_2, M_2) is a quasilocal ring which is not a field with $|R_2| = 8$. Then $(G(R))^c$ is planar.*

Proof. Note that $\text{Max}(R) = \{N_1 = (0) \times R_2, N_2 = F_1 \times M_2\}$. Now $|N_1 \setminus N_2| = |(0) \times (R_2 \setminus M_2)| = |R_2 \setminus M_2| = 4$ and $|N_2 \setminus N_1| = |F_1^* \times M_2| = |F_1^*| |M_2| = |M_2| = 4$. So by Lemma 5, it follows that $(G(R))^c$ is planar. $\square$

Lemma 16. *Let $R = F_1 \times R_2$, where F_1 is a field with $|F_1| = 3$ and (R_2, M_2) is a quasilocal ring which is not a field with $|R_2| = 4$. Then $(G(R))^c$ is planar.*

Proof. Note that $\text{Max}(R) = \{N_1 = (0) \times R_2, N_2 = F_1 \times M_2\}$. Now $|N_1 \setminus N_2| = |(0) \times (R_2 \setminus M_2)| = |R_2 \setminus M_2| = 2$ and $|N_2 \setminus N_1| = |F_1^* \times M_2| = |F_1^*| |M_2| = 2 \times 2 = 4$. So by Lemma 5, we obtain that $(G(R))^c$ is planar. $\square$

Lemma 17. *Let $R = R_1 \times R_2$, where (R_i, M_i) is a quasilocal ring which is not a field for $i = 1, 2$. If $(G(R))^c$ satisfies Ku_1 or Ku_2 , then $|R_1| = |R_2| = 4$.*

Proof. Note that $N_1 = R_1 \times M_2$ and $N_2 = M_1 \times R_2$ are the only maximal ideals of R . As $(G(R))^c$ satisfies Ku_1 or Ku_2 , $|N_i \setminus J(R)| \leq 5$ for $i = 1, 2$. Now, $|N_1 \setminus N_2| = |(R_1 \setminus M_1) \times M_2| \leq 5$ and as $|R_1 \setminus M_1| \geq 2$ and $|M_2| \geq 2$, we obtain that $4 \leq |R_1 \setminus M_1| |M_2| \leq 5$. This implies that $|R_1 \setminus M_1| = 2 = |M_2|$. Hence $|M_1| = 2$ and so $|R_1| = 4$. Similarly from $|N_2 \setminus N_1| \leq 5$, it follows that $|R_2| = 4$. $\square$

Lemma 18. *Let $R = R_1 \times R_2$, where (R_i, M_i) is a quasilocal ring which is not a field with $|R_i| = 4$ for $i = 1, 2$. Then $(G(R))^c$ is planar.*

Proof. Note that $\text{Max}(R) = \{N_1 = R_1 \times M_2, N_2 = M_1 \times R_2\}$. Now $|N_1 \setminus N_2| = |(R_1 \setminus M_1) \times M_2| = 2 \times 2 = 4$ and $|N_2 \setminus N_1| = |M_1 \times (R_2 \setminus M_2)| = 2 \times 2 = 4$. Hence it follows from Lemma 5 that $(G(R))^c$ is planar. $\square$

The following theorem determines necessary and sufficient conditions for $(G(R))^c$ to be planar, where R is a ring with $|\text{Max}(R)| = 2$.

Theorem 19. *Let R be a ring with exactly two maximal ideals. Then the following statements are equivalent.*

- (1) $(G(R))^c$ satisfies Ku_1 .
- (2) R is isomorphic to one of the following rings.
 - (a) $F_1 \times F_2$, where F_1 and F_2 are fields with $|F_i| \leq 5$ for $i = 1, 2$.
 - (b) $F_1 \times R_2$, where F_1 is a field with $|F_1| \leq 3$ and (R_2, M_2) is a local ring which is not a field and

$|R_2| \leq 8$ and if $|F_1| = 3$, then R_2 must be a local ring with $|R_2| = 4$.

(c) $R_1 \times R_2$, where both R_1 and R_2 are local rings which are not fields with $|R_i| = 4$ for $i = 1, 2$.

(3) $(G(R))^c$ satisfies Ku_2 .

