Research Article

On Almost Semiprime Submodules

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We introduce the concept of almost semiprime submodules of unitary modules over a commutative ring with nonzero identity. We investigate some basic properties of almost semiprime and weakly semiprime submodules and give some characterizations of them, especially for (finitely generated faithful) multiplication modules.

1. Introduction

Throughout this paper, all rings are commutative rings with identity and all modules are unitary. Various generalizations of prime (primary) ideals are studied in [1–8]. The class of prime submodules of modules as a generalization of the class of prime ideals has been studied by many authors; see, for example, [9, 10]. Then many generalizations of prime submodules were studied such as weakly prime (primary) [11], almost prime (primary) [12], 2-absorbing [13], classical prime (primary) [14, 15], and semiprime submodules [16]. In this paper, we study weakly semiprime and almost semiprime submodules as the generalizations of semiprime submodules. Weakly semiprime submodules have been already studied in [17]. Here we first define the notion almost semiprime submodules and get a number of propensities of almost semiprime and weakly semiprime submodules. Also, we give some characterizations of such submodules in multiplication modules. Now we define the concepts that we will use.

For any two submodules N and K of an R -module M , the residual of N by K is defined as the set $(N : K) = \{r \in R : rK \subseteq N\}$ which is clearly an ideal of R . In particular, the ideal $(0 : M)$ is called the annihilator of M . Let N be a submodule of M and let I be an ideal of R ; the residual submodule of N by I is defined as $(N :_M I) = \{m \in M : Im \subseteq N\}$. These two residual ideals and submodules were proved to be useful in studying many concepts of modules; see, for example, [18, 19]. A proper submodule N of an R -module M is a prime submodule if, whenever $rm \in N$ for $r \in R$ and

$m \in M$, $m \in N$ or $r \in (N : M)$. An R -module M is called a prime module if its zero submodule is a prime submodule. A proper submodule N of an R -module M is called weakly prime (weakly primary) if $0 \neq rm \in N$, where $r \in R$ and $m \in M$; then $m \in N$ or $r \in (N : M)$ ($m \in N$ or $r \in \text{rad}(N : M)$). A proper submodule N of an R -module is called almost prime (almost primary) if, whenever $rm \in N - (N : M)N$ for $r \in R$ and $m \in M$, $m \in N$ or $r \in (N : M)$ ($m \in N$ or $r \in \text{rad}(N : M)$). A proper ideal I of a commutative ring R is called semiprime if $a^k b \in I$, where $a, b \in R$ and $k \in \mathbb{Z}^+$; then $ab \in I$. A proper submodule N of an R -module M is called semiprime if, whenever $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $r^k m \in N$, $rm \in N$. An R -module M is called a second module provided that, for every element $r \in R$, the R -endomorphism of M produced by multiplication by r is either surjective or zero; this implies that $(0 : M) = p$ is a prime ideal of R and M is said to be p -second [20]. An R -module M is called a multiplication module provided that, for every submodule N of M , there exists an ideal I of R so that $N = IM$ (or equivalently, $N = (N : M)M$). An Ideal I of a ring R is called multiplication if it is multiplication as R -modules. Multiplication modules and ideals have been studied extensively in [21–23]. An R -module M is called a cancellation module if, for all ideals I and J of R , $IM = JM$ implies that $I = J$; see [24]. If R is a ring and M is an R -module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$. Obviously, if R is an integral domain, then $T(M)$ is a submodule of M . If $T(M) = M$, then we say that M is torsion and if $T(M) = 0$, we say that M is torsion-free.

2. Almost Semiprime Submodules

Definition 1. (i) Let R be a commutative ring. A proper ideal I of R is called almost semiprime if, whenever $a^k b \in I - I^2$ for $a, b \in R$ and $k \in \mathbb{Z}^+$, $ab \in I$.

(ii) Let R be a commutative ring and let M be an R -module. A proper submodule N of M is called almost semiprime if, whenever $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $r^k m \in N - (N : M)N$, $rm \in N$.

Let M be an R -module and let N be a submodule of M . Following [25], N is called idempotent in M if $N = (N : M)N$. Thus, any proper idempotent submodule of M is almost semiprime. If M is a multiplication R -module and $N = IM$ and $K = JM$ are two submodules of M , then the product NK of N and K is defined as $NK = (IM)(JM) = (IJ)M$; see [9]. In particular, one has $N^2 = NN = [(N : M)M][(N : M)M] = (N : M)^2 M$.

