Research Article

On the System of High Order Rational Difference Equations

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Received 15 March 2014; Revised 2 July 2014; Accepted 22 July 2014; Published 29 October 2014

Academic Editor: Chunrui Zhang

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This paper is concerned with the boundedness, persistence, and global asymptotic behavior of positive solution for a system of two rational difference equations $x_{n+1} = A + (x_n / \sum_{i=1}^k y_{n-i})$, $y_{n+1} = B + (y_n / \sum_{i=1}^k x_{n-i})$, $n = 0, 1, \dots, k \in \{1, 2, \dots\}$, where $A, B \in (0, \infty)$, $x_{-i} \in (0, \infty)$, and $y_{-i} \in (0, \infty)$, $i = 0, 1, 2, \dots, k$.

1. Introduction

In this paper, we study the global behavior of solutions of the following system:

$$\begin{aligned} x_{n+1} &= A + \frac{x_n}{\sum_{i=1}^k y_{n-i}}, & y_{n+1} &= B + \frac{y_n}{\sum_{i=1}^k x_{n-i}}, \\ n &= 0, 1, \dots, \end{aligned} \quad (1)$$

where A, B are positive constants and initial conditions $x_{-i}, y_{-i} \in (0, \infty)$, $i = 0, 1, 2, \dots, k$.

A pair of sequences of positive real numbers $\{(x_n, y_n)\}$ that satisfies (1) is a positive solution of (1). If a positive solution of (1) is a pair of positive constants (x, y) , then the solution is the equilibrium solution.

A positive solution $\{(x_n, y_n)\}$ of (1) is bounded and persists, if there exist positive constants M, N such that

$$M \leq x_n, \quad y_n \leq N, \quad n = -2, -1, \dots \quad (2)$$

In 1998, DeVault et al. [1] proved that every positive solution of the difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \dots, \quad (3)$$

where $A \in (0, \infty)$, oscillates about the positive equilibrium $c = 1 + A$ of (3). Moreover, every positive solution of (3)

is bounded away from zero and infinity. Also the positive equilibrium of (3) is globally asymptotically stable.

In 2003, Abu-Saris and DeVault [2] studied the following recursive difference equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \dots, \quad (4)$$

where $A \in (1, +\infty)$, $x_{-k}, x_{-k+1}, \dots, x_0$ are positive real numbers.

Papaschinopoulos and Schinas [3] investigated the global behavior for a system of the following two nonlinear difference equations:

$$\begin{aligned} x_{n+1} &= A + \frac{y_n}{x_{n-p}}, & y_{n+1} &= A + \frac{x_n}{y_{n-q}}, \\ n &= 0, 1, \dots, \end{aligned} \quad (5)$$

where A is a positive real number, p, q are positive integers, and $x_{-p}, \dots, x_0, y_{-q}, \dots, y_0$ are positive real numbers.

In 2012, Zhang et al. [4] investigated the global behavior for a system of the following third-order nonlinear difference equations:

$$x_{n+1} = A + \frac{x_n}{y_{n-1} + y_{n-2}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-1} + x_{n-2}}, \quad (6)$$

where $A, B \in (0, \infty)$, and the initial values $x_{-i}, y_{-i} \in (0, \infty)$, $i = 0, 1$. For other related results, the reader can refer to [5–18].

Motivated by the discussion above, we study the global asymptotic behavior of solutions for system (1). More precisely, we prove the following: if $A > 1/k$, $B > 1/k$ then every positive solution $\{(x_n, y_n)\}$ of (1) is persistent and bounded. Moreover, we prove that every positive solution $\{(x_n, y_n)\}$ of (1) converges the unique positive equilibrium (x, y) as $n \rightarrow \infty$.

2. Main Results

In the following lemma, we show boundedness and persistence of the positive solutions of (1).

Lemma 1. Consider (1). Suppose that

$$A > \frac{1}{k}, \quad B > \frac{1}{k} \quad (7)$$

are satisfied. Then, every positive solution (x_n, y_n) of (1) is satisfied, for $n = k + 1, k + 2, \dots$

$$\begin{aligned} A \leq x_n &\leq \frac{1}{(kB)^{n-k}} \left(x_k - \frac{kAB}{kB-1} \right) + \frac{kAB}{kB-1} \\ B \leq y_n &\leq \frac{1}{(kA)^{n-k}} \left(y_k - \frac{kAB}{kA-1} \right) + \frac{kAB}{kA-1}. \end{aligned} \quad (8)$$

Proof. Let $\{(x_n, y_n)\}$ be a positive solution of (1). Since $x_n > 0$ and $y_n > 0$ for all $n \geq 1$, (1) implies that

$$x_n \geq A, \quad y_n \geq B, \quad n = 1, 2, 3, \dots \quad (9)$$

Moreover, using (1) and (9), we have

$$\begin{aligned} x_n &\leq A + \frac{1}{kB} x_{n-1}, \quad y_n \leq B + \frac{1}{kA} y_{n-1}, \\ n &= k + 1, k + 2, \dots \end{aligned} \quad (10)$$

