Research Article

The Matrix Equation $XA - AX = X^\alpha g(X)$ over Fields or Rings

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Let $n, \alpha \in \mathbb{N}_{\geq 2}$ and let K be an algebraically closed field with characteristic 0 or greater than n . We show that if $f \in K[X]$ and $A, B \in M_n(K)$ satisfy $[A, B] = f(A)$, then A, B are simultaneously triangularizable. Let R be a reduced ring such that $n!$ is not a zero divisor and let A be a generic matrix over R ; we show that $X = 0$ is the sole solution of $AX - XA = X^\alpha$. Let R be a commutative ring with unity; let A be similar to $\text{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})$ such that, for every $i \neq j$, $\lambda_i - \lambda_j$ is not a zero divisor. If X is a nilpotent solution of $XA - AX = X^\alpha g(X)$ where $g \in R[X]$, then $AX = XA$.

1. Introduction

Let n be an integer at least 2. In the first part, we consider $n \times n$ matrices with entries in K , a field such that its characteristic $\text{char}(K)$ is 0 or greater than n , and $\bar{K}$ denotes its algebraic closure. In the second part, the entries of the matrices are elements of R , a commutative ring with unity $1 \neq 0$; then, $M_n(R)$ is the ring of endomorphisms of the free R -module R^n .

Let $\alpha \in \mathbb{N}_{\geq 2}$ and let g be a polynomial with coefficients in the complex field $\mathbb{C}$ such that $g(0) \neq 0$. In [1, 2], the matrix $A \in M_n(\mathbb{C})$ is given and the authors study the matrix equations in the unknown $X \in M_n(\mathbb{C})$:

$$XA - AX = X^\alpha, \quad (1)$$

$$XA - AX = X^\alpha g(X), \text{ where } X \text{ is nilpotent.} \quad (2)$$

In the present paper, we extend some results obtained in [1, 2] when we replace $\mathbb{C}$ with K or R .

Two matrices $A, B \in M_n(K)$ are said to be simultaneously triangularizable (abbreviated to *ST*) over K if there exists $P \in GL_n(K)$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices.

The following result is a slight improvement of [1, Theorem 1] or of [3, Theorem 11'].

Theorem 1. *One assumes that $\text{char}(K) > n$ or is 0 and that f is a polynomial with coefficients in K :*

$$\text{Let } A, B \in M_n(K) \text{ be such that } [A, B] = f(A). \quad (3)$$

Then, A, B are ST over $\bar{K}$.

Remark 2. Note that the previous result fails when $\text{char}(K) = n$.

We now consider (1) and (2).

Let R be a commutative ring with unity $1 \neq 0$ and let $A = [a_{ij}]$ be a $n \times n$ matrix where the $\{a_{ij}\}_{1 \leq i, j \leq n}$ are commuting indeterminates. Let $\tilde{R}$ be the ring of the polynomials in the indeterminates $\{a_{ij}\}_{1 \leq i, j \leq n}$ and with coefficients in R . Thus, the algebra generated by A is included in $M_n(\tilde{R})$. In particular, there are no polynomial relations, with coefficients in R , linking the $\{a_{ij}\}_{1 \leq i, j \leq n}$. We say that A is a *generic* matrix over R . When R is reduced (for every $u \in R$, $u^2 = 0$ implies $u = 0$), we obtain a precise result.

Proposition 3. *Let $n \geq 2$, and let R be a reduced ring such that $n!$ is not a zero divisor. Let $A \in M_n(\tilde{R})$ be a generic matrix. Then, $X = 0$ is the sole solution of (1).*

Otherwise, we only obtain a partial result.

Proposition 4. Let R be a commutative ring with unity, $\alpha \geq 2$, and let g be a polynomial with coefficients in R such that $g(0) \neq 0$. Let $X \in M_n(R)$ be a nilpotent solution of (2). Then, all elements of the two-sided ideal, in $M_n(R)$, generated by $XA - AX$ are nilpotent.