Furthermore, M is a cancellation R -module; then by using Lemma 12, $(N : M)N = ((N : M)N : M)M = (N : M)^2 M = N^2$. So in this case, a submodule N is idempotent in M if and only if $N = N^2$. Following [26], a submodule N of an R -module M is called a pure (RD-) submodule if $IN = N \cap IM$ ($rN = N \cap rM$) for any ideal I of R (for any $r \in R$). In [25], it was proved that if N is a pure submodule in a multiplication R -module M with pure annihilator, then N is idempotent in M and so is almost semiprime.

Example 2. (i) It is clear that every semiprime submodule is almost semiprime. But the converse is not true in general. For example, consider $\mathbb{Z}$ -module $M = \mathbb{Z}_{24}$ (the integers modulo 24) and the submodule $N = \langle 8 \rangle$. Then $(N : M)N = N$, and so N is an almost semiprime submodule of M . But N is not semiprime in M , because $2^2 \cdot 2 \in N$, but $2 \cdot 2 \notin N$.

In the semiprime submodules case, N is a semiprime submodule of M , if and only if N/K is so in M/K for any submodule $K \subseteq N$. But the converse part may not be true in the case of almost semiprime submodules. For example, for any non-almost semiprime submodule N of M , we have $N/N = 0$ is an almost semiprime submodule of M/N . For another nontrivial example, we consider the ring $R = K[x, y]$, where K is a field and ideals $P = (x, y^2)$, $I = (x, y)^2$. Then P/I is an almost semiprime submodule of the R -module R/I , while P is not so in R . But we have the following theorem.

Theorem 3. Let N and K be submodules of an R -module M with $K \subseteq (N : M)N$. Then N is an almost semiprime submodule of M if and only if N/K is an almost semiprime submodule of the R -module M/K .

Proof. Let N be an almost semiprime submodule of M and assume that $r \in R$, $m + K \in M/K$, and $k \in \mathbb{Z}^+$ such that $r^k(m + K) \in N/K - (N/K : M/K)N/K$. It is clear that $(N/K :_R M/K) = (N :_R M)$, and so $r^k m \in N - (N : M)N$. Therefore $rm \in N$ since N is almost semiprime. Therefore, $r(m + k) \in N/K$; hence N/K is an almost semiprime submodule. Conversely, let N/K be an almost semiprime submodule of M/K and assume that $r^k m \in N - (N : M)N$

for some $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$. Hence, $r^k(m + K) \in N/K - (N/K : M/K)N/K$, because if $r^k(m + K) \in (N/K : M/K)N/K = (N : M)N/K = ((N : M)N + K)/K = (N : M)N/K$ since $K \subseteq (N : M)N$, $r^k m \in (N : M)N$, a contradiction. Therefore $r(m + K) \in N/K$, so $rm \in N$, as required. $\square$

Theorem 4. Let S be a multiplicative closed subset of R and let N be an almost semiprime submodule of R -module M with $S \cap (N : M) = \emptyset$. Then $S^{-1}N$ is an almost semiprime submodule of the $S^{-1}R$ -module $S^{-1}M$.

Proof. Let N be an almost semiprime submodule of M . Since $(N : M) \cap S = \emptyset$, then $S^{-1}N \neq S^{-1}M$. Assume that $(r/s)^k m/t \in S^{-1}N - (S^{-1} :_{S^{-1}R} S^{-1}M)S^{-1}N$, where $r/s \in S^{-1}R$, $m/t \in S^{-1}M$, and $k \in \mathbb{Z}^+$. Hence, $r^k m/s^k t = n/s'$ for some $n \in N$ and $s' \in S$, and so there exists $t' \in S$ such that $r^k s' t' m = s^k t t' n \in N$. If $r^k s' t' m \in (N : M)N$, then $r^k m/s^k t = r^k s' t' m/s^k t s' t' \in S^{-1}((N : M)N) = S^{-1}(N : M)S^{-1}N \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)S^{-1}N$, a contradiction. So $r^k s' t' m \in N - (N : M)N$, and $rs' t' m \in N$ since N is almost semiprime. Therefore, $rm/st = rs' t' m/sts' t' \in S^{-1}N$; hence $S^{-1}N$ is an almost semiprime submodule of $S^{-1}M$. $\square$

Proposition 5. Let $R = R_1 \times R_2$ where each R_i is a commutative ring with nonzero identity. Let M_i be an R_i -module and let $M = M_1 \times M_2$ be the R -module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$, where $r_i \in R_i$ and $m_i \in M_i$. Then

- (i) N_1 is an almost semiprime submodule of M_1 if and only if $N_1 \times M_2$ is an almost semiprime submodule of M ;
- (ii) N_2 is an almost semiprime submodule of M_2 if and only if $M_1 \times N_2$ is an almost semiprime submodule of M .

