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Research Article

Optimal Foreign Exchange Rate Intervention in Lévy Markets

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This paper considers an exchange rate problem in Lévy markets, where the Central Bank has to intervene. We assume that, in the absence of control, the exchange rate evolves according to Brownian motion with a jump component. The Central Bank is allowed to intervene in order to keep the exchange rate as close as possible to a prespecified target value. The interventions by the Central Bank are associated with costs. We present the situation as an impulse control problem, where the objective of the bank is to minimize the intervention costs. In particular, the paper extends the model by Huang, 2009, to incorporate a jump component. We formulate and prove an optimal verification theorem for the impulse control. We then propose an impulse control and construct a value function and then verify that they solve the quasivariational inequalities. Our results suggest that if the expected number of jumps is high the Central Bank will intervene more frequently and with large intervention amounts hence the intervention costs will be high.

1. Introduction

Exchange rate can be described as the value of foreign nation's currency in terms of home nation's currency. Exchange rate policy is an important tool for the Central Bank in its quest to control volatility of the exchange rate. The Central Bank controls the volatility of the exchange rate by keeping it as close as possible to a prespecified target [1]. According to Kercheval and Moreno [2], there are two types of interventions which can be made by the Central Bank, and these are adjustment of domestic interest rate levels and purchases or sales of foreign currency reserves. Purchasing and selling of foreign currency reserves lead to an impulse stochastic control problem, which is solved using quasivariational inequalities [3].

This research considers intervention by purchasing and selling reserves. The exchange rate should always be maintained within a band or interval around the target rate, determined by the country's Central Bank [2]. If the exchange rate is higher than the target, the Central Bank will release the foreign currency in their reserves to the market and simultaneously hold onto the domestic currency. Such an intervention will create demand for the domestic currency. On the other hand if the exchange rate is lower than the target, the Central Bank will release domestic currency in their reserves to the market and simultaneously buy the foreign currency in the market and this move will create demand for the foreign currency [4].

The exchange rate will always have a tendency to move out of the target interval or target zone. The duty of the Central Bank is to come up with an optimum intervention strategy, that is, to determine the right time to intervene and the appropriate intervention size or amount. The Central Bank may experience very high costs when controlling the exchange rate. These high costs lead to failure to control the exchange rate and as a result the exchange rate may be characterised by fluctuations. Fluctuations create uncertainty in trade and arbitrage opportunities.

The theory of stochastic impulse control in controlling the exchange rate was first applied by Jeanblanc-Picque [5]. She modeled the evolution of the exchange rate as a stochastic process. In order to keep the exchange rate within a given interval; the Central Bank uses impulse control methods.

Korn [6] later extended method of Jeanblanc-Picque [5] and he analysed thoroughly the theory of classical stochastic

control to obtain facts on impulse control problem. He found the minimum intervention cost in the impulse control by an iterative method.

The work in Mundaca and Øksendal [7] also described an optimal intervention policy for Central Banks in order to stabilize the exchange rate. Their paper differs from Jeanblanc-Picque [5] and Korn [6] in the sense that it allows two types of control, namely, intervention at discrete time instants to control the dynamics of the exchange rate and continuous control in the domestic interest rate market. Another difference is that both Jeanblanc-Picque [5] and Korn [6] considered an exogenously (nonchanging) specified target interval within which the exchange rate is to be contained whilst Mundaca and Øksendal [7] considered an endogenous exchange rate target interval. They also used a standard Brownian motion for the underlying exchange rate.

Cadenillas and Zapatero [8] in their first paper only use impulse control as in Jeanblanc-Picque [5] and the difference is that they did not consider the target interval to be exogenous. In their second paper, Cadenillas and Zapatero [9] used both classical and impulse stochastic control methods as in Mundaca and Øksendal [7]. The difference was that they considered the exchange rate to be geometric Brownian motion rather than a pure or standard Brownian motion. Mundaca and Øksendal [7] only gave some general analysis under some general assumptions while Cadenillas and Zapatero [9] gave numerical examples and evaluated a model based on those examples.

Both Mundaca and Øksendal [7] and Cadenillas and Zapatero [9] assumed that investors do not observe or anticipate the interventions of the Central Bank. The work in Kercheval and Moreno [2] extended the work done on optimal impulse control in Cadenillas and Zapatero [9] to incorporate temporary market reactions. They obtained a new explicit optimal impulse control strategy that accounted for these market reactions and showed that they cannot be obtained simply by adjusting the intervention cost in a model without market reactions.

Huang [1] formulated the impulse control problem and introduced the main theorem by illustrating work in Cadenillas and Zapatero [8], but the proof of the theorem is modified using the idea in Kercheval and Moreno [2]. He also demonstrated the numerical algorithm and gave several examples which have different parameters from those in Cadenillas and Zapatero [8]. Huang [1] also assumed that there are no market reactions in his work as done before by Kercheval and Moreno [2] but modified their idea of the proof of the theorem.