(4) $(G(R))^c$ is planar.

Proof. (1) $\Rightarrow$ (2) (resp., (3) $\Rightarrow$ (2))

Now R has exactly two maximal ideals and $(G(R))^c$ satisfies Ku_1 (respectively, Ku_2). Hence from Lemma 10 we obtain that $R \approx R_1 \times R_2$ as rings, where (R_i, M_i) is a quasilocal ring for $i = 1, 2$. If both R_1 and R_2 are fields, then Lemma 11 implies that R is isomorphic to the ring given in (a). If R_1 is a field and R_2 is not a field, then from Lemma 13, we obtain that R is isomorphic to the ring described in (b). If both R_1 and R_2 are not fields, then it follows from Lemma 17 that R is isomorphic to the ring given in (c).

(2) $\Rightarrow$ (4). This follows immediately from Lemmas 12, 14, 15, 16, and 18.

(4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3). Indeed it follows from Kuratowski's theorem [8, Theorem 5.9] that $(G(R))^c$ satisfies both Ku_1^* and Ku_2^* . $\square$

4. Characterization of Rings R with Exactly Three Maximal Ideals Such That $(G(R))^c$ Is Planar

The main result of this section is Theorem 24 which characterizes rings R with $|\text{Max}(R)| = 3$ such that $(G(R))^c$ is planar.

Lemma 20. Let $R = R_1 \times R_2 \times R_3$, where (R_i, M_i) is a quasilocal ring for $i = 1, 2, 3$. If at least one R_i is not a field, then $(G(R))^c$ neither satisfies Ku_1 nor satisfies Ku_2 .

Proof. Without loss of generality we may assume that (R_1, M_1) is not a field. Let $a \in M_1 \setminus \{0\}$. Then the subgraph of $(G(R))^c$ induced on $\{(0, 1, 1), (a, 0, 1), (a, 1, 0), (0, 1, 0), (a, 1, 1), (0, 0, 1)\}$ is a clique. Thus $(G(R))^c$ contains K_6 as a subgraph. Therefore, $(G(R))^c$ neither satisfies Ku_1 nor satisfies Ku_2 . $\square$

Lemma 21. Let $R = F_1 \times F_2 \times F_3$, where F_i is a field for $i = 1, 2, 3$. If $(G(R))^c$ satisfies Ku_1 , then $F_i = \mathbb{Z}_2$ for $i = 1, 2, 3$.

Proof. Note that $\text{Max}(R) = \{M_1 = F_1 \times F_2 \times (0), M_2 = (0) \times F_2 \times F_3, M_3 = F_1 \times (0) \times F_3\}$ and $J(R) = \{(0, 0, 0)\}$. As $(G(R))^c$ satisfies Ku_1 , $|M_1 \setminus J(R)| \leq 4$. This implies that $|F_1 \times F_2 \setminus \{0\}| - 1 \leq 4$. So $|F_1||F_2| \leq 5$. It follows that $2 \times 2 = 4 \leq |F_1||F_2| \leq 5$. Hence $|F_i| = 2$ for $i = 1, 2$. Similarly we obtain that $|F_3| = 2$. Therefore, $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. $\square$

Lemma 22. Let $R = F_1 \times F_2 \times F_3$, where F_i is a field for $i = 1, 2, 3$. If $(G(R))^c$ satisfies Ku_2 , then $F_i = \mathbb{Z}_2$ for $i = 1, 2, 3$.

Proof. Assume that $(G(R))^c$ satisfies Ku_2 . We assert that $|F_i| = 2$ for each $i \in \{1, 2, 3\}$. Suppose that $|F_i| \geq 3$ for some $i \in \{1, 2, 3\}$. Without loss of generality we may assume that $|F_1| \geq 3$. Let $\alpha \in F_1 \setminus \{0, 1\}$. Note that $\text{Max}(R) =$

$\{M_1 = F_1 \times F_2 \times (0), M_2 = (0) \times F_2 \times F_3, M_3 = F_1 \times (0) \times F_3\}$. Let $V_1 = \{(0, 0, 1), (\alpha, 0, 0), (1, 0, 0)\}$ and $V_2 = \{(\alpha, 0, 1), (0, 1, 0), (1, 0, 1)\}$. Note that $V_1 \cap V_2 = \emptyset$ and each vertex of V_1 is adjacent to every vertex of V_2 in $(G(R))^c$. Hence the subgraph of $(G(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $(G(R))^c$ satisfies Ku_2 . Therefore, $|F_i| = 2$ for $i = 1, 2, 3$. $\square$

Lemma 23. If $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $(G(R))^c$ is planar.