Proof. (i) Let N_1 be an almost semiprime submodule of M_1 . Assume that $(r_1, r_2)^k(m_1, m_2) \in N_1 \times M_2 - (N_1 \times M_2 : M)N_1 \times M_2$, where $(r_1, r_2) \in R$, $(m_1, m_2) \in M$, and $k \in \mathbb{Z}^+$. If $r_1^k m_1 \in (N_1 : M_1)N_1$, then $(r_1, r_2)^k(m_1, m_2) \in (N_1 : M_1)N_1 \times (M_2 : M_2)M_2 = ((N_1 : M_1) \times (M_2 : M_2))N_1 \times M_2 = (N_1 \times M_2 : M_1 \times M_2)N_1 \times M_2$, a contradiction. Hence, as N_1 is almost semiprime and $r_1^k m_1 \in N_1 - (N_1 : M_1)N_1$, then $r_1 m_1 \in N_1$, and so $(r_1, r_2)(m_1, m_2) \in N_1 \times M_2$. Conversely, assume that $N_1 \times M_2$ is an almost semiprime submodule of M . Let $r_1^k m_1 \in N_1 - (N_1 : M_1)N_1$ for $r_1 \in R_1$, $m_1 \in M_1$, and $k \in \mathbb{Z}^+$. Then $(r_1, 1)^k(m_1, 0) \in N_1 \times M_2 - (N_1 \times M_2 : M)N_1 \times M_2$ by (i). Therefore $(r_1, 1)(m_1, 0) \in N_1 \times M_2$, since $N_1 \times M_2$ is almost semiprime, so $r_1 m_1 \in N_1$, as needed.

(ii) is similar to (i). $\square$

Let R be a commutative ring with identity and let M be an R -module. Then $R(M) = R(+M)$ with multiplication $(a, m)(b, n) = (ab, an + bm)$ and with addition $(a, m) + (b, n) = (a + b, m + n)$ is a commutative ring with identity $(1, 0)$ and $0(+M)$ is a nilpotent ideal of index 2. The ring $R(+M)$ is said to be the idealization of M or trivial extension of R by M . We view R as a subring of $R(+M)$ via $r \rightarrow (r, 0)$. An ideal J is said to be homogeneous if $J = I(+N)$ for some ideal I of R and some submodule N of M such that $IM \subseteq N$.

Lemma 6. Let $I(+)N$ be an ideal of $R(M)$. Then $(I(+)N)^2 \subseteq I^2(+)IN$.

Proof. The proof is straightforward. $\square$

Theorem 7. Let $I(+)N$ be a homogeneous ideal of $R(M)$. Then, if $I(+)N$ is an almost semiprime ideal of $R(M)$, then I is an almost semiprime ideal of R and N is an almost semiprime submodule of M .

Proof. Assume that $I(+)N$ is an almost semiprime ideal of $R(M)$. Let $a, b \in R$ and $k \in \mathbb{Z}^+$ such that $a^k b \in I - I^2$. Then $(a, 0)^k(b, 0) \in I(+)N - (I(+)N)^2$, because if $(a, 0)^k(b, 0) \in (I(+)N)^2$, then by Lemma 6, $(a, 0)^k(b, 0) \in I^2(+)IN$, hence $a^k b \in I^2$, a contradiction. Therefore $(a, 0)(b, 0) \in I(+)N$, and $ab \in I$, so I is an almost semiprime ideal of R .

Let $r \in R, m \in M$, and $k \in \mathbb{Z}^+$ such that $r^k m \in N - (N : M)N$. Therefore $(r, 0)^k(0, m) \in I(+)N - (I(+)N)^2$, because if $(r, 0)^k(0, m) = (0, r^k m) \in (I(+)N)^2 \subseteq I^2(+)IN$, then $r^k m \in IN$. So $r^k m \in IN \subseteq (N : M)N$ since $I(+)N$ is a homogeneous ideal, a contradiction. Hence $(r, 0)(0, m) \in I(+)N$, so $rm \in N$. Thus, N is an almost semiprime submodule of M . $\square$

Proposition 8. Let M be an R -module and let N be an almost semiprime submodule of M . Then

- (i) if M is a second R -module, then N is a second module;
- (ii) if M is a second R -module, then N is an RD -submodule of M .