Silva [4] adopted the model in Mundaca and Øksendal [7] with a geometric Brownian motion as in Cadenillas and Zapatero [9]. A geometric Brownian motion allowed him to simulate speculative attacks and to check the behaviour of the model in reaction to a disturbance since a drift may be included. They wrote their paper considering a case study of Brazil and concluded that the Brazilian risk management strategy of increasing holdings of international reserves and decreasing short foreign exchange rate exposure in domestic public debt after 2004 gave the country more flexibility to manage foreign exchange rate risk in 2008 and to avoid higher interest rates to attract international capital as was necessary in previous crisis.

Perera [10] studied the Central Bank intervention problem in the foreign exchange market when the market observes and reacts to the bank's interventions. They first modelled an impulse control problem when the controller's action affects the state as well as the dynamics of the state processes for a random amount of time. They then applied their model to solve the Central Bank intervention problem. Their results suggest that the Central Bank would intervene less (more) frequently and the optimal policy is more (less) expensive than its corresponding value without market reactions if the market reactions increase (decrease) the exchange rate volatility.

From the above researches, empirical results are also disappointing regarding the ability to explain future exchange rate movements for currencies. The recent periods of turbulence in foreign exchange markets have renewed interest in the difficult task of identifying the optimal intervention times and sizes. A good example of a country with difficulties in explaining the future trends of their currency is Nigeria. Prior to the structural adjustment program (SAP), Naira enjoyed appreciable value against United States dollar, a factor that created rapid opportunity for economic growth and stability. With the introduction of new economic program, the country began to suffer unstable exchange rate that caused high degree of uncertainty in the Nigeria business environment [11]. Domestic investors face enormous risk as no one, no matter how intelligent, could predict the likelihood of foreign exchange market performance. So a model which includes abrupt changes and turbulence in foreign currency is needed.

This study is extending the model from Huang [1] by including the jump component since exchange rate may have jumps in its course rather than being continuous all the time. Jump diffusion models provide a more realistic description of the evolution of price processes of financial assets. However, explicit solutions are difficult to obtain in jump diffusion models. According to Ayuso and Vega [12], jumps are abrupt changes in the exchange rate (both negative and positive) when no devaluations occur. Barndorff-Nielsen and Shephard [13] suggest that jumps in the foreign exchange market are linked to the arrival of macroeconomic news. Since persistent time-varying diffusion would help forecast diffusion volatility, jumps might contain no predictive information or distort volatility forecasts [14, 15]; therefore it makes sense to model the exchange rate dynamics as a geometric Lévy process. This paper will also provide a numerical solution by using geometric Lévy process. We intend to use some of the ideas from Øksendal and Sulem [16] who used impulse stochastic control with jump components in the areas of forest management and stream of dividends among others.

One major contribution is the formulation of quasi-variational inequalities (QVI) that involve an integrodifferential equation which is not easy to solve. However, by carefully applying the method of undetermined coefficients, we managed to find an explicit solution for the impulse control problem. We also propose a new numerical scheme,

which caters for a jump component in our model and some numerical analysis is done to illustrate the numerical scheme.

The rest of the paper is organised as follows. In Section 2 we present Background and Problem Formulation. The issue of quasivariational inequalities (QVI) and verification theorem will be addressed in Section 3. In Section 4 the solution to the QVI is given. Conclusions and recommendations are covered in Section 5.

2. Background and Problem Formulation

To place our discussion in a rigorous mathematical framework, we consider a probability space $(\Omega, \mathcal{F}, P)$ together with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a one-dimensional Brownian motion W_t.

Let $X(t)$ be the exchange rate in a given economy at time t. In this paper we understand exchange rate to mean domestic currency units per unit of foreign currency at time t.

Motivated by the model in Huang [1], we assume that, in the absence of intervention, $X(t)$ is a jump diffusion of the form

$$dX(t) = X(t^-)\left[\mu\, dt + \sigma\, dW_t + \theta \int_{\mathbb{R}} z\widetilde{N}(dt, dz)\right], \qquad X(0) = x > 0, \tag{1}$$

where μ, σ, θ are positive constants and $\widetilde{N}(dt, dz)$ is a compensated Poison random measure given by $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, where $\nu(\cdot)$ is a Lévy measure.

Now, suppose that the Central Bank is allowed to intervene so as to control the exchange rate with the objective of keeping it as close to a prespecified target value ρ as possible. When the exchange rate is significantly above or below the target value ρ, then the bank has to sell or buy foreign currency, respectively, in order to influence the exchange rate. The selling and buying of foreign currency are associated with transaction costs as indicated before, where the cost of selling foreign currency is higher than the cost of buying foreign currency, since purchasing foreign currency increases international reserves. The objective of the bank is to minimize these transaction costs subject to certain constraints. The problem of the bank is to choose appropriate instants at which they have to effect these transactions and the amount of foreign exchange that has to be transacted. The situation can be described mathematically as an impulse control problem.