Yet, if A is diagonalizable and its spectrum is “good,” then we obtain a complete solution.

Theorem 5. Let $A \in M_n(R)$ be similar over R to $\text{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})$ with $n_1 + \dots + n_r = n$ such that, for every $i \neq j$, $\lambda_i - \lambda_j$ is not a zero divisor. Let X be a nilpotent solution of (2).

(i) Then, there is $P \in GL_n(R)$ such that

$$\begin{aligned} A &= P \text{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}) P^{-1}, \\ X &= P \text{diag}(X_1, \dots, X_r) P^{-1}, \end{aligned} \quad (4)$$

where, for every i , $X_i \in M_{n_i}(R)$ and $X_i^\alpha g(X_i) = 0$.

(ii) If moreover $g(0)$ is a unit, then $X^\alpha = 0$.

2. Equation (1) Over a Field

In this section, K is a field with characteristic not 2. Let J_n denote the nilpotent Jordan-block of dimension n . If A is a square matrix, then χ_A denotes its characteristic polynomial and $C(A)$ denotes its commutant.

We prove Theorem 1.

Proof. Let V be the vector space spanned by $\{B, I_n, A, \dots, A^{n-1}\}$. One checks easily by induction:

$$\forall i \geq 1, \quad A^i B - B A^i = i A^{i-1} f(A). \quad (5)$$

By Cayley-Hamilton's Theorem (that is valid on a commutative ring with unity), $A^i B - B A^i$ belongs to V , and V is Lie's algebra. The derived series of V is

$$V_1 = [V, V] \subset K[A], \quad V_2 = [V_1, V_1] = \{0\}. \quad (6)$$

Thus, V is solvable. According to Lie's Theorem (that is valid when $\text{char}(K) > n$ or is 0, cf. [4, page 38]), V is triangularizable; that is, A, B are ST . $\square$

Remark 6. The hypothesis about $\text{char}(K)$ is necessary; indeed, if $n = 3$ and $\text{char}(K) = 3$, then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad (7)$$

satisfy $[A, B] = A^2$; yet, $A^3 = I_3$ is not nilpotent.

Example 7. Consider the following equation:

$$A^2 - 2AB + B^2 = P(A - B), \quad \text{where } P \text{ is a polynomial.} \quad (8)$$

It is equivalent to

$$N^2 - P(N) = [N, B], \quad \text{where } N = A - B. \quad (9)$$

According to Theorem 1, A, B are ST over $\bar{K}$ and $A - B$ is nilpotent.

Proposition 8. Let $\alpha \in \llbracket 2, n-1 \rrbracket$ and let $A \in M_n(K)$ be such that its eigenvalues in $\bar{K}$ have multiplicity at most α . Then, the solutions of (1) commute with A .

Proof. According to [2, Proposition 2.9], if $X \in M_n(\bar{K})$ is a solution of (1), then each generalized eigenspace of $A \in M_n(\bar{K})$ is X -invariant. Thus, we may assume that A has dimension at most α . Therefore, $X^\alpha = 0$ and we are done. $\square$

We consider the relation:

$$AB - BA = A^\alpha, \quad \text{where } \alpha \in \mathbb{N}_{\geq 2}. \quad (10)$$

Proposition 9. One assumes that $\text{char}(K) > 3$ or is 0. Let $A, B \in M_3(K)$ satisfying (10) such that $AB \neq BA$. Then $\alpha = 2$, A is nilpotent and there is $\lambda \in K$ and a nilpotent matrix N such that $B = \lambda I_3 + N$; moreover, $AN - NA = A^2$. Conversely, there exist such solutions.

