Research Article

Generalizations of Inequalities for Differentiable Co-Ordinated Convex Functions

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A generalized lemma is proved and several new inequalities for differentiable co-ordinated convex and concave functions in two variables are obtained.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function and \( a, b \in I \) with \( a < b \); we have the following double inequality:

\[
\left( a + b \right) \frac{b - a}{2} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}.
\]

(1)

This remarkable result is well known in the literature as the Hermite-Hadamard inequality for convex mappings.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–4]).

A modification for convex functions which is also known as coordinated convex functions was introduced as follows by Dragomir in [5].

Let us consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \); a mapping \( f : \Delta \to \mathbb{R} \) is said to be convex on \( \Delta \) if the inequality

\[
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda tf(x, y) + (1 - \lambda) f(z, w)
\]

holds for all \( (x, y), (z, y), (x, w), (z, w) \in \Delta \), and \( t, \lambda \in [0, 1] \).

Dragomir in [5] established the following Hadamard-type inequalities for coordinated convex functions in a rectangle from the plane \( \mathbb{R}^2 \).

Theorem 2. Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the coordinates on \( \Delta \). Then one has the inequalities as follows:

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx
\]

(4)

Some new integral inequalities that are related to the Hermite-Hadamard type for coordinated convex functions are also established by many authors.
In ([6], 2008), Alomari and Darus defined coordinated s-convex functions and proved some inequalities based on this definition. In ([7], 2009), analogous results for h-convex functions on the coordinates were proved by Latif and Alomari. In ([8], 2009), Alomari and Darus established some Hadamard-type inequalities for coordinated log-convex functions.

In ([9], 2012), Latif and Dragomir obtained some new Hadamard type inequalities for differentiable coordinated convex and concave functions in two variables which are related to the left-hand side of Hermite-Hadamard type inequality for coordinated convex functions in two variables based on the following lemma.

**Lemma 3.** Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \in L(\Delta) \), then the following equality holds:

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)
\]

\[= \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx
\]

\[- \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy
\]

\[= (b-a)(d-c)
\]

\[
\times \int_0^1 K(u, v) \frac{\partial^2 f}{\partial u \partial v} \left(ua+(1-u)b, vc+(1-v)d\right) du \, dv,
\]

(5)

where

\[K(u, v) = \begin{cases}
u v, & (u, v) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \\
u (v-1), & (u, v) \in \left[0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right] \\
(u-1) v, & (u, v) \in \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \\
(u-1) (v-1), & (u, v) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right].
\]

\[\text{Theorem 4 (see [9]).} \text{ Let } f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \text{ be a partial differentiable mapping on } \Delta := [a, b] \times [c, d] \text{ in } \mathbb{R}^2 \text{ with } a < b \text{ and } c < d. \text{ If } \left|\frac{\partial^2 f}{\partial u \partial v}\right|^q \text{ is convex on the coordinates on } \Delta \text{ and } q \geq 1, \text{ then the following equality holds:}
\]

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A
\]

\[\leq \frac{(b-a)(d-c)}{16}
\]

\[\times \left(\left|\frac{\partial^2 f}{\partial u \partial v}(a, c)\right|^q + \left|\frac{\partial^2 f}{\partial u \partial v}(a, d)\right|^q + \left|\frac{\partial^2 f}{\partial u \partial v}(b, c)\right|^q
\]

\[+ \left|\frac{\partial^2 f}{\partial u \partial v}(b, d)\right|^q\right)^{1/q},
\]

(9)

where \( A \) is as given in Theorem 4.

**Theorem 5 (see [9]).** Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \) is convex on the coordinates on \( \Delta \) and \( 1/p + 1/q = 1 \), then the following equality holds:

\[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A
\]

\[\leq \frac{(b-a)(d-c)}{4(p+1)^{2/p}}
\]

\[\times \left(\left|\frac{\partial^2 f}{\partial u \partial v}(a, c)\right|^q + \left|\frac{\partial^2 f}{\partial u \partial v}(a, d)\right|^q + \left|\frac{\partial^2 f}{\partial u \partial v}(b, c)\right|^q
\]

\[+ \left|\frac{\partial^2 f}{\partial u \partial v}(b, d)\right|^q\right)^{1/q},
\]

(8)

where \( A \) is as given in Theorem 4.