Proof. Note that $(G(R))^c$ is the union of cycles:

$$C_1: (0, 1, 1) - (0, 0, 1) - (1, 0, 1) - (1, 0, 0) - (1, 1, 0) - (0, 1, 0) - (0, 1, 1),$$

$$C_2: (0, 0, 1) - (1, 0, 0) - (0, 1, 0) - (0, 0, 1).$$

Observe that C_1 is a cycle of length 6 and it can be represented by means of a hexagon and C_2 is a cycle of length 3. Note that each edge of C_2 is a diagonal of the hexagon representing C_1 and hence it can be represented by means of a triangle which can be drawn inside the hexagon without any crossing over of the edges. This shows that $(G(R))^c$ is planar. $\square$

Theorem 24. Let R be a ring with exactly three maximal ideals. Then the following statements are equivalent.

(1) $(G(R))^c$ satisfies Ku_1 .

(2) $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

(3) $(G(R))^c$ satisfies Ku_2 .

(4) $(G(R))^c$ is planar.

Proof. (1) $\Rightarrow$ (2) (resp., (3) $\Rightarrow$ (2)).

Now by hypothesis R has exactly three maximal ideals and $(G(R))^c$ satisfies Ku_1 (respectively, Ku_2). Hence we obtain from Lemma 10 that $R \approx R_1 \times R_2 \times R_3$ as rings, where (R_i, M_i) is a quasilocal ring for $i = 1, 2, 3$. Now it follows from Lemmas 20 and 21 (respectively, Lemmas 20 and 22) that $R_i = \mathbb{Z}_2$ for $i = 1, 2, 3$. Therefore, $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

(2) $\Rightarrow$ (4)

This follows from Lemma 23.

(4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3).

It follows from Kuratowski's theorem [8, Theorem 5.9] that $(G(R))^c$ satisfies both Ku_1^* and Ku_2^* . $\square$

5. Characterization of Rings R with at Least Two Maximal Ideals Such That $(\Omega(R))^c$ Is Planar

The aim of this section is to characterize rings R with at least two maximal ideals such that $(\Omega(R))^c$ is planar. If $(\Omega(R))^c$ is planar, then $(G(R))^c$ being a subgraph of $(\Omega(R))^c$ must also be planar. Hence we obtain from Lemma 10 that R is isomorphic to the direct product of either two or three finite local rings.

Lemma 25. Let $R = F_1 \times F_2$, where F_1 and F_2 are fields. If at least one F_i contains five elements for some $i \in \{1, 2\}$, then $(\Omega(R))^c$ does not satisfy Ku_1 .

Proof. Note that $M_1 = F_1 \times (0)$ and $M_2 = (0) \times F_2$ are the only maximal ideals of R . Without loss of generality we may assume that $|F_1| \geq 5$. Now $|M_1| = |F_1 \times (0)| = |F_1| \geq 5$. It is clear that the subgraph of $(\Omega(R))^c$ induced on M_1 is a clique. So it follows that $(\Omega(R))^c$ contains K_5 as a subgraph. Therefore, $(\Omega(R))^c$ does not satisfy Ku_1 . $\square$

Lemma 26. *Let $\{M_1, M_2\}$ be the set of all maximal ideals of a ring R with $|M_i| \leq 4$ for $i = 1, 2$. If $M_1 \cap M_2 = (0)$, then $(\Omega(R))^c$ is planar.*