Proof. If $\alpha \geq 3$, then, according to Proposition 8, $AB = BA$ that is contradictory. Thus, $\alpha = 2$ and B has a triple eigenvalue $\lambda = 3^{-1} \text{trace}(B) \in K$. An instance of such a solution is

$$A = J_3, \quad N = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

$\square$

Let $\alpha \in \llbracket 2, n-1 \rrbracket$. According to [2, Proposition 2.9], if $X \in M_n(\bar{K})$ is a solution of (1), then each generalized eigenspace of $A \in M_n(\bar{K})$ is X -invariant. Thus, we may assume that A is nilpotent.

We consider the algebraic variety:

$$\begin{aligned} S_{n,\alpha} = \{ (A, B) \in M_n(\bar{K})^2 \mid B \text{ is nilpotent} \\ \text{and } (A, B) \text{ satisfies (10)} \}. \end{aligned} \quad (12)$$

Let $\dim(S_{n,\alpha})$ denote the dimension of the maximal component of $S_{n,\alpha}$. Recall that the algebraic variety N_n of nilpotent matrices in $M_n(\bar{K})$ has dimension $n^2 - n$ and is irreducible (cf. [5, Section: The nilpotent cone]). Note that the algebraic variety

$$\begin{aligned} W_n = \{ (A, B) \in M_n(\bar{K})^2 \mid A, B \text{ are nilpotent} \\ \text{and } AB = BA \} \end{aligned} \quad (13)$$

has dimension $\dim(N_n) + (n-1) = n^2 - 1$ and is irreducible when $\text{char}(K) > n$ (cf. [6]).

Proposition 10. *Let $\alpha \in \llbracket 2, n-1 \rrbracket$. Then, $\dim(S_{n,\alpha}) \geq n^2 - 1$.*

Proof. A generic nilpotent matrix B is similar to J_n . Put $B = J_n$ and consider the equation $XJ_n - J_nX = X^\alpha$. According to [2, Remark 3.4], X is strictly upper triangular and we can express the entries $\{x_{ij}\}_{1 \leq i, j \leq n}$ of X as functions of the free variables $x_{1,2}, \dots, x_{1,n}$ (the only condition occurs when $n = 2$ and is $x_{1,2} \notin \{-1/k \mid k \in \mathbb{N}^*\}$). Then, the algebraic variety $Y_{n,\alpha} = \{X \mid XJ_n - J_nX = X^\alpha\}$ has dimension $n - 1$. We deduce that $\dim(S_{n,\alpha}) \geq \dim(N_n) + \dim(Y_{n,\alpha}) = n^2 - 1$. $\square$

Remark 11. We show that $\dim(S_{3,2}) = 8$ (cf. Proposition 9).

Case 1. $B = 0$. Then, $A^2 = 0$; that is, $A = 0$ or A is similar to $U = \text{diag}(0, J_2)$; we obtain a variety of dimension $n^2 - \dim(C(U)) = 9 - 5 = 4$.

Case 2. B is similar to U ; that is, B goes through a variety of dimension 4. From the Gröbner basis method (implemented in Maple), we deduce that the Hilbert dimension of $\{X \mid XU - UX = X^2\}$ is 3. Thus, the dimension of the component of $S_{n,\alpha}$ associated to this choice of B is 7.

Computational evidence leads us to conjecture that $\dim(S_{n,\alpha}) = n^2 - 1$, an expression that does not depend on α .

We consider the equation:

$$A^3 - 3A^2B + 3AB^2 - B^3 = 0_n. \quad (14)$$

A family of such equations is studied in [7]. As in [1, 2], an essential argument used by the authors is that the sequence of iterated kernels $(\ker(A - \lambda I)^k)_k$ is ultimately stationary. Moreover, the properties about the generalized eigenspaces are similar; A and B have same spectrum $(\lambda_i)_i$ over $\bar{K}$; moreover, for every i , the generalized eigenspaces $E_{\lambda_i}(A)$ and $E_{\lambda_i}(B)$ are equal. Thus, to study the solutions (A, B) of (14) can be reduced to study the restrictions of A, B to a generalized eigenspace $E_\lambda(A) = E_\lambda(B)$. Moreover, if (A, B) is a solution of (14), then, for every $\mu \in K$, $(A - \mu I_n, B - \mu I_n)$ is also a solution of (14). Finally, it suffices to solve (14) when A, B are assumed to be nilpotent matrices.