In ([10], 2012), analogous results which are related to the right-hand side of Hermite-Hadamard type inequality for coordinated convex functions in two variables were proved by Sarikaya et al. based on the following lemma.
Lemma 7. Let \( f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f/\partial u \partial v \in L(\Delta) \), then the following equality holds:

\[
\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx
\]

\[
= \frac{1}{2} \left[ \int_a^b \left[ f(x,c) + f(x,d) \right] \, dx + \int_c^d \left[ f(a,y) + f(b,y) \right] \, dy \right]
\]

\[
+ \frac{1}{d-c} \int_c^d \left[ f(a,y) + f(b,y) \right] \, dy
\]

\[
= \frac{(b-a)(d-c)}{4}
\]

\[
\times \int_0^1 (1-2u)(1-2v) \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(b, c)} + \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(b, d)} \times (4)^{-1}
\]

\[
\times (ua + (1-u)b, vc + (1-v)d) \, du \, dv.
\]

Theorem 8 (see [10]). Let \( f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f/\partial u \partial v \) is convex on the coordinates on \( \Delta \), then the following equality holds:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right|
\]

\[
\leq \frac{(b-a)(d-c)}{16}
\]

\[
\times \left( \left| \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(a,c)} + \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(a,d)} + \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(b,c)} \right) \times (4)^{-1},
\]

where \( A \) is as given in Theorem 8.

Theorem 9 (see [10]). Let \( f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f/\partial u \partial v \) is convex on the coordinates on \( \Delta \) and \( p, q > 1, 1/p + 1/q = 1 \), then the following equality holds:

\[
\left| f(a,c) + f(a,d) + f(b,c) + f(b,d)
\right|
\]

\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A
\]

\[
\leq \frac{(b-a)(d-c)}{16}
\]

\[
\times \left( \left| \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(a,c)} + \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(a,d)} + \frac{\partial^2 f}{\partial u \partial v} \left. \right|_{(b,c)} \right) \times (4)^{-1},
\]

where \( A \) is as given in Theorem 8.

In [11], Ozdemir et al. established some Simpson’s inequalities for coordinated convex functions based on the following lemma.

Lemma 11. Let \( f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f/\partial u \partial v \in L(\Delta) \), then the following equality holds:

\[
\left( f \left( a, \frac{c+d}{2} \right) + f \left( b, \frac{c+d}{2} \right) + 4f \left( a+b, \frac{c+d}{2} \right) + f \left( a+b, \frac{c+d}{2} \right) \right)
\]

\[
+ \left( f \left( a+b, \frac{c+d}{2} \right) + f \left( a+b, \frac{c+d}{2} \right) \right) \times (9)^{-1}
\]
\[ + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{36} \]

\[ - \frac{1}{6 (b-a)} \int_a^b \left( f(x,c) + 4f\left(x, \frac{c+d}{2}\right) + f(x,d) \right) dx \]

\[ - \frac{1}{6 (d-c)} \int_c^d \left( f(a,y) + 4f\left(a+b \frac{2}{2}, y\right) + f(b,y) \right) dy \]

\[ + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy \ dx \]

\[ = (b-a)(d-c) \]

\[ \times \left( \int_0^1 p(x,u)p(y,v) \frac{\partial^2 f}{\partial u \partial v} \right. \]

\[ \times (ua + (1-u)b, vc + (1-v)d) du \ dv, \]  

\[ \text{(16)} \]

where

\[ p(x,u) = \begin{cases} 
(u - \frac{1}{6}), & t \in [0, \frac{1}{2}] \\
(u - \frac{5}{6}), & t \in (\frac{1}{2}, 1], 
\end{cases} \]

\[ p(y,v) = \begin{cases} 
(v - \frac{1}{6}), & s \in [0, \frac{1}{2}] \\
(v - \frac{5}{6}), & s \in (\frac{1}{2}, 1]. 
\end{cases} \]  

\[ \text{(17)} \]