Proof. Note that the vertex set of $(\Omega(R))^c = (M_1 \setminus (0)) \cup (M_2 \setminus (0)) \cup (0) \cup U(R)$. Let $V_i = M_i \setminus (0)$ for $i = 1, 2$, $V_3 = (0)$, $V_4 = U(R)$. It is clear that $V_i \cap V_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3, 4\}$. Observe that the subgraph of $(\Omega(R))^c$ induced on V_i is a complete graph on at most three vertices for $i = 1, 2$ and each vertex of V_4 is an isolated vertex. The vertex 0 is adjacent to every vertex of $V_1 \cup V_2$ and $|V_1 \cup V_2| \leq 6$. Now it is clear that $(\Omega(R))^c$ can be drawn in a plane in such a way that there is no crossing over of the edges. This proves that $(\Omega(R))^c$ is planar. $\square$

Corollary 27. *Let $R = F_1 \times F_2$, where F_1 and F_2 are fields with $|F_i| \leq 4$ for $i = 1, 2$. Then $(\Omega(R))^c$ is planar.*

Proof. Observe that $M_1 = F_1 \times (0)$ and $M_2 = (0) \times F_2$ are the only maximal ideals of R with $|M_i| = |F_i| \leq 4$ for $i = 1, 2$ and $M_1 \cap M_2 = \{(0, 0)\}$. Hence it follows immediately from Lemma 26 that $(\Omega(R))^c$ is planar. $\square$

Lemma 28. *Let $R = R_1 \times R_2$, where (R_i, M_i) is a quasilocal ring for $i = 1, 2$ and (R_2, M_2) is not a field. If $(\Omega(R))^c$ satisfies Ku_1 or Ku_2 , then $|R_1| = 2$ and $|R_2| = 4$.*

Proof. Note that $\{N_1 = R_1 \times M_2, N_2 = M_1 \times R_2\}$ is the set of all maximal ideals of R . If $(\Omega(R))^c$ satisfies Ku_1 or Ku_2 , then $|N_i| \leq 5$ for $i = 1, 2$. Since R_2 is not a field, $|M_2| \geq 2$. Thus if $|R_1| \geq 3$, then $|N_1| = |R_1||M_2| \geq 6$. This is impossible. Therefore, $|R_1| = 2$. Thus $M_1 = (0)$ and so $|M_1| = 1$. We next verify that $|R_2| = 4$. Since R_2 is not a field, it follows that $|R_2| \geq 4$. As $|N_2| \leq 5$, we obtain that $|M_1 \times R_2| = |R_2| \leq 5$. Therefore, $|R_2| = 4$. $\square$

Lemma 29. *Let $R = F_1 \times R_2$, where F_1 is a field with $|F_1| = 2$ and (R_2, M_2) is a quasilocal ring which is not a field with $|R_2| = 4$. Then $(\Omega(R))^c$ is planar.*

Proof. Now $F_1 = \{0, 1\}$. As $|R_2| = 4$, it follows that $|M_2| = 2$. Let $M_2 = \{0, a\}$. Note that $R_2 = \{0, 1, a, 1+a\}$. Observe that the vertex set of $(\Omega(R))^c = R = V_1 \cup V_2 \cup V_3$, where $V_1 = \{(1, 0), (1, a), (0, a), (0, 0)\}$, $V_2 = \{(0, 1), (0, 1+a), (0, a), (0, 0)\}$, and $V_3 = \{(1, 1), (1, 1+a)\}$. Note that the subgraph g_i of $(\Omega(R))^c$ induced on V_i is complete for $i = 1, 2$. Each vertex of V_3 is an isolated vertex. Thus $(\Omega(R))^c$ is the union of g_1, g_2 and two isolated vertices. Since K_4 is planar and as the edge $(0, 0) - (0, a)$ is the only edge which is common to both g_1 and g_2 , $(\Omega(R))^c$ can be drawn in a plane in such a way that there is no crossing over of the edges. This proves that $(\Omega(R))^c$ is planar. $\square$

Lemma 30. *Let $R = R_1 \times R_2 \times R_3$, where (R_i, M_i) is a quasilocal ring for $i = 1, 2, 3$. If $(\Omega(R))^c$ satisfies Ku_1 or Ku_2 , then $|R_i| = 2$ for $i = 1, 2, 3$.*