Definition 1. An impulse control

$$u = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots) \tag{2}$$

is a double sequence of intervention times τ_i and intervention sizes ξ_i, where $\{\tau_i\}_{n=1}^{\infty} = \{\tau_1, \tau_2, \ldots\}$ is an infinite sequence of stopping times with respect to the filtration $\mathcal{F}_t$ satisfying

$$0 \leq \tau_1 < \tau_2 < \tau_3 < \tau_4 \cdots \tag{3}$$

and each $\xi_i : \Omega \to \mathbb{R}$ is $\mathcal{F}_{\tau_i}$, measurable.

Suppose that the controlled exchange rate process with an initial value of x is denoted by $X_x^u(t)$ and is defined in two cases as follows.

Case 1. If $\tau_i > 0$,

$$\begin{aligned} X_x^u(t) &= X_x(t), \quad 0 \leq t < \tau_1 \\ X_x^u(\tau_i) &= X_x^u(\tau_i^-) + \xi_i, \quad i = 1, 2, 3, \ldots. \end{aligned} \tag{4}$$

Case 2. If $\tau_i = 0$,

$$\begin{aligned} X_x^u(\tau_i^-) &= x \\ X_x^u(\tau_i) &= X_x^u(\tau_i^-) + \xi_i, \quad i = 1, 2, 3, \ldots. \end{aligned} \tag{5}$$

Here the drift and volatility are not affected when the exchange rate process is shifted by control. So after the exchange rate process is controlled it follows the jump diffusion dynamics up to the time the Central Bank decides to control again. The process is given by

$$\begin{aligned} dX_x^u(t) = X_x^u(t^-)\Big[&\mu dt + \sigma X_x^u(t)\, dW_t \\ &+ \theta X_x^u(t) \int_{\mathbb{R}} z\widetilde{N}(dt, dz)\Big], \\ &X_x^u(0) = x. \end{aligned} \tag{6}$$

Define the performance functional $J^u(x)$ by

$$J^u(x) = E\left[\int_0^{\infty} e^{-rt} f(X_x^u(t))\, dt + \sum_{i=1}^{\infty} e^{-r\tau_i} g(\xi_i)\, I_{\{\tau_i < \infty\}}\right], \tag{7}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the running cost function, $g : \mathbb{R} \to \mathbb{R}$ is the intervention cost function, r is the discount rate, and $I(\cdot)$ is the indicator function.

We take $f(\cdot)$ as

$$f(x) = (x - \rho)^2 \tag{8}$$

and $g(\cdot)$ as follows:

$$g(\xi) = \begin{cases} C + c\xi, & \xi > 0 \\ \min(C, D), & \xi = 0 \\ D + d\xi, & \xi < 0 \end{cases} \qquad C, c, D, d, \rho \in (0, \infty), \tag{9}$$

where f denotes running costs incurred when the exchange rate x moves away from the target exchange rate ρ. When the Central Bank controls the exchange rate by pushing it upwards, C denotes fixed costs of controlling and c denotes proportional costs of controlling while D and d represent the fixed and proportional costs, respectively, of controlling when the Central Bank pushes the exchange rate downwards.

The case $\xi > 0$ implies that the Central Bank releases domestic currency into the market simultaneously buying foreign currency from the market, thus pushing the exchange

rate upwards. It also means that if the intervention amount $\xi < 0$, then the Central Bank holds onto the domestic currency and releases foreign markets from their reserves to the market, thus pushing the exchange rate downwards. Lastly, when the intervention amount $\xi = 0$, no amount is released or bought from the market, and the Central Bank only incurs the minimum of the fixed costs C or D.

Definition 2. An impulse control $u = (\tau_1, \tau_2, \ldots, \tau_n \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$ is called admissible if we have

$$X_x^u(\tau_i) > 0 \quad \forall i, \tag{10}$$

$$E\left[\int_0^\infty e^{-rt} f\left(X_x^u(t)\right) dt\right] < \infty, \tag{11}$$

$$P\left(\left[\lim_{t\to\infty} \tau_i \le t\right]\right) = 0, \quad \forall t \ge 0, \tag{12}$$

$$\lim_{t\to\infty} E\left[e^{-rt}\left(X_x^u(t)\right)\right] = 0. \tag{13}$$

We will denote the set of all admissible controls by Γ.

The problem is to find the value function $V(x)$ and the associated optimal control u^* such that

$$V(x) = J^{u^*}(x) = \inf_{u\in\Gamma} J^u(x). \tag{14}$$

3. Quasivariational Inequalities (QVI) and Verification Theorem

Definition 3. For a function $\phi : \mathbb{R}^+ \to \mathbb{R}$ and $x \in \mathbb{R}^+$, $\xi \in \mathbb{R}$, define the optimal intervention operator $\mathscr{M}$ as follows:

$$\mathscr{M}\phi(x) = \inf_\xi \left\{g(\xi) + \phi(x+\xi) : \xi \in \mathbb{R},\ \xi + x \in \mathbb{R}^+\right\}, \tag{15}$$

where $g(\cdot)$ is the intervention cost function defined as in (7).