Let (A, B) be a solution of (14). We may wonder whether A, B are ST. The answer is no, as we can see with the following solution of (14) when $n = 4$ and $\text{char}(K) \geq 5$ or 0 :

$$\left(\begin{pmatrix} 0 & \frac{4}{3} & -\frac{1}{3} & -1 \\ 1 & 0 & \frac{4}{3} & -\frac{3}{4} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, J_4 \right). \quad (15)$$

Clearly A, B are nilpotent and $[A, B]$ is invertible. We say that a pair (U, V) has property L (cf. [8]) if there are orderings of the eigenvalues $(\lambda_i), (\mu_i)$ of U, V such that, for every $a \in K$, the eigenvalues of $U + aV$ are $(\lambda_i + a\mu_i)$; if U, V are ST, then

they have property L . In our instance, (A, B) has not property L because, if $a \neq 0$, then $A + aB$ is not nilpotent.

Let α, n be integers ≥ 2 .

Proposition 12. *Let K be a field such that $\text{char}(K) > n$ or is 0. If $A \in M_n(K)$ has n distinct eigenvalues in $\bar{K}$, then $X = 0$ is the sole solution of (1).*

Proof. Note that A satisfies the property:

$$\begin{aligned} \mathcal{P} : & \text{for every } Y \in M_n(K), \\ \{AY = YA, Y^\alpha = 0\} & \text{ imply } Y = 0. \end{aligned} \quad (16)$$

Indeed, we may assume that A is a diagonal matrix over $\bar{K}$. Since Y commutes with A , Y is also diagonal and clearly $Y = 0$. According to Theorem 1, $[A, X]$ is nilpotent and X too; assume that k , the nilindex of X , is greater than α . According to (5), $[A, X^{k-\alpha+1}] = 0$; by the property $\mathcal{P}$ and $X^{(k-\alpha+1)\alpha} = 0$, we deduce $X^{k-\alpha+1} = 0$, that is contradictory and therefore $k \leq \alpha$. Thus, $X^\alpha = 0$ and $AX = XA$; by Property $\mathcal{P}$, we conclude that $X = 0$. $\square$

Remark 13. The previous result is shown, when K is a field of characteristic 0, in [2, Proposition 2.5].

3. Equations (1) and (2) Over a Ring

- (i) Let $(R_i)_{i \in I}$ be commutative rings with unity. Their ring subdirect product R is defined if there is $f : R \rightarrow \prod_{i \in I} R_i$ an injective ring homomorphism such that, for every $j \in I$, the projection of f on R_j is onto.
- (ii) A commutative ring R with unity is reduced if, for every $u \in R$, $u^2 = 0$ implies $u = 0$. That is equivalent to say that R is isomorphic to a subring of a direct product of fields or isomorphic to a subdirect product of domains (cf. [9, Theorem 11.6.7]). For instance, $R = \mathbb{Z} \times \mathbb{Z}$ is a reduced ring with $\text{char}(R) = 0$. More generally, $\text{char}(R)$ is 0 or a product of distinct primes. Note that $R = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}$ is reduced with $\text{char}(R) = 0$ and yet, $3 = (0, 3)$ is a zero divisor.

We show Proposition 3.