Proof. Let N be an almost semiprime submodule of M . Let $r \in R$. If $rM = 0$, then $rN \subseteq rM = 0$. Let $rM = M$. Now It is enough to show that $N \subseteq rN$. First, we show that $(N : M)N = 0$. Since N is a proper submodule of M , for any $r \in (N : M)$, we have $rM = 0$. Therefore $(N : M)N = 0$. Let $n \in N$. We may assume that $n \neq 0$. Since $rM = M$, $n = rm$ for some $m \in M$, and $m = rm'$ for some $m' \in M$. Hence $n = r^2 m' \in N - (N : M)N$, as N is almost semiprime so $m = rm \in N$. Hence $n = rm \in rN$, so $N \subseteq rN$. Therefore $rN = N$ and N is second.

(ii) Let $r \in R$. If $rM = 0$, then $rN = 0$, so $rN = 0 = N \cap rM$. Suppose that $rM = M$, so by (i), $rN = N$; therefore $rN = N \cap rM$. $\square$

In the following theorems, we give other characterizations of almost semiprime submodules.

Theorem 9. Let M be an R -module and let N be a proper submodule of M . Then the following are equivalent:

- (i) N is an almost semiprime submodule of M .
- (ii) For $r \in R$ and $k \in \mathbb{Z}^+$, $(N :_M \langle r^k \rangle) = (N :_M \langle r \rangle) \cup ((N : M)N :_M \langle r^k \rangle)$.
- (iii) For $r \in R$ and $k \in \mathbb{Z}^+$, $(N :_M \langle r^k \rangle) = (N :_M \langle r \rangle)$ or $(N :_M \langle r^k \rangle) = ((N : M)N :_M \langle r^k \rangle)$.

Proof. (i) $\Rightarrow$ (ii) Let $m \in (N :_M \langle r^k \rangle)$; then $r^k m \in N$. If $r^k m \notin (N :_R M)N$, as N is almost semiprime, $rm \in N$,

so $m \in (N :_M \langle r \rangle)$. Let $r^k m \in (N :_R M)N$; then $m \in ((N :_R M)N :_M \langle r^k \rangle)$; hence $(N :_M \langle r^k \rangle) \subseteq (N :_M \langle r \rangle) \cup ((N : M)N :_M \langle r^k \rangle)$. The other containment holds for any submodule N .

(ii) $\Rightarrow$ (iii) It is well known that if a submodule is the union of two submodules, then it is equal to one of them.

(iii) $\Rightarrow$ (i) Let $r^k m \in N - (N :_R M)N$ for some $r \in R, m \in M$, and $k \in \mathbb{Z}^+$. Hence $m \in (N :_M \langle r^k \rangle)$ and $m \notin ((N :_R M)N :_M \langle r^k \rangle)$, so by assumption, $m \in (N :_M \langle r \rangle)$ and $rm \in N$. Therefore N is almost semiprime. $\square$

The following theorem gives from Theorem 9.

Theorem 10. Let M be an R -module and let N be a proper submodule of M . Then N is almost semiprime in M if and only if for any submodule K of M , $a \in R$, and $k \in \mathbb{Z}^+$ with $\langle a \rangle^k K \subseteq N$ and $\langle a \rangle^k K \not\subseteq (N :_R M)N$, one has $\langle a \rangle K \subseteq N$.

We know that if N is a semiprime submodule of M , then $(N :_R M)$ is a semiprime ideal of R . But it may not be true in the case of almost semiprime submodules.

Example 11. Let M denote the cyclic $\mathbb{Z}$ -module $\mathbb{Z}_4$ (the integers modulo 4). Take $N = \{0\}$. Certainly, N is almost semiprime, but $(N :_R M) = 4\mathbb{Z}$ is not an almost semiprime ideal of $\mathbb{Z}$, because $2^2 \in (N : M) - (N : M)^2$, but $2 \notin (N : M)$.

Now in the following theorem, we give a characterization of almost semiprime submodules in (finitely generated faithful) multiplication modules. We first need the following lemma.

Lemma 12. Let N be a submodule of a finitely generated faithful multiplication (so cancellation) R -module. Then $(IN : M) = I(N : M)$ for every ideal I of R .

Proof. The proof is by [12, Lemma 3.4]. $\square$

Theorem 13. Let M be a finitely generated faithful multiplication R -module and let N be a proper submodule of M . Then the following are equivalent.