We define the Operator $\mathscr{L}$ as follows:

$$\mathscr{L}\phi(x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\phi(x) + \mu x \frac{\partial}{\partial x}\phi(x) - r\phi(x) + \int_{\mathbb{R}} \left\{\phi(x + x\theta z) - \phi(x) - \phi'(x)\theta z x\right\} \nu(dz). \tag{16}$$

The notion of quasivariational inequalities is defined below.

Definition 4. One says that a function $\phi \in C^2(\mathbb{R})$ satisfies the quasivariational inequalities (QVI) associated with problem (14) if ϕ satisfies the following three conditions:

$$\mathscr{L}\phi(x) + f(x) \ge 0, \tag{17}$$

$$\phi(x) \le \mathscr{M}\phi(x), \tag{18}$$

$$\left(\mathscr{L}\phi(x) + f(x)\right)\left(\phi(x) - \mathscr{M}\phi(x)\right) = 0. \tag{19}$$

Note that the solution ϕ of the QVI divides the space $(0, +\infty)$ into two subspaces: a continuation/nonintervention region

$$D = \left\{x \in (0, +\infty) : \phi(x) \le \mathscr{M}\phi(x),\ \mathscr{L}\phi(x) = 0\right\} \tag{20}$$

and the intervention region

$$\Sigma = \left\{x \in (0, +\infty) : \phi(x) = \mathscr{M}\phi(x),\ \mathscr{L}\phi(x) \le 0\right\}. \tag{21}$$

We will show that the solution to the QVI above is exactly the solution to the optimal problem (14) if the inequalities are satisfied. Before proving it, we can construct the following impulse control from the solution to the QVI.

Definition 5. Let ϕ be a continuous solution of the QVI defined above. Then the following impulse control is called the QVI-control associated with ϕ (if it exists):

$$\tau_1 = \inf\left\{t \ge 0 : \phi\left(X_x^u(t^-)\right) = \mathscr{M}\phi\left(X_x^u(t^-)\right)\right\} \tag{22}$$

$$\tau_i = \inf\left\{t > \tau_{i-1} : \phi\left(X_x^u(t^-)\right) = \mathscr{M}\phi\left(X_x^u(t^-)\right)\right\}, \quad i = 2, 3, 4 \ldots \tag{23}$$

$$\xi_i = \arg\inf_\xi \left\{g(\xi) + \phi\left(X_x^u(\tau_i^-) + \xi\right) : \xi \in \mathbb{R},\ \xi + X_x^u(\tau_i^-) \in \mathbb{R}^+\right\}, \quad i \in \mathbb{N}. \tag{24}$$

This means that the Central Bank intervenes whenever ϕ and $\mathscr{M}\phi$ coincide and the size of the intervention corresponds to $\mathscr{M}\phi$. Note that $X_x^u(0^-) = x$ in (22).

Theorem 6. *Let $\phi \in C^1(\mathbb{R})$ be a solution of the QVI associated with the problem (14), and suppose that there is a finite subset $\mathscr{N} \subset \mathbb{R}$ such that $\phi \in C^2(\mathbb{R}^+ - \mathscr{N})$. If ϕ satisfies the growth conditions*

$$E\left[\int_0^\infty \left(e^{-rt}\sigma\left(X_x^u(t)\right)\phi'\left(X_x^u(t)\right)\right)^2 dt\right] < \infty, \tag{25}$$

$$\lim_{t\to\infty} E\left[e^{-rt}\phi\left(X_x^u(t)\right)\right] = 0, \tag{26}$$

for every process $X_x^u(t)$ corresponding to an admissible impulse control u, then for every $x \in \mathbb{R}^+$

$$V(x) \ge \phi(x). \tag{27}$$

Moreover, if the QVI-control corresponding to ϕ is admissible then it is an optimal impulse control, and for every $x \in \mathbb{R}^+$

$$V(x) = \phi(x), \tag{28}$$

where $V(x)$ is the value function defined in (14).

Proof. Consider any admissible control $u = \{(\tau_n, \xi_n)\}_{n\in\mathbb{N}}$. Define $\tau^*(t) = \max\{\tau_i : \tau_i \le t\}$; note that, almost surely, $\tau^*(t) \to \infty$ as $t \to \infty$, due to the admissibility condition (12). We can write

$$\begin{aligned} &e^{-r\tau*(t)}\phi\left(X_x^u(\tau^*(t))\right) - \phi(x) \\ &= \sum_{i=1}^\infty I_{\{\tau_i \le t\}}\left[e^{-r\tau_i}\phi\left(X_x^u(\tau_i^-)\right) - e^{-r\tau_{i-1}}\phi\left(X_x^u(\tau_{i-1})\right)\right] \\ &\quad + \sum_{i=1}^\infty I_{\{\tau_i \ge t\}} e^{-r\tau_i}\left[\phi\left(X_x^u(\tau_i)\right) - \phi\left(X_x^u(\tau_i^-)\right)\right]. \end{aligned} \tag{29}$$