Proof. Note that, if R is reduced, then $\bar{R}$ also is. Since $\bar{R}$ is a subring of a direct product of algebraic closed fields $\prod_{i \in I} K_i$, we may assume $\bar{R} = \prod_{i \in I} K_i$ where, for every $i \in I$, K_i is a field such that $\text{char}(K_i) > n$ or is 0. Let $X = (X_i)_i$ and $A = (A_i)_i$ be the associated decompositions. Thus, for any $i \in I$, $X_i A_i - A_i X_i = X_i^\alpha$, where the i th component $A_i \in M_n(K_i)$ of A is generic; then, for every i , the discriminant of χ_{A_i} is not 0 and the matrix A_i has n distinct eigenvalues. According to Proposition 12, for every $i \in I$, $X_i = 0$ and consequently $X = 0$. $\square$

Proposition 14. *Let R be a commutative ring with unity such that $n!$ is not a zero divisor and let $A \in M_n(R)$. If $X \in M_n(R)$ is a solution of (1), then X is a nilpotent matrix.*

Proof. Note that $[A, X]$ and X commute and that the Cayley-Hamilton theorem is true over R . According to the proof

of Jacobson lemma (cf. introduction of [10] and also [11] where an improvement of this result is stated within the framework of the algebraic operators on a complex Banach space), $n![A, X]^{2^n-1} = 0$ that implies $[A, X]^{2^n-1} = 0$ and we are done. $\square$

If A is generic over R , then we have a more precise result for small n, α .

Proposition 15. *Let $n = 2, 2 \leq \alpha \leq 4$. Let R be a commutative ring with unity such that, if $\alpha = 2, 3$, or 4 , then $2, 3!$ or $5!$ is not a zero divisor. Let $A \in M_2(\tilde{R})$ be a generic matrix. If $X = [x_{ij}] \in M_2(\tilde{R})$ is a solution of (1) then*

$$\begin{aligned} AX - XA &= X^\alpha = 0, \\ \text{for every } (i, j), \quad x_{ij}^{2\alpha-1} &= 0. \end{aligned} \quad (17)$$

Proof. The 4 parameters are the (a_{ij}) . We have a system of 4 equations in the 4 unknowns (x_{ij}) . Using Gröbner basis theory in any specified characteristic, we obtain the required result. $\square$

When $n = 3, 4, 5$ and $2 \leq \alpha \leq 4$, the calculations have great complexity; thus, we carry out specializations of the (a_{ij}) in the ring R . Then, we randomly choose the matrix A (in order to simulate the generic nature of the matrix) and we formally solve (1) in the n^2 unknowns (x_{ij}) . Numerical experiments, again using Gröbner basis theory in characteristic great enough, lead to the following result: $X^\alpha = 0_n$ and for every (i, j) , $x_{ij}^{(\alpha-1)n+1} = 0$; for instance, if $n = 5, \alpha = 4$, then the supplementary condition is “13! is not a zero divisor.” Therefore, we conjecture the following.

Conjecture. Let $n \geq 2, R$ be a commutative ring with unity satisfying a condition in the form: “the integer $\phi(n, \alpha)!$ is not a zero divisor.” Let $A \in M_n(\tilde{R})$ be a generic matrix. If $X = [x_{ij}] \in M_n(\tilde{R})$ is a solution of (1), then $AX - XA = X^\alpha = 0_n$ and for every (i, j) , $x_{ij}^{(\alpha-1)n+1} = 0$.

Remark 16. (i) The instance $R = \mathbb{Z}/27\mathbb{Z}, U = 3I_2$ shows that if U is a nilpotent $n \times n$ matrix, then we have not necessarily $U^n = 0$.

(ii) In the previous conjecture, note that the exponent $(\alpha-1)n+1$ is very special; indeed, if $U^\alpha = 0_n$, then we have

$$(\text{trace}(U))^{(\alpha-1)n+1} = 0, \quad (18)$$

and we cannot do better (cf. [12]).

