Research Article

On Generalized ϕ -Recurrent (ϵ, δ) -Trans-Sasakian Manifolds

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We study generalized ϕ -recurrent (ϵ, δ) -trans-Sasakian manifolds. A relation between the associated 1-forms A and B and relation between characteristic vector field ξ and the vector fields ρ_1, ρ_2 for a generalized ϕ -recurrent.

1. Introduction

As a generalization of both Sasakian and Kenmotsu manifolds, Oubiña [1] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kahler manifolds. Trans-Sasakian manifolds of types $(0, 0)$, $(\alpha, 0)$, and $(0, \beta)$ are, respectively, called the cosymplectic, α -Sasakian, and β -Kenmotsu manifolds, with α, β being scalar functions. In particular, if $\alpha = 0, \beta = 1; \alpha = 1, \beta = 0$; then a trans-Sasakian manifold will be Kenmotsu and Sasakian manifolds respectively. The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [2]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or β -Kenmotsu or α -Sasakian manifold.

The investigation of manifolds with indefinite metrics was taken by several authors. In 1993, Bejancu and Duggal [3] introduced the concept of (ϵ) -Sasakian manifolds and Xufeng and Xiaoli [4] established that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. Kumar et al. [5] studied the curvature conditions of these manifolds. Tripathi et al. [6] introduced and studied (ϵ) -almost paracontact manifolds.

The paper is organised as follows: Section 2 contains preliminaries of (ϵ, δ) -trans-Sasakian manifolds. In Sections 3 and 4 we obtain results for generalized ϕ -recurrent and generalized concircular ϕ -recurrent (ϵ, δ) -Trans-Sasakian Manifolds.

2. Preliminaries

Let M be the connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) ; that is, ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form, and g is compatible Riemannian metric, such that

$$\phi^2 X = -X + \eta(X)\xi, \quad (1)$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0. \quad (2)$$

Almost contact metric manifold M is called (ϵ) -almost contact metric manifold if

$$g(\xi, \xi) = \epsilon, \quad \eta(X) = \epsilon g(X, \xi), \quad (3)$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y),$$

for all vector fields X, Y on M , where $\epsilon = g(\xi, \xi) = \pm 1$.

An (ϵ) -almost contact metric manifold is called a (ϵ, δ) -trans-Sasakian manifold if

$$(\nabla_X \phi)(Y) = \alpha \{g(X, Y)\xi - \epsilon \eta(Y)X\} + \beta \{g(\phi X, Y)\xi - \delta \eta(Y)\phi X\} \quad (4)$$

holds for some smooth functions α and β on M and $\epsilon = \pm 1, \delta = \pm 1$. For $\beta = 0, \alpha = 1$, (ϵ, δ) -trans-Sasakian manifold reduces to (ϵ) -Sasakian and for $\alpha = 0, \beta = 1$

it reduces to (ϵ) -Kenmotsu manifold. (ϵ, δ) -Trans-Sasakian manifolds have been studied by authors [7] and they obtained the following results:

$$\nabla_X \xi = -\epsilon \alpha \phi X - \delta \beta \{X - \eta(X) \xi\}, \quad (5)$$

$$(\nabla_X \eta)(Y) = \delta \beta [\epsilon g(X, Y) - \eta(X) \eta(Y)] - \alpha g(\phi X, Y), \quad (6)$$

$$\begin{aligned} R(X, Y) \xi &= (\alpha^2 - \beta^2) [\eta(Y) X - \eta(X) Y] \\ &\quad - \epsilon [(X\alpha) \phi Y - (Y\alpha) \phi X] \\ &\quad - \delta [(X\beta) \phi^2(Y) - (Y\beta) \phi^2(X)] \\ &\quad + 2\alpha\beta(\delta - \epsilon) g(\phi X, Y) \xi \\ &\quad + 2\epsilon\delta\alpha\beta [\eta(Y) \phi X - \eta(X) \phi Y], \end{aligned} \quad (7)$$

$$\begin{aligned} S(X, \xi) &= -(\phi X) \alpha \\ &\quad + [2n(\epsilon\alpha^2 - \beta^2\delta) - (\xi\beta)] \eta(X) - (2n-1) X\beta, \end{aligned} \quad (8)$$