**Theorem 12** (see [11]). Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \) is convex on the coordinates on \( \Delta \), then the following equality holds:

\[ \left| \int \int \left[ \frac{\partial^2 f}{\partial u \partial v} (a,c) + \frac{\partial^2 f}{\partial u \partial v} (a,d) + \frac{\partial^2 f}{\partial u \partial v} (b,c) + \frac{\partial^2 f}{\partial u \partial v} (b,d) \right] \right| \]

\[ \times (ua + (1-u)b, vc + (1-v)d) du \ dv, \]  

\[ \text{(18)} \]

where

\[ A = \frac{1}{6 (b-a)} \int_a^b \left( f(x,c) + 4f\left(x, \frac{c+d}{2}\right) + f(x,d) \right) dx \]

\[ + \frac{1}{6 (d-c)} \int_c^d \left( f(a,y) + 4f\left(a+b \frac{2}{2}, y\right) + f(b,y) \right) dy. \]  

\[ \text{(19)} \]

For recent results and generalizations concerning Hermite-Hadamard type inequality for differentiable coordinated convex functions see ([12], 2012) and the references given therein.

In this paper, a generalized lemma is proved and several new inequalities for differentiable coordinated convex and concave functions in two variables are obtained.

**2. Lemmas**

To establish our results, we need the following lemma.

**Lemma 13.** Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \in L(\Delta) \) and \( \lambda \in [0,1] \), then the following equality holds:

\[ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy \ dx \]

\[ + (1-\lambda)^2 f\left(a+b \frac{2}{2}, c+d \frac{2}{2}\right) + \frac{\lambda (1-\lambda)}{2} \]

\[ \times \left[ f\left(a, c+d \frac{2}{2}\right) + f\left(a, c+d \frac{2}{2}\right) \right] \]

\[ + \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - \frac{1}{2 (b-a)} \]

\[ \times \left[ \int_a^b \left( \frac{\partial f}{\partial x} (x,c) + 2 (1-\lambda) f\left(x, \frac{c+d}{2}\right) + \lambda f(x,d) \right) dx \right. \]

\[ - \frac{1}{2 (d-c)} \]

\[ \times \left. \int_c^d \left( \frac{\partial f}{\partial y} (a,y) + 2 (1-\lambda) f\left(a+b \frac{2}{2}, y\right) + \lambda f(b,y) \right) dy \right. \]

\[ = (b-a)(d-c) \]

\[ \times \left( \int_0^1 M(u,v) \frac{\partial^2 f}{\partial u \partial v} \right. \]

\[ \times (ua + (1-u)b, vc + (1-v)d) du \ dv, \]  

\[ \text{(20)} \]
where
\[
M(u, v) = \begin{cases} 
(u - \lambda \frac{1}{2})(v - \lambda \frac{1}{2}), \\
(u, v) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]
\end{cases}
\]
\[
(u - \left(\frac{1}{2} - \lambda\right))(v - \lambda \frac{1}{2}), \\
(u, v) \in \left(\frac{1}{2}, 1\right] \times [0, \frac{1}{2}]
\]
\[
(u - \left(\frac{1}{2} - \lambda\right))(v - \left(1 - \lambda\right) \frac{1}{2}), \\
(u, v) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right]
\]
\[
(\frac{1}{2}, 1] \times \left(\frac{1}{2}, 1\right]
\]

**Proof.** Since
\[
\int_{0}^{1} M(u, v) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv
\]
\[
= \int_{0}^{1/2} \left( u - \frac{\lambda}{2} \right) \left( v - \frac{\lambda}{2} \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv
\]
\[
+ \int_{1/2}^{1} \int_{0}^{1/2} \left( u - \frac{\lambda}{2} \right) \left( v - \left(1 - \frac{\lambda}{2}\right) \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv
\]
\[
+ \int_{1/2}^{1} \left( u - \left(1 - \frac{\lambda}{2}\right) \right) \left( v - \frac{\lambda}{2} \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv
\]
\[
+ \int_{1/2}^{1} \left( u - \left(1 - \frac{\lambda}{2}\right) \right) \left( v - \left(1 - \frac{\lambda}{2}\right) \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv,
\]
thus, by integration by parts, it follows that
\[
\int_{0}^{1/2} \left( u - \frac{\lambda}{2} \right) \left( v - \frac{\lambda}{2} \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv
\]
\[
= \int_{0}^{1/2} \left( u - \frac{\lambda}{2} \right) \frac{\partial f}{\partial v} \left( \frac{v - \lambda}{2} \right) \frac{1}{c - d} \, du
\]
\[
\times (ua + (1 - u)b, vc + (1 - v)d) \]
\[- \frac{\lambda}{2(c - d)(a - b)} \int_{0}^{1/2} f(ua + (1 - u)b, d) \, du \]
\[- \frac{1 - \lambda}{2(c - d)(a - b)} \int_{0}^{1/2} f \left( \frac{a + b}{2}, wc + (1 - v) d \right) \, dv \]
\[- \frac{\lambda}{2(c - d)(a - b)} \int_{0}^{1/2} f(b, wc + (1 - v)d) \, dv \]
\[+ \frac{1}{(c - d)(a - b)} \times \int_{0}^{1/2} f(ua + (1 - u)b, wc + (1 - v)d) \, dv. \]