Note that, here, we make a convention that $X_x^u(0^-) = x$ and $e^{-r\tau_0}\phi(X_x^u(\tau_0)) = \phi(x)$. Between τ_{i-1} and τ_i^-, $X_x^u(t)$ actually follows the Lévy process (1), so an application of Ito's formula (see Ruijter [17], Theorem 2.5, and Protter [18], Theorem 32) gives

$$\begin{aligned} e^{-r\tau_i}\phi(X_x^u(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^u(\tau_{i-1})) &= \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\mathscr{L}\phi(X_x^u(s))\,ds \\ &\quad + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi'(X_x^u(s))\,\sigma(X_x^u(s))\,dW_s \\ &\quad + \sum_{0<\tau_{i-1}<\tau_i}\left[\phi(X_x^u(s)) - \phi(X_x^u(s^-))\right]. \end{aligned} \tag{30}$$

By inequality (17), this expression becomes

$$\begin{aligned} e^{-r\tau_i}\phi(X_x^u(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^u(\tau_{i-1})) &\geq \int_{\tau_{i-1}}^{\tau_i} e^{-rs}(-f(X_x^u(s)))\,ds \\ &\quad + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi'(X_x^u(s))\,\sigma(X_x^u(s))\,dW_s \\ &\quad + \sum_{0<\tau_{i-1}<\tau_i}\left[\phi(X_x^u(s)) - \phi(X_x^u(s^-))\right]. \end{aligned} \tag{31}$$

If τ_{i-1} and τ_i are the intervention times defined in (22) and (23), then $\phi(X_x^u(s)) < \mathscr{M}\phi(X_x^u(s))$ for $\tau_{i-1} \leq s < \tau_i$, so $\mathscr{L}\phi(X_x^u(s)) + f(X_x^u(s)) = 0$ by Definition 4 of QVI. So the inequality above becomes an equality for the QVI-control associated with ϕ. Note that $\xi_i = X_x^u(\tau_i) - X_x^u(\tau_i^-)$; according to inequality (18), we have

$$e^{-r\tau_i}\left[\phi(X_x^u(\tau_i)) - \phi(X_x^u(\tau_i^-))\right] \geq -e^{-r\tau_i}g(\xi_i). \tag{32}$$

Also, this inequality becomes an equality for the QVI-control associated with ϕ, since $g(\xi_i) + \phi(X_x^u(\tau_i^-)) = \mathscr{M}\phi(X_x^u(\tau_i^-)) = \phi(X_x^u(\tau_i))$ if (τ_i, ξ_i) is the impulse control defined in Definition 5 of the QVI-control. Therefore combining the above two inequalities, we obtain

$$\begin{aligned} \phi(x) - e^{-r\tau^*(t)}\phi(X_x^u(\tau^*(t))) &\leq \sum_{i=1}^{\infty} I_{\{\tau_i\leq t\}}\Bigg(e^{-r\tau_i}g(\xi_i) + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}f(X_x^u(s))\,ds \\ &\quad - \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi'(X_x^u(s))\,\sigma(X_x^u(s))\,dW_s \\ &\quad + \sum_{0<\tau_{i-1}<\tau_i}\left[\phi(X_x^u(s)) - \phi(X_x^u(s^-))\right]\Bigg) \\ &= \sum_{i=1}^{\infty} e^{-r\tau_i}g(\xi_i)\,I_{\{\tau_i\leq t\}} + \int_0^{\tau^*(t)} e^{-rs}f(X_x^u(s))\,ds \\ &\quad - \int_{\tau_0}^{\tau^*(t)} e^{-rs}\phi'(X_x^u(s))\,\sigma(X_x^u(s))\,dW_s \\ &\quad + \sum_{i=1}^{\infty}\sum_{0<\tau_{i-1}<\tau_i}\left[\phi(X_x^u(s)) - \phi(X_x^u(s^-))\right]. \end{aligned} \tag{33}$$

Taking expectation, we have

$$\begin{aligned} \phi(x) - E\left[e^{-r\tau^*(t)}\phi(X_x^u(\tau^*(t)))\right] &\leq E\Bigg(\sum_{i=1}^{\infty} e^{-r\tau_i}g(\xi_i)\,I_{\{\tau_i\leq t\}} \\ &\quad + \int_0^{\tau^*(t)} e^{-rs}f(X_x^u(s))\,ds \\ &\quad - \int_{\tau_0}^{\tau^*(t)} e^{-rs}\phi'(X_x^u(s))\,\sigma(X_x^u(s))\,dW_s \\ &\quad + \sum_{i=1}^{\infty}\sum_{0<\tau_{i-1}<\tau_i}\left[\phi(X_x^u(s)) - \phi(X_x^u(s^-))\right]\Bigg). \end{aligned} \tag{34}$$