$$(\xi\alpha) + 2\delta\alpha\beta = 0, \quad (9)$$

$$S(\xi, \xi) = 2n(\epsilon\alpha^2 - \beta^2\delta - \xi\beta), \quad (10)$$

$$\phi \operatorname{grad} \alpha = \operatorname{grad} \beta, \quad (11)$$

$$\begin{aligned} \eta(R(X, Y) Z) &= 2\alpha\beta(\delta - \epsilon) g(\phi X, Y) \eta(Z) \\ &\quad - (X\alpha) g(Y, \phi Z) \\ &\quad + (Y\alpha) g(X, \phi Z) - \alpha\beta(\delta + \epsilon) \\ &\quad \times [g(Y, \phi Z) \eta(X) - g(X, \phi Z) \eta(Y)] \\ &\quad - (\epsilon\alpha^2 - \delta\beta^2) \\ &\quad \times [g(X, Z) \eta(Y) - g(Y, Z) \eta(X)] \\ &\quad - (X\beta) [g(Y, Z) - \epsilon\eta(Y) \eta(Z)] \\ &\quad + (Y\beta) [g(X, Z) - \epsilon\eta(X) \eta(Z)]. \end{aligned} \quad (12)$$

3. Generalized ϕ -Recurrent (ϵ, δ) -Trans-Sasakian Manifolds

Definition 1. A (ϵ, δ) -trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called generalized ϕ -recurrent if its curvature tensor R satisfies the condition

$$\begin{aligned} \phi^2((\nabla_W R)(X, Y) Z) \\ = A(W) R(X, Y) Z \\ + B(W) [g(Y, Z) X - g(X, Z) Y], \end{aligned} \quad (13)$$

where A and B are two 1-forms, B is nonzero, and these are defined by

$$A(W) = g(W, \rho_1), \quad B(W) = g(W, \rho_2), \quad (14)$$

and ρ_1, ρ_2 are vector fields associated with 1-forms A and B , respectively.

Let us consider a generalized ϕ -recurrent (ϵ, δ) -trans-Sasakian manifold. Then by virtue of (1) and (13) we have

$$\begin{aligned} -(\nabla_W R)(X, Y) Z + \eta((\nabla_W R)(X, Y) Z) \xi \\ = A(W) R(X, Y) Z + B(W) [g(Y, Z) X - g(X, Z) Y], \end{aligned} \quad (15)$$

from which it follows that

$$\begin{aligned} -g((\nabla_W R)(X, Y) Z, U) + \eta((\nabla_W R)(X, Y) Z) \eta(U) \\ = A(W) g(R(X, Y) Z, U) \\ + B(W) [g(Y, Z) g(X, U) - g(X, Z) g(Y, U)]. \end{aligned} \quad (16)$$

Let $\{e_i\}$, $i = 1, 2, \dots, (2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (16) and taking summation over i , $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} -(\nabla_W S)(Y, Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y) Z) \eta(e_i) \\ = A(W) S(Y, Z) + 2nB(W) g(Y, Z). \end{aligned} \quad (17)$$

Put $Z = \xi$ in (17) and by virtue of (8) and the equation $g((\nabla_W R)(e_i, Y) \xi, \xi) = 0$, we have

$$\begin{aligned} (\nabla_W S)(Y, \xi) &= -A(W) \\ &\quad \times [- (\phi Y) \alpha \\ &\quad + [2n(\epsilon\alpha^2 - \beta^2\delta) - \xi\beta] \eta(Y) - Y\beta] \\ &\quad - 2nB(W) \frac{\eta(Y)}{\epsilon}. \end{aligned} \quad (18)$$

Now we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \quad (19)$$

Using (5) and (6) in the above relation it follows that

$$\begin{aligned} (\nabla_W S)(Y, \xi) &= -W(\phi Y) \alpha - \phi Y(W\alpha) \\ &\quad + [4n\epsilon\alpha(W\alpha) - 4n\beta\delta(W\beta) \\ &\quad - W(\xi\beta)] \eta(Y) \\ &\quad + [2n(\epsilon\alpha^2 - \beta^2\delta) - \xi\beta] \\ &\quad \times \{\delta\beta[\epsilon g(W, Y) - \eta(W) \eta(Y)] \\ &\quad - \alpha g(\phi W, Y)\} \\ &\quad - (2n-1) W(Y\beta) + (\phi \nabla_W Y) \alpha \\ &\quad + (2n-1) (\nabla_W Y) \beta + \epsilon\alpha S(Y, \phi W) \end{aligned}$$

$$\begin{aligned}
& -\delta\beta S(Y, W) + \delta\beta\eta(W) \\
& \times [- (\phi Y)\alpha + [2n(\epsilon\alpha^2 - \beta^2\delta) - \xi\beta]\eta(Y) \\
& - (2n-1)Y\beta].
\end{aligned} \tag{20}$$