- (i) N is almost semiprime in M .
- (ii) $(N :_R M)$ is almost semiprime in R .
- (iii) $N = PM$ for some almost semiprime ideal P of R .

Proof. (i) $\Rightarrow$ (ii) Suppose that N is an almost semiprime submodule of M . Let $a, b \in R$ and $k \in \mathbb{Z}^+$ such that $a^k b \in (N : M) - (N : M)^2$. Then $\langle a \rangle^k(bM) \subseteq N$ and $\langle a \rangle^k(bM) \not\subseteq (N : M)N$. Indeed, if $\langle a \rangle^k(bM) \subseteq (N : M)N$, then, by Lemma 12, $a^k b \in ((N : M)N : M) = (N : M)^2$, a contradiction. Now, N almost semiprime implies that $\langle a \rangle(bM) \subseteq N$ by Theorem 10, so $ab \in (N : M)$; hence $(N : M)$ is almost semiprime in R .

(ii) $\Rightarrow$ (i) In this direction, we need M to be just a multiplication module. Let $r^k m \in N - (N : M)M$, where $r \in R, m \in M$, and $k \in \mathbb{Z}^+$. Then $\langle r \rangle^k(\langle m \rangle : M) \subseteq (\langle r^k m \rangle : M) \subseteq (N : M)$. Moreover, $\langle r \rangle^k(\langle m \rangle : M) \not\subseteq (N : M)^2$ because, otherwise, if $\langle r \rangle^k(\langle m \rangle : M) \subseteq (N : M)^2 \subseteq ((N : M) : M)$

$N : M)$, then $\langle r \rangle^k \langle m \rangle = \langle r \rangle^k (\langle m \rangle : M)M \subseteq (N : M)N$, a contradiction. As $(N : M)$ is an almost semiprime ideal of R , $\langle r \rangle (\langle m \rangle : M) \subseteq (N : M)$. Therefore $\langle r \rangle \langle m \rangle = \langle r \rangle (\langle m \rangle : M)M \subseteq (N : M)M = N$, and so $rm \in N$, as required.

(ii) $\Leftrightarrow$ (iii) We choose $P = (N : M)$. $\square$

Lemma 14. *Let N be a submodule of a faithful multiplication R -module M and let I be a finitely generated faithful multiplication ideal of R . Then*

(i) $N = (IN :_M I)$;

(ii) if $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R .

Proof. It follows from [18]. $\square$

Theorem 15. *Let N be a submodule of a faithful multiplication R -module M and let I be a finitely generated faithful multiplication ideal of R . Then N is an almost semiprime submodule of IM if and only if $(N :_M I)$ is an almost semiprime submodule of M .*

Proof. Assume that N is almost semiprime in IM . Let $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $r^k m \in (N :_M I) - ((N :_M I) :_R M)(N :_M I)$. Then $\langle r \rangle^k Im \subseteq N - (N :_R IM)N$. In fact, if $\langle r \rangle^k (Im) \subseteq (N : IM)N$, then, by Lemma 14, $r^k m \in ((N : IM)N :_M I) = (N : IM)(N :_M I) = ((N :_M I) : M)(N :_M I)$, a contradiction. As N is almost semiprime in IM , then $\langle r \rangle (Im) \subseteq N$, so $rm \in (N :_M I)$; hence $(N :_M I)$ is almost semiprime in M .

Conversely, suppose that $(N :_M I)$ is almost semiprime in M . Let K be a submodule of IM , $a \in R$, and $k \in \mathbb{Z}^+$ such that $\langle a \rangle^n K \subseteq N - (N : M)N$. Then $\langle a \rangle^n (K :_M I) \subseteq (\langle a \rangle^n K :_M I) \subseteq (N :_M I)$. Moreover, if $\langle a \rangle^n (K :_M I) \subseteq ((N :_M I) : M)(N :_M I) = (N :_R IM)(N :_M I)$, then, by Lemma 14, $\langle a \rangle^n K = \langle a \rangle^n (IK :_M I) = \langle a \rangle^n (K :_M I)I \subseteq (N : IM)(N :_M I)I = (N : IM)N$, a contradiction. As $(N :_M I)$ is almost semiprime in M , $\langle a \rangle (K :_M I) \subseteq (N :_M I)$ and so $\langle a \rangle K = \langle a \rangle (K :_M I) \subseteq (N :_M I)I = N$. Therefore N is almost semiprime in IM . $\square$

Lemma 16. *For every proper ideal I of R , $\text{rad}(I)$ is an almost semiprime ideal of R .*

Proof. Since $(\text{rad}(I))^2 = \text{rad}(I)$, the proof is held. $\square$

Let N be a proper submodule of M . Then the M -radical of N , denoted by $M - \text{rad}(N)$, is defined to be the intersection of all prime submodules of M containing N . It is shown in [22] that if N is a proper submodule of a multiplication R -module M , then $M - \text{rad}(N) = \text{rad}(N :_R M)M$.