Let t go to ∞; then $\tau^*(t) \to \infty$, so the left hand side of the above inequality becomes $\phi(x)$ because of the growth condition (26), while the growth condition (25) implies that the expectation of the stochastic integral $\int_0^{\tau^*(t)} e^{-rs}\phi'(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s$ vanishes (see Øksendal [19], Theorem 3.2.1). For all $\epsilon > 0$ we have $E(\sum_{i=1}^{\infty}\sum_{0<\tau_{i-1}<\tau_i}[\phi(X_x^u(s)) - \phi(X_x^u(s^-))]) < \varepsilon$. (This means that there is no delay between the time when a decision for intervention is taken and the time when the intervention is actually carried out.) Now by arbitrariness of ε we obtain

$$\phi(x) \leq E\left[\sum_{i=1}^{\infty} e^{-r\tau_i}g(\xi_i)\,I_{\{\tau_n<\infty\}} + \int_0^{\infty} e^{-rt}f(X_x^u(s))\,ds\right]; \tag{35}$$

that is,

$$\phi(x) \leq J^u(x). \tag{36}$$

As this is true for any control ν, we have

$$\phi(x) \leq V(x). \tag{37}$$

Again, it becomes an equality for the QVI-control associated with ϕ because all the above inequalities become equalities for the QVI-control associated with ϕ. □

4. The Solution to the QVI

In this section we propose an impulse control of (14) and construct a value function $V(x)$ by Theorem 6 and then verify

that they solve QVI inequalities. We assume that if there is no intervention, the exchange rate follows the geometric Lévy process (1) and the running cost and the intervention cost are defined as in (9).

We now propose an optimal impulse control:

$$0 < a < \alpha \leq \beta < b < \infty, \tag{38}$$

where

a represents the lower intervention level,

b represents the upper intervention level,

α is the optimal restarting value when the exchange rate is pushed upwards,

β is the optimal restarting value when the exchange rate is pushed downwards.

This means that the Central Bank intervenes when the exchange rate is below a by pushing it up to α and when the exchange rate is above b the Central Bank intervenes by moving it downwards to β. Hence it is optimal not to control the exchange rate while it is inside the interval (a, b).

That is

$$\begin{gathered}\tau_1 = \inf\{t \geq 0 : X_x^u(t) \notin (a,b)\},\\ \tau_1 = \inf\{t > \tau_{i-1} : X_x^u(t) \notin (a,b)\}, \quad i = 2,3,\ldots,\\ X_x^u(\tau_i) = X_x^u(\tau_i^-) + \xi_i = \beta I_{X_x^u(\tau_i^-)\geq b} + \alpha I_{X_x^u(\tau_i^-)\leq a}, \quad i \in \aleph.\end{gathered} \tag{39}$$

The strategy indicates that the value function $V(x)$ should be of the forms:

$$\begin{aligned}V(x) &= V(\alpha) + C + c(\alpha - x), \quad \text{if } x \in (0,a],\\ V(x) &= V(\beta) + D + d(x-\beta), \quad \text{if } x \in [b,\infty).\end{aligned} \tag{40}$$

Differentiating $V(x)$ at a and b from (40), we get

$$\begin{aligned}V'(a) &= -c,\\ V'(b) &= d.\end{aligned} \tag{41}$$

By the definition of b and β in the conjecture above, we have $V(b) = \mathscr{M}V(b) = V(\beta) + D + d(b-\beta)$, which means that the minimum of $V(y) = \mathscr{M}V(b) = V(\beta) + D + d(b-y)$ is attained at $y = \beta$. So

$$\frac{d}{dy}\left[V(y) + D + d(b-y)\right] = 0, \tag{42}$$

which implies

$$V'(\beta) = d. \tag{43}$$

Similarly, the minimum of $V(y) + C + c(y-a)$ is attained at $y = \alpha$; we have

$$V'(\alpha) = -c. \tag{44}$$

We also propose that, in the region (a, b), $V(x)$ satisfies

$$\mathscr{L}V(x) = -f(x) = -(x-\rho)^2, \quad \text{if } x \in (a,b). \tag{45}$$

This implies

$$\begin{aligned}\mathscr{L}\phi(x) &= \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\phi(x) + \mu x \frac{\partial}{\partial x}\phi(x) - r\phi(x)\\ &\quad + \int_{\mathbb{R}} \left\{\phi(x + x\theta z) - \phi(x) - \phi'(x)\theta z x\right\} \nu(dz)\\ &= -(x-\rho)^2.\end{aligned} \tag{46}$$

Using the method of undetermined coefficients we have

$$\begin{aligned}h_c(x) &: \mathscr{L}\phi(x)\\ &= \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\phi(x) + \mu x \frac{\partial}{\partial x}\phi(x) - r\phi(x)\\ &\quad + \int_{\mathbb{R}} \left\{\phi(x + x\theta z) - \phi(x) - \phi'(x)\theta z x\right\} \nu(dz) = 0.\end{aligned} \tag{47}$$

We propose a solution of the form:

$$\phi(x) = Ax^\gamma, \tag{48}$$

where $\gamma \in \mathbb{R}$ is to be determined and substituting $\phi(x) = Ax^\gamma$ and its derivatives, we have

$$\begin{aligned}&\frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - r\\ &\quad + \int_{\mathbb{R}} \left\{(1+\theta z)^\gamma - 1 - \theta z \gamma\right\} \nu(dz) = 0,\end{aligned} \tag{49}$$

after some algebraic simplifications.