From (18) and (20) we obtain

$$\begin{aligned}
& -W(\phi Y)\alpha - \phi Y(W\alpha) \\
& + [4n\epsilon\alpha(W\alpha) - 4n\beta\delta(W\beta) - W(\xi\beta)]\eta(Y) \\
& + [2n(\epsilon\alpha^2 - \beta^2\delta) - \xi\beta] \\
& \times \{\delta\beta[\epsilon g(W, Y) - \eta(W)\eta(Y)] - \alpha g(\phi W, Y)\} \\
& - (2n-1)W(Y\beta) + (\phi\nabla_W Y)\alpha + (2n-1)(\nabla_W Y)\beta \\
& + \epsilon\alpha S(Y, \phi W) - \delta\beta S(Y, W) + \delta\beta\eta(W) \\
& \times [- (\phi Y)\alpha \\
& + [2n(\epsilon\alpha^2 - \beta^2\delta) - \xi\beta]\eta(Y) - (2n-1)Y\beta] \\
& = -A(W) \\
& \times [- (\phi Y)\alpha \\
& + [2n(\epsilon\alpha^2 - \beta^2\delta) - \xi\beta]\eta(Y) - (2n-1)Y\beta] \\
& - 2nB(W)\frac{\eta(Y)}{\epsilon}.
\end{aligned} \tag{21}$$

Replacing $Y = \xi$ in (21) and then using (2) we get

$$\begin{aligned}
& 4n\epsilon\alpha(W\alpha) - 4n\beta\delta(W\beta) - 2nW(\xi\beta) \\
& + [\epsilon\alpha(W - \eta(W)\xi) + \delta\beta\phi W]\alpha \\
& + (2n-1)[- \epsilon\alpha\phi W - \delta\beta(W - \eta(W)\xi)]\beta \\
& + \epsilon\alpha[- (\phi^2 W)\alpha - (2n-1)(\phi W)\beta] \\
& - \delta\beta[- (\phi W)\alpha - (2n-1)W\beta + (2n-1)\eta(W)\xi\beta] \\
& = -A(W)[2n(\epsilon\alpha^2 - \beta^2\delta - \xi\beta)] - \frac{2nB(W)}{\epsilon}.
\end{aligned} \tag{22}$$

Theorem 2. In a generalized ϕ -recurrent (ϵ, δ) -trans-Sasakian manifold, the 1-forms A and B are related as in (22).

If $\beta = 0$, then, from (22), we obtain

$$-2\alpha(W\alpha) = \alpha^2 A(W) + B(W). \tag{23}$$

Corollary 3. In a generalized ϕ -recurrent ϵ - α -Sasakian manifold, the 1-forms A and B are related as in (23).

(i) If $\epsilon = 1$, then ϵ - α -Sasakian manifold reduces as α -Sasakian manifold.

(ii) If $\epsilon = 1$ and $\alpha = 1$, then ϵ - α -Sasakian manifold reduces as Sasakian manifold.

If $\alpha = 0$, then, from (22), we obtain

$$\begin{aligned}
& (1-2n)\delta\beta\epsilon\eta(W)(\xi\beta) - 4n\delta\beta\epsilon(W\beta) - 2n\epsilon W(\xi\beta) \\
& = A(W)[2n\epsilon(\beta^2\delta + \xi\beta)] - 2nB(W).
\end{aligned} \tag{24}$$

Corollary 4. In a generalized ϕ -recurrent ϵ - β -Kenmotsu manifold, the 1-forms A and B are related as in (24).

If $\alpha = \beta = 0$, then, from (22), we obtain

$$B(W) = 0. \tag{25}$$

From definition (13) this leads to the following corollary.

Corollary 5. In cosymplectic manifold, generalized ϕ -recurrent reduces to ϕ -recurrent manifold.