(23)

Similarly, we can get
\[\int_{0}^{1/2} \int_{1/2}^{1} \left( u - \frac{\lambda}{2} \right) \left( v - \left( 1 - \frac{\lambda}{2} \right) \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, wc + (1 - v)d) \, du \, dv \]
\[= \frac{(1 - \lambda)^2}{4(c - d)(a - b)} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \]
\[+ \frac{\lambda (1 - \lambda)}{4(c - d)(a - b)} f \left( \frac{b}{2}, \frac{c + d}{2} \right) - \frac{1 - \lambda}{2(c - d)(a - b)} \]
\[\times \int_{0}^{1/2} f \left( ua + (1 - u)b, \frac{c + d}{2} \right) \, du \]
\[+ \frac{\lambda (1 - \lambda)}{4(c - d)(a - b)} f \left( \frac{a + b}{2}, c \right) \]
\[+ \frac{\lambda^2}{4(c - d)(a - b)} f(b, c) - \frac{\lambda}{2(c - d)(a - b)} \]
\[\times \int_{0}^{1/2} f \left( ua + (1 - u)b, c \right) \, du \]
\[- \frac{1 - \lambda}{2(c - d)(a - b)} \]
\[\times \int_{1/2}^{1} f \left( \frac{a + b}{2}, wc + (1 - v)d \right) \, dv - \frac{\lambda}{2(c - d)(b - a)} \]
\[\times \int_{1}^{1/2} f(a, wc + (1 - v)d) \, dv \]
\[+ \frac{1}{(c - d)(a - b)} \]
\[\times \int_{0}^{1/2} \int_{1/2}^{1} f(ua + (1 - u)b, wc + (1 - v)d) \, du \, dv, \]
\[\int_{1/2}^{1} \int_{0}^{1/2} \left( u - \left( 1 - \frac{\lambda}{2} \right) \right) \left( v - \left( 1 - \frac{\lambda}{2} \right) \right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, wc + (1 - v)d) \, du \, dv \]
\[= \frac{(1 - \lambda)^2}{4(c - d)(a - b)} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \]
\[+ \frac{\lambda (1 - \lambda)}{4(c - d)(a - b)} f \left( a, \frac{c + d}{2} \right) \]
\[+ \frac{\lambda (1 - \lambda)}{4(c - d)(a - b)} f \left( \frac{a + b}{2}, c \right) \]
\[+ \frac{\lambda^2}{4(c - d)(a - b)} f(a, c) - \frac{\lambda}{2(c - d)(a - b)} \]
\[\times \int_{0}^{1/2} f \left( ua + (1 - u)b, c \right) \, du \]
\[- \frac{1 - \lambda}{2(c - d)(b - a)} \]
\[\times \int_{1/2}^{1} f \left( \frac{a + b}{2}, wc + (1 - v)d \right) \, dv \]
\[-\frac{\lambda}{2(c-d)(a-b)} \int_{1/2}^{1} f(a, v, (1-v)d) \, dv + \frac{1}{(c-d)(a-b)} \times \int_{1/2}^{1} f(ua + (1-u)b, v, (1-v)d) \, du \, dv. \]  