Theorem 17. *Let M be a finitely generated faithful multiplication R -module. Then for every proper submodule N of M , $M - \text{rad}(N)$ is an almost semiprime submodule of M .*

Proof. Let N be a proper submodule of M . Hence by Lemma 16, $\text{rad}(N :_R M)$ is an almost semiprime ideal of R . Therefore by Theorem 13, $M - \text{rad}(N) = \text{rad}(N :_R M)M$ is an almost semiprime submodule of M . $\square$

3. Weakly Semiprime Submodules

Definition 18. (i) Let R be a commutative ring. A proper ideal I of R is called weakly semiprime if, whenever $0 \neq a^k b \in I$ for some $a, b \in R$ and $k \in \mathbb{Z}^+$, $ab \in I$.

(ii) Let M be an R -module. A proper submodule N of M is called weakly semiprime if, whenever $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$, $rm \in N$.

Remark 19. Let M be a module over a commutative ring R . Then semiprime submodules $\Rightarrow$ weakly semiprime submodules $\Rightarrow$ almost semiprime submodules.

Example 20. Consider the $\mathbb{Z}$ -module $M = \mathbb{Z}_{24}$ and the proper submodule $N = \langle 8 \rangle = \{0, 8, 16\}$. Then $0 = 0.8$, $8 = 16.8$, and $16 = 16.16$, so $(N : M)N = N$. Therefore N is almost semiprime. On the other hand, $0 \neq 2^2 \cdot 2 \in N$, but $2.2 \notin N$, and so N is not weakly semiprime.

Theorem 21. *Let M be an R -module and let N be a proper submodule of M . Then N is an almost semiprime submodule of M if and only if $N/(N : M)N$ is a weakly semiprime submodule of the R -module $M/(N : M)N$.*

Proof. Assume that N is an almost semiprime submodule of M . Let $r \in R$, $m + (N : M)N \in M/(N : M)N$, and $k \in \mathbb{Z}^+$ such that $0 \neq r^k(m + (N : M)N) \in N/(N : M)N$. Hence $r^k m \in N - (N : M)N$, and so $rm \in N$. Therefore $r(m + (N : M)N) \in N/(N : M)N$, as needed.

Conversely, assume that $N/(N : M)N$ is weakly semiprime in $M/(N : M)N$. Let $r^k m \in N - (N : M)N$, where $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$. Then $0 \neq r^k(m + (N : M)N) \in N/(N : M)N$, and hence $r(m + (N : M)N) \in N/(N : M)N$. Therefore $rm \in N$, as required. $\square$

Proposition 22. *Let R be an integral domain and let M be a torsion-free R -module. Then every weakly semiprime submodule of M is semiprime.*

Proof. Let N be a weakly semiprime submodule of M . Let $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $r^k m \in N$. If $0 \neq r^k m$, then N weakly semiprime gives that $rm \in N$. Suppose that $r^k m = 0$. If $r^k \neq 0$, then $m \in T(M) = 0$, so $rm \in N$. If $r^k = 0$, then $r = 0$, and hence $rm \in N$. Therefore N is semiprime. $\square$

Proposition 23. *Let M be a prime R -module. Then every weakly semiprime submodule of M is semiprime.*

Proof. Let N be a weakly semiprime submodule of M . Let $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $r^k m \in N$. If $0 \neq r^k m$, then N weakly semiprime gives that $rm \in N$. Suppose that $r^k m = 0$; then $rm = 0$ or $r^{k-1}M = 0$ since M is a prime module. By following this method, we get $rm = 0 \in N$; hence N is a semiprime submodule of M . $\square$

Proposition 24. *Let M be a second R -module and let N be a proper submodule of M . Then N is almost semiprime if and only if N is weakly semiprime.*

Proof. We know that every weakly semiprime is almost semiprime. Let N be an almost semiprime submodule of M and $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$. By Proposition 8, we have $(N : M)N = 0$; hence $r^k m \in N - (N : M)N$, and so $rm \in N$. Therefore N is weakly semiprime submodule of M . $\square$

Now we get other characterizations of weakly semiprime submodule.