Let v_n, w_n be the solution of the system, respectively,

$$v_n = A + \frac{1}{kB} v_{n-1}, \quad w_n = B + \frac{1}{kA} w_{n-1}, \quad n \geq k + 1, \quad (11)$$

such that

$$v_i = x_i, \quad w_i = y_i, \quad i = 1, 2, \dots, k. \quad (12)$$

We prove by induction that

$$x_n \leq v_n, \quad y_n \leq w_n, \quad n \geq k + 1. \quad (13)$$

Suppose that (13) is true for $n = m \geq k + 1$. Then, from (10), we get

$$\begin{aligned} x_{m+1} &\leq A + \frac{1}{kB} x_m \leq A + \frac{1}{kB} v_m = v_{m+1}, \\ y_{m+1} &\leq B + \frac{1}{kA} y_m \leq B + \frac{1}{kA} w_m = w_{m+1}. \end{aligned} \quad (14)$$

Therefore, (13) is true. From (11) and (12), we have

$$\begin{aligned} v_n &= \frac{1}{(kB)^{n-k}} \left(x_k - \frac{kAB}{kB-1} \right) + \frac{kAB}{kB-1}, \\ w_n &= \frac{1}{(kA)^{n-k}} \left(y_k - \frac{kAB}{kA-1} \right) + \frac{kAB}{kA-1}, \\ n &\geq k. \end{aligned} \quad (15)$$

Then, from (9), (13), and (15), the proof of the relation (8) follows immediately. $\square$

Theorem 2. Consider the system of difference equation (1). If relation (7) is satisfied, then the following statements are true.

(i) Equation (1) has a unique positive equilibrium (x, y) given by

$$x = \frac{k^2 AB - 1}{k(kB - 1)}, \quad y = \frac{k^2 AB - 1}{k(kA - 1)}. \quad (16)$$

(ii) Every positive solution (x_n, y_n) of system (1) tends to the positive equilibrium (x, y) of (1) as $n \rightarrow \infty$.

Proof. (i) Let x and y be positive numbers such that

$$x = A + \frac{x}{ky}, \quad y = B + \frac{y}{kx}. \quad (17)$$

Then, from (7) and (17), we have that the positive solution (x, y) is given by (16). This completes the proof of Part (i).

(ii) From (1) and (8), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &= L_1, & \liminf_{n \rightarrow \infty} x_n &= l_1, \\ \limsup_{n \rightarrow \infty} y_n &= L_2, & \liminf_{n \rightarrow \infty} y_n &= l_2, \end{aligned} \quad (18)$$

where $l_i, L_i \in (0, \infty)$, $i = 1, 2$. Then, from (1) and (18), we get

$$\begin{aligned} L_1 &\leq A + \frac{L_1}{kL_2}, & l_1 &\geq A + \frac{l_1}{kL_2}, \\ L_2 &\leq B + \frac{L_2}{kL_1}, & l_2 &\geq B + \frac{l_2}{kL_1}, \end{aligned} \quad (19)$$

from which we have

$$L_1(kB - 1) \leq l_2(kA - 1), \quad L_2(kA - 1) \leq l_1(kB - 1). \quad (20)$$

Then, relations (7) and (20) imply that $L_1 L_2 \leq l_1 l_2$, from which it follows that

$$L_1 L_2 = l_1 l_2. \quad (21)$$

We claim that

$$L_1 = l_1, \quad L_2 = l_2. \quad (22)$$

Suppose on the contrary that $l_1 < L_1$. Then, from (21), we have $L_1 L_2 = l_1 l_2 < L_1 l_2$ and so $L_2 < l_2$ which is a contradiction. So $L_1 = l_1$. Similarly, we can prove that $L_2 = l_2$. Therefore, (22) is true. Hence, from (1) and (22), there exist the $\lim x_n$ and $\lim y_n$, as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} x_n = x, \quad \lim_{n \rightarrow \infty} y_n = y, \quad (23)$$

where (x, y) is the unique positive equilibrium of (1). This completes the proof of Part (ii). The proof of Theorem 2 is completed. $\square$

Theorem 3. Consider the system of difference equation (1). If relation (7) is satisfied and assuming that

$$\frac{k^2 AB - 1}{kA - 1} + \frac{k^2 AB - 1}{kB - 1} < 1, \quad (24)$$

then the unique positive equilibrium (x, y) is locally asymptotically stable.