Now from (15) we have

$$\begin{aligned}
& (\nabla_W R)(X, Y)Z = \eta((\nabla_W R)(X, Y)Z)\xi - A(W)R(X, Y)Z \\
& - B(W)[g(Y, Z)X - g(X, Z)Y].
\end{aligned} \tag{26}$$

From (26) and second Bianchi identity we get

$$\begin{aligned}
& A(W)\eta(R(X, Y)Z) \\
& + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) \\
& + B(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\
& + B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\
& + B(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)] = 0.
\end{aligned} \tag{27}$$

By virtue of (12) we obtain from (27) that

$$\begin{aligned}
& A(W)[2\alpha\beta(\delta - \epsilon)g(\phi X, Y)\eta(Z) - (X\alpha)g(Y, \phi Z) \\
& + (Y\alpha)g(X, \phi Z) - \alpha\beta(\delta + \epsilon) \\
& \times [g(Y, \phi Z)\eta(X) - g(X, \phi Z)\eta(Y)] - (\epsilon\alpha^2 - \delta\beta^2) \\
& \times [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \\
& - (X\beta)[g(Y, Z) - \epsilon\eta(Y)\eta(Z)] \\
& + (Y\beta)[g(X, Z) - \epsilon\eta(X)\eta(Z)] \\
& + A(X)[2\alpha\beta(\delta - \epsilon)g(\phi Y, W)\eta(Z) - (Y\alpha)g(W, \phi Z) \\
& + (W\alpha)g(Y, \phi Z) - \alpha\beta(\delta + \epsilon) \\
& \times [g(W, \phi Z)\eta(Y) - g(Y, \phi Z)\eta(W)] \\
& - (\epsilon\alpha^2 - \delta\beta^2)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)] \\
& - (Y\beta)[g(W, Z) - \epsilon\eta(W)\eta(Z)] \\
& + (W\beta)[g(Y, Z) - \epsilon\eta(Y)\eta(Z)]
\end{aligned}$$

$$\begin{aligned}
& + A(Y) [2\alpha\beta(\delta - \epsilon)g(\phi W, X)\eta(Z) - (W\alpha)g(X, \phi Z) \\
& \quad + (X\alpha)g(W, \phi Z) - \alpha\beta(\delta + \epsilon) \\
& \quad \times [g(X, \phi Z)\eta(W) - g(W, \phi Z)\eta(X)] \\
& \quad - (\epsilon\alpha^2 - \delta\beta^2) \\
& \quad \times [g(W, Z)\eta(X) - g(X, Z)\eta(W)] \\
& \quad - (W\beta)[g(X, Z) - \epsilon\eta(X)\eta(Z)] \\
& \quad + (X\beta)[g(W, Z) - \epsilon\eta(W)\eta(Z)] \\
& + B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\
& + B(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)] = 0.
\end{aligned} \tag{28}$$

Putting $Y = Z = e_i$ in (28) and taking summation over $1 \leq i \leq 3$, we get

$$\begin{aligned}
& A(W) [\{\epsilon\alpha^2 - \delta\beta^2 - \epsilon(\xi\beta)\}\eta(X) - (\phi X)\alpha] \\
& \quad + A(X) [\{-\epsilon\alpha^2 - \delta\beta^2 + \epsilon(\xi\beta)\}\eta(W) + (\phi W)\alpha] \\
& \quad + A(\xi) \\
& \quad \times [2\alpha\beta(\delta - \epsilon)g(\phi W, X) + \epsilon(W\beta)\eta(X) - \epsilon(X\beta)\eta(W)] \\
& \quad + A(\phi X) [(W\alpha) + \alpha\beta(\delta + \epsilon)\eta(W)] \\
& \quad + A(\phi W) [-(X\alpha) - \alpha\beta(\delta + \epsilon)\eta(X)] \\
& \quad + B(W)\eta(X) - B(X)\eta(W) = 0.
\end{aligned} \tag{29}$$

If $\beta = 0$, then, from (29), we obtain

$$\begin{aligned}
\epsilon\alpha^2 A(W)\eta(X) &= \epsilon\alpha^2 A(X)\eta(W), \\
B(W)\eta(X) &= B(X)\eta(W)
\end{aligned} \tag{30}$$

for all vector fields X, W . Replacing X by ξ in (30), we get

$$\begin{aligned}
A(W) &= \eta(W)\eta(\rho_1), \\
B(W) &= \eta(W)\eta(\rho_2)
\end{aligned} \tag{31}$$

for all vector field W .

From (30) and (31), we can state the following.