Now

\[
\int_{0}^{1} M(u, v) \frac{\partial^2 f}{\partial u \partial v} (ua + (1-u)b, v, (1-v)d) \, du \, dv = (1-\lambda)^2 \frac{1}{2(c-d)(b-a)} \int_{1/2}^{1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{\lambda}{2} \left(\frac{c-d}{b-a}\right) \times \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + f\left(a + \frac{b}{2}, c\right) + f\left(a + \frac{b}{2}, d\right)\right] + \frac{\lambda^2}{4} \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + f\left(a, c\right) + f\left(b, d\right)\right] \]

\[
+ \frac{1}{2(d-c)(b-a)} \times \left[f\left(a, c\right) + f\left(b, c\right) + f\left(a, d\right) + f\left(b, d\right)\right] \times \left[1 \left| f\left(a, \frac{c+d}{2}\right) + f\left(a, \frac{c+d}{2}\right) + f\left(a + \frac{b}{2}, c\right) + f\left(a + \frac{b}{2}, d\right)\right| \right] \times \left[1 \left| \frac{\partial f}{\partial u} \right| + \left| \frac{\partial f}{\partial v} \right|\right] \times \left[\left| \frac{\partial^2 f}{\partial u \partial v} \right| + \left| \frac{\partial^2 f}{\partial u \partial v} \right|\right],
\]

where

\[
A = \frac{1}{2(b-a)} \int_{a}^{b} \left(\lambda f(x, c) + 2(1-\lambda) f\left(x, \frac{c+d}{2}\right) + \lambda f(x, d)\right) \, dx + \frac{1}{2(d-c)} \int_{c}^{d} \left(\lambda f(a, y) + 2(1-\lambda) f\left(a + \frac{b}{2}, y\right) + \lambda f(b, y)\right) \, dy - \frac{\lambda}{2} \left[f\left(a, \frac{c+d}{2}\right) + f\left(a, \frac{c+d}{2}\right) + f\left(a + \frac{b}{2}, c\right) + f\left(a + \frac{b}{2}, d\right)\right].
\]

Proof. From Lemma 13, we obtain

\[
\left| \frac{1}{(b-a)(d-c)} \int_{a}^{b} f(x, y) \, dy \right| dx + (1-\lambda)^2 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \]

Remark 14. Applying Lemma 13 for \(\lambda = 0, 1, 1/3\), we get the results of Lemmas 3, 7, and 11, respectively.

3. Main Results

Theorem 15. Let \(f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}\) be a partial differentiable mapping on \(\Delta := [a, b] \times [c, d] \in \mathbb{R}^2\) with \(a < b\) and \(c < d\). If \(|\frac{\partial^2 f}{\partial u \partial v}|\) is convex on the coordinates on \(\Delta\) and \(\lambda \in [0, 1]\), then the following equality holds:

\[
\left| \int_{a}^{b} f(x, y) \, dy \right| dx + (1-\lambda)^2 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\]

\[
+ \frac{\lambda^2}{4} \left[f\left(a, c\right) + f\left(b, d\right) + f\left(a, d\right) + f\left(b, c\right)\right] - A
\]

\[
\leq (b-a)(d-c) \left(\frac{2\lambda^2 - 2\lambda + 1}{8}\right)^2 \times \left[\left| \frac{\partial^2 f}{\partial u \partial v} \right| + \left| \frac{\partial^2 f}{\partial u \partial v} \right| + \left| \frac{\partial^2 f}{\partial u \partial v} \right| + \left| \frac{\partial^2 f}{\partial u \partial v} \right|\right],
\]

where

\[
A = \frac{1}{2(b-a)} \int_{a}^{b} \left(\lambda f(x, c) + 2(1-\lambda) f\left(x, \frac{c+d}{2}\right) + \lambda f(x, d)\right) \, dx + \frac{1}{2(d-c)} \int_{c}^{d} \left(\lambda f(a, y) + 2(1-\lambda) f\left(a + \frac{b}{2}, y\right) + \lambda f(b, y)\right) \, dy - \frac{\lambda}{2} \left[f\left(a, \frac{c+d}{2}\right) + f\left(a, \frac{c+d}{2}\right) + f\left(a + \frac{b}{2}, c\right) + f\left(a + \frac{b}{2}, d\right)\right].
\]
Because \( \frac{\partial^2 f}{\partial u \partial v} \) is a convex function on the coordinates on \( \Delta \), then one has