Proof. Note that R is a ring with exactly three maximal ideals. If $(\Omega(R))^c$ satisfies Ku_1 (resp., Ku_2), then $(G(R))^c$ satisfies Ku_1 (resp., Ku_2). Hence it follows from (1) $\Rightarrow$ (2) (resp., (3) $\Rightarrow$ (2)) of Theorem 24 that $|R_i| = 2$ for $i = 1, 2, 3$. $\square$

Lemma 31. *If $R \approx \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, then $(\Omega(R))^c$ is planar.*

Proof. Observe that $V((\Omega(R))^c) = V((G(R))^c) \cup \{(0, 0, 0), (1, 1, 1)\}$. It was already noted in the proof of Lemma 23 that $(G(R))^c$ is the union of the cycles $C_1: (0, 1, 1) - (0, 0, 1) - (1, 0, 1) - (1, 0, 0) - (1, 1, 0) - (0, 1, 0) - (0, 1, 1)$ and $C_2: (0, 0, 1) - (1, 0, 0) - (0, 1, 0) - (0, 0, 1)$. Note that C_1 is a cycle of length 6 and it can be represented by means of a hexagon. The vertex $(0, 0, 0)$ is adjacent to all the six vertices of C_1 in $(\Omega(R))^c$. The vertex $(0, 0, 0)$ can be plotted inside the hexagon representing C_1 and it can be joined to all the vertices of C_1 without any crossing over of the edges. The edges of C_2 , each of which is a diagonal of the hexagon representing C_1 , can be drawn outside the hexagon in such a way that there is no crossing over of the edges. The vertex $(1, 1, 1)$ is an isolated vertex of $(\Omega(R))^c$. This proves that $(\Omega(R))^c$ is planar. $\square$

Theorem 32. *Let R be a ring with exactly two maximal ideals. Then the following statements are equivalent.*

- (1) $(\Omega(R))^c$ satisfies Ku_1 and Ku_2 .
- (2) $(\Omega(R))^c$ satisfies Ku_1 .
- (3) R is isomorphic to one of the following rings:
 - (a) $F_1 \times F_2$, where F_1 and F_2 are fields with $|F_i| \leq 4$ for $i = 1, 2$;
 - (b) $F_1 \times R_2$, where F_1 is a field with $|F_1| = 2$ and (R_2, M_2) is a local ring which is not a field and $|R_2| = 4$.
- (4) $(\Omega(R))^c$ is planar.

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3)

Now by hypothesis R has exactly two maximal ideals and $(\Omega(R))^c$ satisfies Ku_1 . As $(G(R))^c$ is a subgraph of $(\Omega(R))^c$, it follows that $(G(R))^c$ satisfies Ku_1 . Hence from Lemma 10 it follows that $R \approx R_1 \times R_2$ as rings, where (R_i, M_i) is a local ring for $i = 1, 2$. Now if both R_1 and R_2 are fields, then it follows from Lemma 25 that R is isomorphic to the ring given in (a). If either R_1 or R_2 is not a field, then it follows from Lemma 28 that R is isomorphic to the ring given in (b).

(3) $\Rightarrow$ (4).

This follows from Corollary 27 and Lemma 29:

(4) $\Rightarrow$ (1).

It follows from Kuratowski's theorem [8, Theorem 5.9] that $(\Omega(R))^c$ satisfies both Ku_1^* and Ku_2^* . $\square$

Lemma 33. *Let $R = F_1 \times F_2$, where F_1 and F_2 are fields. Then the following statements are equivalent.*

- (1) $(\Omega(R))^c$ satisfies Ku_2 .
- (2) $|F_i| \leq 5$ for $i = 1, 2$.

Proof. (1) $\Rightarrow$ (2)

As $(G(R))^c$ is a subgraph of $(\Omega(R))^c$, we obtain that $(G(R))^c$ satisfies Ku_2 . So, by Lemma 11, it follows that $|F_i| \leq 5$ for $i = 1, 2$.

(2) $\Rightarrow$ (1).