Let R be a commutative ring with unity and $A \in M_n(R)$. We look for the *nilpotent* solutions $X \in M_n(R)$ of (2), where $\alpha \geq 2$ and g is a polynomial in X , with coefficients in R , such that $g(0) \neq 0$. Then, according to (5), for every i ,

$$X^i A - A X^i = i X^{\alpha+i-1} g(X). \quad (19)$$

Let $\text{val}(T)$ denote the valuation of the polynomial T , with the following convention: $\text{val}(0) = +\infty$. In the sequel, X is a nilpotent solution of (2).

Lemma 17. *Let u be a polynomial in X . Then, for every $l, uA^l = \sum_{i=0}^l A^i v_i$, where for every i, v_i is a polynomial in X such that $\text{val}(v_i) \geq \text{val}(u)$.*

Proof. For every $i, X^i A = A X^i + v_0(X)$ with $\text{val}(v_0) \geq i$. Then $uA = Au + v_1(X)$ with $\text{val}(v_1) \geq \text{val}(u)$. In the same way, $uA^2 = A^2 u + 2Av_1 + v_2$ and $uA^3 = A^3 u + 3A^2 v_1 + 3Av_2 + v_3$ with $\text{val}(v_3) \geq \text{val}(v_2) \geq \text{val}(v_1)$, and so on. $\square$

Lemma 18. *Let $k \geq 2$ and $(v_i)_i$ be polynomials in X with, for every $i, \text{val}(v_i) \geq \alpha$. Then*

$$\left(\sum_i A^i v_i \right)^k \text{ is in the form } \sum_i A^i w_i, \quad (20)$$

where for every i, w_i is a polynomial in X with $\text{val}(w_i) \geq k\alpha$.

Proof. We use a reasoning by recurrence. Let

$$E = \left(\sum_i A^i v_i \right)^k = \left(\sum_i A^i z_i \right) \left(\sum_j A^j v_j \right), \quad (21)$$

where for every i, z_i is a polynomial in X such that $\text{val}(z_i) \geq (k-1)\alpha$. Using Lemma 17, $E = \sum_{i,j} A^i \sum_{r \leq j} A^r y_{ri} v_j$, where $\text{val}(y_{ri}) \geq (k-1)\alpha$. $\square$

Lemma 19. *Let P, Q be polynomials in X, A . Then, for every k ,*

$$(P(AX - XA)Q)^k \text{ is in the form } \sum_i A^i v_i, \quad (22)$$

where for every i, v_i is a polynomial in X such that $\text{val}(v_i) \geq k\alpha$.

Proof. Let $E = (P(AX - XA)Q)^k = (PX^\alpha gQ)^k$. Then $E = \sum \Pi_{i=1}^k (P_i X^\alpha g Q_i)$, where for every i, P_i, Q_i are monomials in the form $A^{i_1} X^{j_1} A^{i_2} X^{j_2} \dots$. Assume, for instance, that $P_i = A^{i_1} X^{j_1} A^{i_2} X^{j_2} A^{i_3} X^{j_3}$. By Lemma 17,

$$\begin{aligned} P_i &= A^{i_1} X^{j_1} A^{i_2} \left(\sum_r A^r u_r \right) X^{j_3} = \sum A^{i_1} X^{j_1} A^{i_2+r} u_r X^{j_3} \\ &= \sum \sum A^{i_1} A^s w_s u_s X^{j_3} \end{aligned} \quad (23)$$

and finally P_i, Q_i are in the form $\sum_j A^j p_j, \sum_j A^j q_j$, where p_j, q_j are polynomials in X . Now

$$\begin{aligned} E &= \sum \Pi_{i=1}^k \left(\sum_j A^j p_j \right) X^\alpha g \left(\sum_j A^j q_j \right) \\ &= \sum (A^{i_1} p_1 X^\alpha g A^{j_1} q_1) \dots (A^{i_k} p_k X^\alpha g A^{j_k} q_k) \\ &= \sum (A^{i_1} \tilde{v}_1) \dots (A^{i_k} \tilde{v}_k), \end{aligned} \quad (24)$$

where for every $i, \text{val}(\tilde{v}_i) \geq \alpha$. Using Lemma 18, $E = \sum_i A^i v_i$, where for every $i, \text{val}(v_i) \geq k\alpha$. $\square$

We deduce Proposition 4.