Suppose that

$$\begin{aligned}h(\gamma) &= \frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - r\\ &\quad + \int_{\mathbb{R}} \left\{(1+\theta z)^\gamma - 1 - \theta z \gamma\right\} \nu(dz);\end{aligned} \tag{50}$$

then $h(0) = -r$, $\lim_{\gamma\to+\infty} h(\gamma) = +\infty$, $\lim_{\gamma\to-\infty} h(\gamma) = +\infty$, and the coefficient of γ^2 is $(1/2)\sigma^2 > 0$. We see that there exist two solutions γ_1, γ_2 of $h(\gamma) = 0$ such that

$$\gamma_2 < 0 < \gamma_1, \tag{51}$$

and we get a complementary solution of the form:

$$h_c(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \int_{\mathbb{R}} z^2 \nu(dz), \tag{52}$$

where $\gamma_1, \gamma_2 < 0$ are solutions to the above equation.

Now to get the particular solution we consider

$$h_p(x) = k_1 x^2 + k_2 x + k_3; \tag{53}$$

substituting $h_p(x)$ and its derivatives in $\mathscr{L}\phi(x) = -f(x)$ yields

$$\left(\sigma^2 + 2\mu - r\right) k_1 x^2 + (\mu - r) k_2 x - r k_3 = -x^2 + 2\rho x - \rho^2. \tag{54}$$

Comparing coefficients of x^2, x, and x^0, we have $k_1 = -1/(\sigma^2 + 2\mu - r)$, $k_2 = 2\rho/(\mu - r)$, and $k_3 = \rho^2/r$ and substituting k_1, k_2, and k_3 in $h_p(x)$ we get

$$h_p(x) = \frac{x^2}{-\sigma^2 - 2\mu + r} - \frac{2\rho x}{r - \mu} + \frac{\rho^2}{r}. \tag{55}$$

The solution is $h(x) = h_c(x) + h_p(x)$ and we get

$$\begin{aligned} h(x) = {} & A x^{\gamma_1} + B x^{\gamma_2} + \frac{x^2}{-\sigma^2 - 2\mu + r} - \frac{2\rho x}{r - \mu} + \frac{\rho^2}{r} \\ & + \int_{\mathbb{R}} z^2 \nu(dz), \end{aligned} \tag{56}$$

where A and B are constants.

Note that the value function has properties (40)-(41) when $x \in [0, a) \cup [b, \infty)$ and $V(x)$ is exactly $h(x)$ when $x \in (a, b)$ by the continuity of $V(x)$ and $V'(x)$ and at the connecting points a and b we have

$$\begin{aligned} h(a) &= h(\alpha) + C + c(\alpha - a), \\ h(b) &= h(\beta) + D + d(b - \beta), \\ h'(a) &= -c, \\ h'(b) &= d. \end{aligned} \tag{57}$$

Since $V(x)$ is exactly $h(x)$ when $x \in (a, b)$ so from the properties (44)-(43) we have

$$\begin{aligned} h'(\alpha) &= -c, \\ h'(\beta) &= d. \end{aligned} \tag{58}$$

We can obtain the six unknowns $A, B, a, b, \alpha, \beta$ from the above system of (57)-(58).

The theorem below is used to prove the conjecture stated above.

Theorem 7. *Let $h(x)$ be defined as in (56) and let $A, B, a, b, \alpha, \beta$ with $0 < a < \alpha \le \beta < b < \infty$ be a solution of the system of (57)-(58). Define the function $V : (0, \infty) \to [0, \infty)$ by*

$$V(x) = \begin{cases} h(\alpha) + C + c(\alpha - x), & x \le a \\ h(x), & a < x < b \\ h(\beta) + D + d(x - \beta), & x \ge b, \end{cases} \tag{59}$$

and if

$$\begin{aligned} a < {} & \frac{1}{2}\left[c(\mu - r) + 2\rho\right] \\ & - \frac{1}{2}\left[\left((c(\mu - r) + 2\rho)^2 \right.\right. \\ & \quad - 4\left(\rho^2 + r\int_{\mathbb{R}} z^2 \nu(dz)\right. \\ & \quad \left.\left. \left. - rV(\alpha) - rC - c\alpha\right)\right)^{1/2}\right], \end{aligned} \tag{60}$$

$$\begin{aligned} b > {} & \frac{1}{2}\left[d(\mu - r) + 2\rho\right] \\ & - \frac{1}{2}\left[\left((d(\mu - r) + 2\rho)^2 \right.\right. \\ & \quad - 4\left(\rho^2 + r\int_{\mathbb{R}} z^2 \nu(dz)\right. \\ & \quad \left.\left. \left. - rV(\beta) - rD - d\beta\right)\right)^{1/2}\right], \end{aligned} \tag{61}$$

$$-c < V'(x) < d, \quad \forall \alpha < x < \beta, \tag{62}$$

$$V'(x) \ge d \quad \forall a \le x \le \beta, \tag{63}$$

$$V'(x) \ge -c \quad \forall \beta \le x \le b, \tag{64}$$

then $V(x)$ is the value function of the problem (14) and the strategy (39) is the corresponding optimal impulse control.