Theorem 25. *Let M be an R -module and let N be a proper submodule of M . Then the following are equivalent.*

- (i) N is a weakly semiprime submodule of M .
- (ii) For $r \in R$ and $k \in \mathbb{Z}^+$, $(N :_M \langle r^k \rangle) = (0 :_M \langle r^k \rangle) \cup (N :_M \langle r \rangle)$.
- (iii) For $r \in R$ and $k \in \mathbb{Z}^+$, $(N :_M \langle r^k \rangle) = (0 :_M \langle r^k \rangle)$ or $(N :_M \langle r^k \rangle) = (N :_M \langle r \rangle)$.

Proof. (i) $\Rightarrow$ (ii) Let $m \in (N :_M \langle r^k \rangle)$; then $r^k m \in N$. If $r^k m \neq 0$, as N is weakly semiprime, $rm \in N$, so $m \in (N :_M \langle r \rangle)$. Let $r^k m = 0$; then $m \in (0 :_M \langle r^k \rangle)$, and hence $(N :_M \langle r^k \rangle) \subseteq (N :_M \langle r \rangle) \cup (0 :_M \langle r^k \rangle)$. Clearly, $(N :_M \langle r \rangle) \cup (0 :_M \langle r^k \rangle) \subseteq (N :_M \langle r^k \rangle)$; therefore $(N :_M \langle r^k \rangle) = (0 :_M \langle r^k \rangle) \cup (N :_M \langle r \rangle)$.

(ii) $\Rightarrow$ (iii) It is straightforward.

(iii) $\Rightarrow$ (i) Let $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$. Hence $m \in (N :_M \langle r^k \rangle)$ and $m \notin (0 :_M \langle r^k \rangle)$, so by assumption, $m \in (N :_M \langle r \rangle)$. Therefore N is weakly semiprime. $\square$

Theorem 26. *Let M be an R -module and let N be a proper submodule of M . Then N is weakly semiprime in M if and only if for any submodule K of M , $a \in R$, and $k \in \mathbb{Z}^+$ with $0 \neq \langle a \rangle^k K \subseteq N$, one has $\langle a \rangle K \subseteq N$.*

Theorem 27. *Let N be a weakly semiprime submodule of an R -module M with $T(M) = 0$. Then for any nonzero ideal I of R , $(N :_M I)$ is a weakly semiprime submodule of M .*

Proof. Let $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $0 \neq r^k m \in (N :_M I)$. Hence $\langle r \rangle^k (mI) \subseteq N$. If $0 \neq \langle r \rangle^k (mI) \subseteq N$, then, by Theorem 26, N weakly semiprime gives that $\langle r \rangle (mI) \in N$, so $rm \in (N :_M I)$, as needed. Suppose that $\langle r \rangle^k (mI) = 0$, so $r^k ma = 0$ for some nonzero $a \in I$. Hence $r^k m \in T(M) = 0$, which is a contradiction. Therefore $(N :_M I)$ is weakly semiprime. $\square$

In Theorem 27, the assumption $T(M) = 0$ is necessary. To see this, consider $\mathbb{Z}$ -module $\mathbb{Z}_{16}$. Let $N = \{0\}$ and $I = 2\mathbb{Z}$. Clearly, N is weakly semiprime submodule of M , but $(N :_M I) = \{0, 8\}$ is not weakly semiprime.

Theorem 28. *Let I be an ideal of R and let N be a submodule of M such that $I(+)N$ is a weakly semiprime ideal of $R(M)$. Then I is a weakly semiprime ideal of R and N is a weakly semiprime submodule of M .*

Proof. Assume that $I(+)N$ is a weakly semiprime ideal of $R(M)$. Let $a, b \in R$ and $k \in \mathbb{Z}^+$ such that $0 \neq a^k b \in I$. Then $(0, 0) \neq (a, 0)^k (b, 0) \in I(+)N$. Therefore $(a, 0)(b, 0) \in I(+)N$, and $ab \in I$, so I is a weakly semiprime ideal of R .

Let $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$ such that $0 \neq r^k m \in N$. Therefore $(0, 0) \neq (r, 0)^k (0, m) \in I(+)N$; hence $(r, 0)(0, m) \in I(+)N$, so $rm \in N$. Thus, N is a weakly semiprime submodule of M . $\square$

In [17], the authors have proved that if N is a weakly semiprime submodule of a faithful cyclic R -module M , then the ideal $(N : M)$ is weakly semiprime. But we show that the assumption faithful R -module for this theorem is sufficient.