Proof. Use Lemmas 18 and 19 and the fact that X is a nilpotent matrix. $\square$

Remark 20. (i) When R is an arbitrary algebraically closed field, the previous result is equivalent to say that A and X are ST over R (cf. [13] and compare with Theorem 1).

(ii) When R is a ring, McCoy, in [14], gave an equivalent condition that, unfortunately, seems almost useless. In fact, if $AB = BA$ and A, B are triangularizable over R , then they have not necessarily a common eigenvector; the following example, for $n = 2$, is due to J. Starr and uses the ring of dual numbers:

$$R = \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}, \quad A = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}. \quad (25)$$

Moreover, let $A = \begin{pmatrix} -\epsilon & 2\epsilon+1 \\ 0 & 5\epsilon \end{pmatrix}$; then, unlike the case where the entries of the matrices are in a field, the equation $XA - AX = X^2$ admits a solution $X = \begin{pmatrix} 4\epsilon & -3 \\ 0 & \epsilon \end{pmatrix}$ that does not commute with A ; yet $X^3 = 0$.

Note that we can reduce (theoretically) the resolution of (10) to the case $\alpha = 2$. Indeed, let (A, B) be satisfying (10). According to (5), $A^{\alpha-1}B - BA^{\alpha-1} = (\alpha - 1)A^{2\alpha-2}$. If $\alpha - 1$ is a unit, then put $B = (\alpha - 1)B_1$, $A_1 = A^{\alpha-1}$; therefore, $A_1B_1 - B_1A_1 = A_1^2$.

We have a more precise result when A is diagonalizable and its spectrum is “good.”

Lemma 21. Let $A \in M_n(R)$ be similar to $\text{diag}(\lambda_1, \dots, \lambda_n)$ and $B \in M_n(R)$ such that $AB = BA$.

- (i) Assume that, for every $i \neq j$, $\lambda_i - \lambda_j$ is not a zero divisor in R . Then A and B are simultaneously diagonalizable.
- (ii) Assume that, for every $i \neq j$, $\lambda_i - \lambda_j$ is a unit. Then B is a polynomial in A of degree at most $n - 1$ and with coefficients in R .

Proof. We may assume that $A = \text{diag}(\lambda_1, \dots, \lambda_n)$.

- (i) If $B = [b_{ij}]$, then $AB - BA = [c_{ij}]$ with $c_{ij} = (\lambda_i - \lambda_j)b_{ij}$; if $i \neq j$, then $b_{ij} = 0$.
- (ii) According to (i), we may assume that $B = \text{diag}(\mu_1, \dots, \mu_n)$. We must solve the linear system, in the unknowns $(\alpha_i)_{0 \leq i < n}$:

$$\text{for every } j, \quad \mu_j = \sum_i \alpha_i \lambda_j^i. \quad (26)$$

Since the determinant of the associated Vandermonde matrix is a unit, we are done. $\square$

Proposition 22. Let $A \in M_n(R)$ be similar to $\text{diag}(\lambda_1, \dots, \lambda_n)$ such that, for every $i \neq j$, $\lambda_i - \lambda_j$ is not a zero divisor and $B \in M_n(R)$. If A and $[A, B]$ commute, then $AB = BA$.

Proof. We may assume that $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ and put $B = [b_{ij}]$, $D = [d_{ij}] = A[A, B] - [A, B]A$. We obtain $d_{i,j} = (\lambda_i - \lambda_j)^2 b_{ij} = 0$; therefore, if $i \neq j$, then $b_{ij} = 0$. $\square$

The proof of the following is interesting in itself; yet, Theorem 5 is a stronger result.