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right|
+ (1 - \lambda)^2 \left( \frac{a + b}{2}, \frac{c + d}{2} \right)
+ \frac{\lambda^2}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A
\leq (b-a)(d-c)
\]

\[
\times \int_0^1 |M(u, v)|
\times \left\{ u \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right| + u (1 - v) \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right|
+ (1 - u) \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|
+ (1 - u) (1 - v) \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right| \right\} \, du \, dv
\]

\[= (b-a)(d-c)
\]

\[
\times \left[ \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right| \int_0^1 |M(u, v)| \, uv \, du \, dv
\right.
\]

\[+ \left. \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right| \int_0^1 |M(u, v)| \, u (1 - v) \, du \, dv \right.
\]

\[+ \left. \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right| \int_0^1 |M(u, v)| \, (1 - u) \, vdu \, dv \right.
\]

\[+ \left. \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right| \int_0^1 |M(u, v)| \, (1 - u) (1 - v) \, du \, dv \right]\]

\[= (b-a)(d-c) \left( \frac{2\lambda^2 - 2\lambda + 1}{8} \right)^2
\]

On the other hand, we have

\[
\int_0^1 |M(u, v)| \, uv \, du \, dv
\]

\[= \int_0^1 |M(u, v)| \, u (1 - v) \, du \, dv
\]

\[= \int_0^1 |M(u, v)| \, (1 - u) \, vdu \, dv
\]

\[= \int_0^1 |M(u, v)| \, (1 - u) (1 - v) \, du \, dv,
\]

\[
\int_0^1 |M(u, v)| \, uv \, du \, dv
\]

\[= \int_{1/2}^{1/2} \left| (u - \frac{\lambda}{2}) \left( v - \frac{\lambda}{2} \right) \right| \, uv \, du \, dv
\]

\[+ \int_{1/2}^{1/2} \left| (u - \frac{\lambda}{2}) \left( v - (1 - \frac{\lambda}{2}) \right) \right| \, uv \, du \, dv
\]

\[+ \int_{1/2}^{1/2} \left| (u - (1 - \frac{\lambda}{2})) \left( v - \frac{\lambda}{2} \right) \right| \, uv \, du \, dv
\]

\[+ \int_{1/2}^{1/2} \left| (u - (1 - \frac{\lambda}{2})) \left( v - (1 - \frac{\lambda}{2}) \right) \right| \, uv \, du \, dv
\]

\[= \left( \frac{2\lambda^2 - 2\lambda + 1}{8} \right)^2,
\]

which completes the proof. \( \Box \)

**Remark 16.** Applying Theorem 15 for \( \lambda = 0, 1, 1/3 \), we get the results of Theorems 4, 8, and 12, respectively.

**Theorem 17.** Let \( f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \) is convex on the coordinates on \( \Delta \) and \( q > 1 \), one gets the following inequality:

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right|
+ (1 - \lambda)^2 \left( \frac{a + b}{2}, \frac{c + d}{2} \right)
\]

\[+ \frac{\lambda^2}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A
\]

\[\leq \frac{(b-a)(d-c)}{4} \left( \frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{p+1} \right)^{2/p}
\]

Results of Theorems 4, 8, and 12, respectively.
where $\lambda \in [0,1]$ and $A$ is as given in Theorem 15 and $1/p + 1/q = 1$.

**Proof.** From Lemma 13, we obtain

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx$$

$$+ (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)$$

$$+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A$$

$$\leq (b-a)(d-c) \left( \int_0^1 |M(u,v)|^p \, du \, dv \right)^{1/p}$$

$$\times \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} (ua + (1-u)b, vc + (1-v)d) \right|^q \, du \, dv \right)^{1/q}.$$  

(31)

By using the well-known Hölder inequality for double integrals, then one has

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx$$

$$+ (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)$$

$$+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A$$

$$\leq (b-a)(d-c) \left( \int_0^1 |M(u,v)|^p \, du \, dv \right)^{1/p}$$

$$\times \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} (ua + (1-u)b, vc + (1-v)d) \right|^q \, du \, dv \right)^{1/q}.$$  

(32)

Hence, it follows that

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx$$

$$+ (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)$$

$$+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A$$

$$\leq (b-a)(d-c) \left( \int_0^1 |M(u,v)|^p \, du \, dv \right)^{1/p}$$

$$\times \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} (ua + (1-u)b, vc + (1-v)d) \right|^q \, du \, dv \right)^{1/q}.$$  

(33)

Remark 18. Applying Theorem 17 for $\lambda = 0, 1$, we get the results of Theorems 5 and 9, respectively.