Now $R = F_1 \times F_2$, where $|F_i| \leq 5$ for $i = 1, 2$. Note that $\{M_1 = (0) \times F_2, M_2 = F_1 \times (0)\}$ is the set of all maximal ideals of R with $|M_i| \leq 5$ for $i = 1, 2$. Let $V_1 = M_1 \setminus \{(0, 0)\}$, $V_2 = M_2 \setminus \{(0, 0)\}$, $V_3 = \{(0, 0)\}$, and $V_4 = F_1 \setminus (0) \times F_2 \setminus (0)$. Let g_i ($i = 1, 2$) be the subgraph of $(\Omega(R))^c$ induced on V_i . Observe that g_i is a complete graph on at most four vertices for $i = 1, 2$ and each vertex of V_4 is an isolated vertex, and $(0, 0)$ is adjacent to each vertex of $V_1 \cup V_2$. Note that the vertex set of $(\Omega(R))^c = V_1 \cup V_2 \cup V_3 \cup V_4$ and $V_i \cap V_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3, 4\}$. It is clear from the above description of $(\Omega(R))^c$ that it satisfies Ku_2 . $\square$

Remark 34. Let R be a ring with exactly two maximal ideals. If $(\Omega(R))^c$ satisfies Ku_1 , then we obtain from (2) $\Rightarrow$ (1) of Theorem 32 that $(\Omega(R))^c$ satisfies Ku_2 . We mention below an example of a ring R with exactly two maximal ideals such that $(\Omega(R))^c$ satisfies Ku_2 but it does not satisfy Ku_1 .

Let $R = F_1 \times F_2$, where F_1 is a field with $|F_1| = 5$ and F_2 is a field with $|F_2| \leq 5$. Note that $\{M_1 = (0) \times F_2, M_2 = F_1 \times (0)\}$ is the set of all maximal ideals of R with $|M_1| \leq 5$ and $|M_2| = 5$. As the subgraph of $(\Omega(R))^c$ induced on M_2 is a clique on five vertices, it follows that $(\Omega(R))^c$ does not satisfy Ku_1 but by Lemma 33, we obtain that $(\Omega(R))^c$ satisfies Ku_2 .

Theorem 35. Let R be a ring with exactly two maximal ideals. Then the following statements are equivalent.

- (1) $(\Omega(R))^c$ satisfies Ku_2 .
- (2) R is isomorphic to one of the following rings.
 - (a) $F_1 \times F_2$, where F_1 and F_2 are fields with $|F_i| \leq 5$ for $i = 1, 2$.
 - (b) $F_1 \times R_2$, where F_1 is a field with $|F_1| = 2$ and (R_2, M_2) is a local ring which is not a field and $|R_2| = 4$.

Proof. (1) $\Rightarrow$ (2)

As $(G(R))^c$ is a subgraph of $(\Omega(R))^c$, it follows that $(G(R))^c$ satisfies Ku_2 . Hence from Lemma 10 we obtain that $R \approx R_1 \times R_2$ as rings, where (R_i, M_i) is a quasilocal ring for $i = 1, 2$. If both R_1 and R_2 are fields, then Lemma 33 implies that R is isomorphic to the ring given in (a). If R_2 is not a field, then from Lemma 28, we obtain that R is isomorphic to the ring described in (b).

(2) $\Rightarrow$ (1)

(2) $\Rightarrow$ (1) follows immediately from Theorem 32 and Lemma 33. $\square$

Theorem 36. Let R be a ring with exactly three maximal ideals. Then the following statements are equivalent.

- (1) $(\Omega(R))^c$ satisfies Ku_1 .

(2) $R \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

(3) $(\Omega(R))^c$ satisfies Ku_2 .

(4) $(\Omega(R))^c$ is planar.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2).

As $(G(R))^c$ is a subgraph of $(\Omega(R))^c$, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) follow from Theorem 24.

(2) $\Rightarrow$ (4)

(2) $\Rightarrow$ (4) follows immediately from Lemma 31.

(4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3).

Indeed it follows from Kuratowski's theorem [8, Theorem 5.9] that $(\Omega(R))^c$ satisfies both Ku_1^* and Ku_2^* . $\square$

6. Characterization of Quasilocal Rings R Which Are Not Fields Such That $(\Omega(R))^c$ Is Planar

Let (R, M) be a quasilocal ring. If $M = (0)$, then R is a field. In such a case, each vertex of $(\Omega(R))^c$ is an isolated vertex and hence it is clear that $(\Omega(R))^c$ is planar. So, we assume that $M \neq (0)$. The main result of this section is Theorem 37 which characterizes quasilocal rings (R, M) which are not fields such that $(\Omega(R))^c$ is planar.