Proposition 29. *Let M be a faithful R -module and let N be a weakly semiprime submodule of M . Then $(N : M)$ is a weakly semiprime ideal of R .*

Proof. Suppose N is weakly semiprime, $a, b \in R$, and $k \in \mathbb{Z}^+$ such that $0 \neq a^k b \in (N : M)$. Then $0 \neq \langle a \rangle^k (bM) \subseteq N$. Indeed, if $\langle a \rangle^k (bM) = 0$, then $a^k b \in (0 : M) = 0$, a contradiction. Now, Theorem 25 implies that $\langle a \rangle (bM) \subseteq N$, so $ab \in (N : M)$, and $(N : M)$ is weakly semiprime in R . $\square$

Now we give characterizations of weakly semiprime submodules in (finitely generated faithful) multiplication modules.

Theorem 30. *Let M be a finitely generated faithful multiplication R -module and let N be a proper submodule of M . Then the following are equivalent.*

- (i) N is weakly semiprime in M .
- (ii) $(N :_R M)$ is weakly semiprime in R .
- (iii) $N = QM$ for some weakly semiprime ideal Q of R .

Proof. (i) $\Rightarrow$ (ii) It follows from Proposition 29.

(ii) $\Rightarrow$ (i) In this direction, we need M to be just a multiplication module. Let $0 \neq r^k m \in N$, where $r \in R$, $m \in M$, and $k \in \mathbb{Z}^+$. Then $\langle r \rangle^k (\langle m \rangle : M) \subseteq (\langle r^k m \rangle : M) \subseteq (N : M)$. Moreover, $\langle r \rangle^k (\langle m \rangle : M) \neq 0$ because, otherwise, if $\langle r \rangle^k (\langle m \rangle : M) = 0$, then $\langle r \rangle^k \langle m \rangle = \langle r \rangle^k (\langle m \rangle : M)M = 0$, a contradiction. As $(N : M)$ is a weakly semiprime ideal of R , $\langle r \rangle (\langle m \rangle : M) \subseteq (N : M)$. Therefore $\langle r \rangle \langle m \rangle = \langle r \rangle (\langle m \rangle : M)M \subseteq (N : M)M = N$, and so $rm \in N$, as required.

(ii) $\Leftrightarrow$ (iii) We choose $Q = (N : M)$. $\square$

Definition 31. A proper ideal I of an integral domain R is said to be divided if $I \subset Rx$ for all $x \in R - I$ (see [27]). Generalizing this idea to modules one says that a proper submodule N of an R -module M is divided if $N \subset Rm$ for all $m \in M - N$.

Lemma 32. *Let R be a commutative ring and let M be a finitely generated faithful multiplication R -module. If P is a divided prime submodule of M , then $(P : M)$ is a divided prime ideal of R .*

Proof. The proof is by [28, Proposition 6]. $\square$

Theorem 33. Let R be a commutative ring, let M be a finitely generated faithful multiplication R -module, and let N be a proper submodule of M such that $M - \text{rad}(N) = P$, where P is a divided prime submodule of M . Then

- (i) if N is a semiprime submodule of M , then N is a primary submodule of M ;
- (ii) if N is an almost semiprime submodule of M , then N is an almost primary submodule of M ;
- (iii) if N is a weakly semiprime submodule of M , then N is a weakly primary submodule of M .

Proof. Note first that $P = (P : M)M$, where $(P : M) = p$ is a prime ideal of R . Also, $M - \text{rad}(N) = \text{rad}(N : M)M$. On the other hand, $M - \text{rad}(N) = P = pM = \text{rad}(N : M)M$. Moreover, every finitely generated faithful multiplication module is a cancellation, so that $p = (P : M) = \text{rad}(N : M)$.

(i) Assume that $am \in N$ and $a \notin \text{rad}((N : M)) = p$. Then from $am \in P$ and $a \notin p$, we get $m \in P$ since P is prime. By Lemma 32, p is the divided prime ideal of R . So $p \subset Ra$ since $a \notin p$. Therefore $P = pM \subset Ma$, and hence $m = am'$ for some $m' \in M$. Now it follows from $am = a^2m' \in N$, and so $m = am' \in N$ since N is assumed to be semiprime submodule. This shows that N is a primary submodule of M .

The proofs of (ii) and (iii) follow from (i). $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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