**Theorem 19.** Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $|\partial^2 f/\partial u \partial v|^q$ is convex on the coordinates on $\Delta$ and $q \geq 1$, then

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx$$

$$+ (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)$$

$$+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A$$

$$\leq \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} (a,c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (a,d) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b,c) \right|^q$$

$$\left| \frac{\partial^2 f}{\partial u \partial v} (b,d) \right|^q \right)^{1/q}.$$  

(34)
+ \lambda^2/4 \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \right| \
\leq (b - a)(d - c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^2 \
\times \left( \left[ \frac{\partial^2 f}{\partial u \partial v} (a, c)^q + \frac{\partial^2 f}{\partial u \partial v} (a, d)^q + \frac{\partial^2 f}{\partial u \partial v} (b, c)^q \
+ \frac{\partial^2 f}{\partial u \partial v} (b, d)^q \right] \times (4)^{-1} \right)^{1/q} = (b - a)(d - c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^{2(1/q)} \
\times \left[ \frac{\partial^2 f}{\partial u \partial v} (ua + (1 - u)b, vc) \
+ (1 - v)d \right]^{1/q}.
\quad (37)

Because $|\partial^2 f/\partial u \partial v|^q$ is a convex function on the coordinates on $\Delta$, then one has

\left| \frac{\partial^2 f}{\partial u \partial v} (ua + (1 - u)b, vc + (1 - v)d) \right|^{1/q} \
\leq u \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + u(1 - v) \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right|^q \
+ (1 - u) \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q + (1 - u)(1 - v) \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right|^q.
\quad (40)

Thus, it follows that

\left| \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \
+ (1 - \lambda)^2 \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right| \
+ \lambda^2/4 \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \right| \
\leq (b - a)(d - c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^{2(1/q)} \
\times \left( \int_0^1 |M(\mu, \nu)|^{1-(1/q)} \right) \
\times \left( \int_0^1 |M(\mu, \nu)| \right). 
\quad (41)
On the other hand, we obtain
\[
\iint_0^1 |M(u, v)| \times u (1 - v) \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + (1 - u) \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q \\
+ (1 - u) (1 - v) \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right|^q \, du \, dv
\]
\[
= \left( \frac{2\lambda^2 - 2\lambda + 1}{8} \right)^2 \times \left( \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right|^q \right). 
\]

Thus, we get the following inequality:
\[
\left| \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right| \\
+ (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
+ \frac{\lambda^2}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \\
\leq (b-a)(d-c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^2 \\
\times \left( \left( \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right|^q \right. \right.
\left. + \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right|^q \left. \right) \times (4)^{-1} \right)^{1/q}, 
\]

which completes the proof.

Remark 20. Applying Theorem 19 for \( \lambda = 0, 1 \), we get the result of Theorems 6 and 10, respectively.

Theorem 21. Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( |\partial^2 f/\partial u \partial v|^q \) is concave on the coordinates on \( \Delta \) and \( q > 1 \), then
\[
\left| \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right| \\
+ (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \frac{\lambda^2}{4} \\
\times \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \\
\leq \frac{(b - a) (d - c) \left( \frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{p + 1} \right)^{2/p}}{4} \\
\times \left| \frac{\partial^2 f}{\partial u \partial v} \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right|^q, 
\]

where \( \lambda \in [0, 1] \) and \( A \) is as given in Theorem 15 and \( 1/p + 1/q = 1 \).

Proof. Similarly as in Theorem 17, because \( |\partial^2 f/\partial u \partial v|^q \) is a concave function on the coordinates on \( \Delta \), by the reversed direction of (4), we get
\[
\left| \int_0^1 \frac{\partial^2 f}{\partial u \partial v} (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv \right|^q \\
\leq \left| \frac{\partial^2 f}{\partial u \partial v} \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right|^q. 
\]

Hence, it follows that
\[
\left| \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right| \\
+ (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \frac{\lambda^2}{4} \\
\times \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \\
\leq \frac{(b - a) (d - c) \left( \frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{p + 1} \right)^{2/p}}{4} \\
\times \left| \frac{\partial^2 f}{\partial u \partial v} \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right|^q, 
\]

which yields the desired result.

Conflict of Interests

☐ The author has declared that no conflict of interests exists.
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References