Theorem 37. Let (R, M) be a quasilocal ring which is not a field. Then the following statements are equivalent.

- (1) $(\Omega(R))^c$ satisfies Ku_1 and Ku_2 .
- (2) $(\Omega(R))^c$ satisfies Ku_1 .
- (3) $|M| \leq 4$, $|R/M| \leq 4$, and $|R| \leq 16$.
- (4) $(\Omega(R))^c$ is planar.

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3).

As $(\Omega(R))^c$ satisfies Ku_1 , it follows that $|M| \leq 4$. Now $|R/M| \leq 4$ follows from Lemma 6. Hence $|R| = |M||R/M| \leq 4 \times 4 = 16$.

(3) $\Rightarrow$ (4).

Now, $|M| \leq 4$, $|R/M| \leq 4$, and $|R| \leq 16$. Note that $(\Omega(R))^c$ is a graph with vertex set $R = U(R) \cup NU(R)$. Each unit of R is an isolated vertex of $(\Omega(R))^c$ and as $|NU(R)| \leq 4$, it is clear that $(\Omega(R))^c$ is planar:

(4) $\Rightarrow$ (1)

It follows from Kuratowski's theorem [8, Theorem 5.9] that $(\Omega(R))^c$ satisfies both Ku_1^* and Ku_2^* . $\square$

Lemma 38. Let (R, M) be a quasilocal ring which is not a field. Then the following statements are equivalent.

- (1) $(\Omega(R))^c$ satisfies Ku_2 .
- (2) $|M| \leq 5$, $|R/M| \leq 5$, and $|R| \leq 25$.

Proof. (1) $\Rightarrow$ (2)

As $(\Omega(R))^c$ satisfies Ku_2 , it follows that $|M| \leq 5$. Now $|R/M| \leq 5$ follows from Lemma 6. Hence $|R| = |M||R/M| \leq 5 \times 5 = 25$.

(2) $\Rightarrow$ (1)

Now $R = U(R) \cup NU(R)$ and $|NU(R)| = |M| \leq 5$. As each unit of R is an isolated vertex of $(\Omega(R))^c$ and any graph on at most five vertices satisfies Ku_2 , it follows that $(\Omega(R))^c$ satisfies Ku_2 . $\square$

Example 39. Let $R = \mathbb{Z}/25\mathbb{Z}$. Note that R is a local ring with unique maximal ideal $M = 5\mathbb{Z}/25\mathbb{Z} = \{0 + 25\mathbb{Z}, 5 + 25\mathbb{Z}, 10 + 25\mathbb{Z}, 15 + 25\mathbb{Z}, 20 + 25\mathbb{Z}\}$. As $|M| = 5$, it is clear that $(\Omega(R))^c$ does not satisfy Ku_1 . Moreover, from Lemma 38, it follows that $(\Omega(R))^c$ satisfies Ku_2 .

Remark 40. With the help of theorems proved by Corbas and Williams in [9], Belshoff and Chapman in [10] have listed up to isomorphism, the local rings of order p^2, p^3 and all the local rings of order 16. From the above list, we mention below up to isomorphism of rings, all local rings which are not fields of order 4, 8, 9, 25 and all local rings (R, M) with $|R| = 16$ and $|M| = 4$.

- (1) Local rings of order 4 which are not fields are $\mathbb{Z}_4$ and $\mathbb{Z}_2[X]/(X^2)$.
- (2) Local rings of order 8 which are not fields are $\mathbb{F}_2[X]/(X^3)$, $\mathbb{F}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_4[X]/(2X, X^2)$, and $\mathbb{Z}_4[X]/(2X, X^2 - 2)$.
- (3) Local rings of order 9 which are not fields are $\mathbb{Z}_9$ and $\mathbb{Z}_3[X]/(X^2)$.
- (4) Local rings (R, M) with $|R| = 16$ and $|M| = 4$ are $\mathbb{F}_4[X]/(X^2)$ and $\mathbb{Z}_4[X]/(1 + X + X^2)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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