Proposition 23. Let $A \in M_n(R)$ be similar in R to $\text{diag}(\lambda_1, \dots, \lambda_n)$ such that, for every $i \neq j$, $\lambda_i - \lambda_j$ is not a zero divisor. If $n!$ is not a zero divisor and $X \in M_n(R)$ is a solution of (1), then there are $P \in GL_n(R)$ and $\{\mu_i\}_{1 \leq i \leq n} \in R^n$ such that

$$\begin{aligned} A &= P \text{diag}(\lambda_1, \dots, \lambda_n) P^{-1}; \\ X &= P \text{diag}(\mu_1, \dots, \mu_n) P^{-1}; \end{aligned} \quad (27)$$

for every i , $\mu_i^\alpha = 0$.

Proof. According to Proposition 14, X is nilpotent. Assume that the nilindex of X is $i + \alpha - 1$. According to (5), X^i and A commute. Since $X^{i-\alpha+1}A - AX^{i-\alpha+1} = (i - \alpha + 1)X^i$, $[X^{i-(\alpha-1)}, A]$ and A commute. According to Proposition 22, $[X^{i-(\alpha-1)}, A] = 0$. Then we obtain a finite sequence of matrices

$$(X^{i-k(\alpha-1)})_{0 \leq k < i/(\alpha-1)} \quad (28)$$

that commute with A and where, at each step, the exponent decreases by $\alpha - 1$. Finally, we obtain a matrix X^β , with $\beta \in \llbracket 1, \alpha - 1 \rrbracket$, that commutes with A . Since $X^{\beta+\alpha-1} = 0$, $[X^\alpha, A] = \alpha X^{\beta+\alpha-1} X^{\alpha-\beta} = 0$. Finally $[X, A]$ and A commute that implies $[A, X] = 0$. $\square$

We show Theorem 5.

Proof. We may assume that $A = \text{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}) = \text{diag}(\mu_1, \dots, \mu_n)$.

(i)

- (a) For every $k \geq 1$, $\ker((A - \lambda_1 I)^k) = [e_1, \dots, e_{n_1}]$, where $(e_i)_{1 \leq i \leq n}$ is the canonical basis of R^n . Indeed, if $x = [x_1, \dots, x_n]^T$ and $(A - \lambda_1 I)^k x = 0$, then, for every i , $(\mu_i - \lambda_1)^k x_i = 0$. Thus, for every $i > n_1$, $x_i = 0$.
- (b) $X(\ker(A - \lambda_1 I)) \subset \ker(A - \lambda_1 I)$. Indeed, this comes from the proof of the following.

Theorem 24 (cf. [1, Theorem 3]). Let K be an algebraically closed field with characteristic 0 and $A \in M_n(K)$. Assume that the matrix $X \in M_n(K)$ satisfies (2) and that X is a nilpotent matrix. Then, the generalized eigenspaces of A are X -invariant.

The formal proof remains valid when K is replaced with a ring and the generalized eigenspaces simply are eigenspaces. Therefore, $AX = XA$.

- (c) In the same way as in the proof of Lemma 21(i), we obtain that X has the required form.

(ii) In $M_n(R)$, $g(0)I_n$ is a unit and $g(X) - g(0)I_n$ is nilpotent; therefore, $g(X)$ is a unit and we are done. $\square$

Remark 25. In the previous proposition, consider a matrix X_i . According to [15, Theorem 8.54], $\det(g(X_i))$ is the resultant $\text{Res}(\chi_{X_i}, g)$. Therefore, if $\text{Res}(\chi_{X_i}, g)$ is a unit, then X_i^α is zero again; in general, it is not, as we see in the following instance. Let $\tau \in R$ such that $\tau^2 \neq 0$, $\tau^3 = 0$, $X_i = \tau I_2$, $g(x) = \tau$, and $\alpha = 2$. Then, $X_i^2 \neq 0$, $X_i^3 = 0$, and $X_i^2 g(X_i) = 0$.

Disclosure

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Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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