Proof. The proof is in two parts (a) and (b).

(a) We show that V satisfies the QVI (17)–(19).

(i) First inequality:

$$\begin{aligned} & \mathscr{L}V(x) + f(x) \\ & = \begin{cases} -c\mu x - r\left[h(\alpha) + C + c(\alpha - x)\right] \\ \quad + (x - \rho)^2, & x \le a, \\ \mathscr{L}h(x) + (x - \rho)^2, & a < x < b, \\ d\mu x - r\left[h(\beta) + D + d(x - \beta)\right] \\ \quad + (x - \rho)^2, & x \ge b. \end{cases} \end{aligned} \tag{65}$$

We have $\mathscr{L}V(x) + f(x) = 0$ in interval (a, b) by construction of $h(x)$. Condition (60) implies that $\mathscr{L}V(x) + f(x) > 0$ in $(0, a]$ and (61) implies that $\mathscr{L}V(x) + f(x) > 0$ in $[b, \infty)$.

(ii) Second inequality:

$$\mathscr{M}V(x) = \begin{cases} h(\alpha) + C + c(\alpha - x), & x \le \alpha, \\ h(x) + \min(C, D), & \alpha < x < \beta, \\ h(\beta) + D + d(x - \beta), & x \ge \beta. \end{cases} \tag{66}$$

$\mathscr{M}V$ in the interval (α, β) using condition (62). Thus in the intervention region $(0, a] \cup [b, \infty)$,

$V(x) - \mathscr{M}V(x) = 0$ and in the region (a, b), $V(x) - \mathscr{M}V(x) < 0$ because of the conditions (63)-(64).

(iii) Third inequality: $(\mathscr{L}V(x) + f(x))(V(x) - \mathscr{M}V(x)) = 0$ follows automatically from the inequalities in (65) and (66).

(b) In this part we show that V satisfies the growth conditions (25) and (26) such that $V'(x)$ is continuous in $[a, b]$ and is constant in $(0, a]$ and $[b, \infty)$, so $V(x)$ is bounded. Besides, note that σ is a constant in this example, so $V(x)$ satisfies the growth condition (25). Let u be any admissible control, so it satisfies the admissibility condition (13); that is, $\lim_{t\to\infty} E[e^{-rt}(X_x^u(t))] = 0$, so $V(x)$ satisfies the growth condition (26), since $V(x)$ is bounded in $[a, b]$ and is just linear in $(0, a]$ and $[b, \infty)$.

So by Theorem 6, $V(x)$ is the value function and the strategy (39) is the corresponding optimal impulse control. □

5. Numerical Algorithm

In this section Newton's method to solve the nonlinear system of ((57)-(58)) for $a, \alpha, \beta, b, A, B$ using Matlab is outlined. Explanations for the effects for the changes of different parameters on optimal intervention strategy are also given.

Step 1. Define $F = (f_1, f_2, f_3, f_4, f_5, f_6) : \mathbb{R}^6 \to \mathbb{R}^6$, and the components are defined below.

Let

$$\begin{aligned} a &= x_1, \qquad \alpha = x_2, \qquad \beta = x_3, \\ b &= x_4, \qquad A = x_5, \qquad B = x_6 \end{aligned} \tag{65}$$

such that

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4, x_5, x_6) &= h(x_1) - h(x_2) \\ &\quad - C - c(x_2 - x_1), \\ f_2(x_1, x_2, x_3, x_4, x_5, x_6) &= h(x_4) - h(x_3) \\ &\quad - D - d(x_4 - x_3), \\ f_1(x_1, x_2, x_3, x_4, x_5, x_6) &= h'(x_1) + c, \\ f_1(x_1, x_2, x_3, x_4, x_5, x_6) &= h'(x_4) - d, \\ f_1(x_1, x_2, x_3, x_4, x_5, x_6) &= h'(x_2) + c, \\ f_1(x_1, x_2, x_3, x_4, x_5, x_6) &= h'(x_3) - d. \end{aligned} \tag{66}$$

Step 2. Define JF which is a 6×6 Jacobian matrix of F at x.

Step 3. Use a Matlab function for Newton's method for nonlinear systems in Fausett [20] page 142 to evaluate J and JF at initial guess (x_0), preset tolerance (tol), and maximum number of iterations (max-it).

Step 4. Output x.

The algorithm above gives us the results for $a, \alpha, \beta, b, A, B$ with different parameters.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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