Theorem 6. *A generalized ϕ -recurrent ϵ - α -Sasakian manifold, the characteristic vector field ξ , and the vector fields ρ_1, ρ_2 associated with the 1-forms A, B , respectively, are in the same direction and the 1-forms A, B are given by (31).*

4. Generalized Concircular ϕ -Recurrent (ϵ, δ) -Trans-Sasakian Manifolds

In this section we study three-dimensional generalized concircular ϕ -recurrent (ϵ, δ) -trans-Sasakian manifold.

Definition 7. (ϵ, δ) -Trans-Sasakian manifold (M^3, g) is called generalized concircular ϕ -recurrent if its concircular curvature tensor $\bar{C}$

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{6}[g(Y, Z)X - g(X, Z)Y] \tag{32}$$

satisfies the condition

$$\begin{aligned}
\phi^2((\nabla_W \bar{C})(X, Y)Z) &= A(W)\bar{C}(X, Y)Z \\
&+ B(W)[g(Y, Z)X - g(X, Z)Y],
\end{aligned} \tag{33}$$

where $A(W)$ and $B(W)$ are defined as in (14) and r is the scalar curvature of the manifold (M^3, g) .

Let us consider a generalized concircular ϕ -recurrent (ϵ, δ) -trans-Sasakian manifold. Then by virtue of (1) we have

$$\begin{aligned}
& -(\nabla_W \bar{C})(X, Y)Z + \eta((\nabla_W \bar{C})(X, Y)Z)\xi \\
& = A(W)\bar{C}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],
\end{aligned} \tag{34}$$

from which it follows that

$$\begin{aligned}
& -g((\nabla_W \bar{C})(X, Y)Z, U) + \eta((\nabla_W \bar{C})(X, Y)Z)\eta(U) \\
& = A(W)g(\bar{C}(X, Y)Z, U) \\
& \quad + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)].
\end{aligned} \tag{35}$$

Let e_1, e_2, e_3 be a local orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (35) and taking summation over $i, 1 \leq i \leq 3$, we get

$$\begin{aligned}
& -(\nabla_W S)(X, U) + \frac{\nabla_W r}{3}g(X, U) \\
& \quad + (\nabla_W S)(X, \xi)\eta(U) - \frac{\nabla_W r}{3}\eta(X)\eta(U) \\
& = A(W)\left[S(X, U) - \frac{r}{3}g(X, U)\right] + 2B(W)g(X, U).
\end{aligned} \tag{36}$$

Putting $X = U = \xi$ in (36) and using (11) we have

$$A(W)\left[2(\beta^2\delta - \epsilon\alpha^2 + \xi\beta) + \frac{r\epsilon}{3}\right] - 2\epsilon B(W) = 0. \tag{37}$$

This leads to the following.

Theorem 8. *In a generalized concircular ϕ -recurrent (ϵ, δ) -trans-Sasakian manifold $M^3(\phi, \xi, \eta, g)$, the 1-forms A and B are related as in (37).*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] J. A. Oubiña, “New classes of almost contact metric structures,” *Publicationes Mathematicae Debrecen*, vol. 32, no. 3-4, pp. 187–193, 1985.
- [2] J. C. Marrero, “The local structure of trans-Sasakian manifolds,” *Annali di Matematica Pura ed Applicata*, vol. 162, pp. 77–86, 1992.
- [3] A. Bejancu and K. L. Duggal, “Real hypersurfaces of indefinite Kaehler manifolds,” *International Journal of Mathematics and Mathematical Sciences*, vol. 16, no. 3, pp. 545–556, 1993.
- [4] X. Xufeng and C. Xiaoli, “Two theorems on (ϵ) -Sasakian manifolds,” *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 2, pp. 249–254, 1998.
- [5] R. Kumar, R. Rani, and R. K. Nagaich, “On sectional curvatures of (ϵ) -Sasakian manifolds,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 93562, 8 pages, 2007.
- [6] M. M. Tripathi, E. Kılıç, S. Y. Perktas, and S. Keleş, “Indefinite almost paracontact metric manifolds,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 846195, 19 pages, 2010.
- [7] H. G. Nagaraja, R. C. Premalatha, and G. Somashekara, “On an (ϵ, δ) -trans-Sasakian structure,” *Proceedings of the Estonian Academy of Sciences*, vol. 61, no. 1, pp. 20–28, 2012.