Proof. From Theorem 2, the system of difference equation (1) has a unique equilibrium (x, y) . The linearized equation of system (1) about the equilibrium point (x, y) is

$$\Psi_{n+1} = B\Psi_n, \quad (25)$$

where $\Psi_n = (x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k})^T$, and

$$B_{(2k+2) \times (2k+2)} = \begin{pmatrix} \frac{1}{ky} & 0 & \cdots & 0 & 0 & 0 & -\frac{x}{k^2 y^2} & \cdots & -\frac{x}{k^2 y^2} & -\frac{x}{k^2 y^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{y}{k^2 x^2} & \cdots & -\frac{y}{k^2 x^2} & -\frac{y}{k^2 x^2} & \frac{1}{kx} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \quad (26)$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{2k+2}$ denote the eigenvalues of matrix B and let $D = \text{diag}(d_1, d_2, \dots, d_{2k+2})$ be a diagonal matrix, where $d_1 = d_{k+2} = 1$, $d_i = d_{k+1+i} = 1 - i\varepsilon$ ($i = 2, \dots, k+1$), and

$$0 < \varepsilon < \min \left\{ \frac{1}{k+1} \left(1 - \frac{x+y}{ky^2} \right), \frac{1}{k+1} \left(1 - \frac{x+y}{kx^2} \right) \right\}. \quad (27)$$

Clearly, D is invertible. Computing matrix DBD^{-1} , we obtain that

$$DBD^{-1} = \begin{pmatrix} \frac{1}{ky} & 0 & \cdots & 0 & 0 & 0 & -\frac{x}{k^2 y^2} d_1 d_{k+3}^{-1} & \cdots & -\frac{x}{k^2 y^2} d_1 d_{2k+1}^{-1} & -\frac{x}{k^2 y^2} d_1 d_{2k+2}^{-1} \\ d_2 d_1^{-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & d_{k+1} d_k^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{y}{k^2 x^2} d_{k+2} d_2^{-1} & \cdots & -\frac{y}{k^2 x^2} d_{k+2} d_k^{-1} & -\frac{y}{k^2 x^2} d_{k+2} d_{k+1}^{-1} & \frac{1}{kx} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & d_{k+3} d_{k+2}^{-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & d_{2k+2} d_{2k+1}^{-1} & 0 \end{pmatrix}. \quad (28)$$

From $d_1 > d_2 > \dots > d_{k+1} > 0$ and $d_{k+2} > d_{k+3} > \dots > d_{2k+2} > 0$, imply that

$$\begin{aligned} d_2 d_1^{-1} < 1, \quad d_3 d_2^{-1} < 1, \dots, d_{k+1} d_k^{-1} < 1, \\ d_{k+3} d_{k+2}^{-1} < 1, \dots, d_{2k+2} d_{2k+1}^{-1} < 1. \end{aligned} \quad (29)$$

Furthermore, noting (7), (24), and (27), we have

$$\begin{aligned} & \frac{1}{ky} + \frac{x}{k^2 y^2} d_1 d_{k+3}^{-1} + \dots + \frac{x}{k^2 y^2} d_1 d_{2k+2}^{-1} \\ &= \frac{1}{ky} + \frac{x}{k^2 y^2} \left(\frac{1}{1-2\varepsilon} + \dots + \frac{1}{1-(k+1)\varepsilon} \right) \\ &< \left(\frac{1}{ky} + \frac{x}{ky^2} \right) \frac{1}{1-(k+1)\varepsilon} < 1, \\ & \frac{1}{kx} + \frac{y}{k^2 x^2} d_{k+2} d_2^{-1} + \dots + \frac{y}{k^2 x^2} d_{k+2} d_{k+1}^{-1} \\ &= \frac{1}{kx} + \frac{y}{k^2 x^2} \left(\frac{1}{1-2\varepsilon} + \dots + \frac{1}{1-(k+1)\varepsilon} \right) \\ &< \left(\frac{1}{kx} + \frac{y}{kx^2} \right) \frac{1}{1-(k+1)\varepsilon} < 1. \end{aligned} \quad (30)$$

It is well known that B has the same eigenvalues as DBD^{-1} ; we have that

$$\begin{aligned} & \max_{1 \leq i \leq 2k+2} |\lambda_i| \\ &\leq \|DBD^{-1}\|_{\infty} \\ &= \max \left\{ d_2 d_1^{-1}, \dots, d_{k+1} d_k^{-1}, d_{k+3} d_{k+2}^{-1}, \dots, d_{2k+2} d_{2k+1}^{-1}, \frac{1}{ky} \right. \\ &\quad \left. + \frac{x}{k^2 y^2} \left(\frac{1}{1-2\varepsilon} + \dots + \frac{1}{1-(k+1)\varepsilon} \right), \frac{1}{kx} \right. \\ &\quad \left. + \frac{y}{k^2 x^2} \left(\frac{1}{1-2\varepsilon} + \dots + \frac{1}{1-(k+1)\varepsilon} \right) \right\} \\ &< 1. \end{aligned} \quad (31)$$

This implies that the equilibrium (x, y) of (1) is locally asymptotically stable. $\square$

Combining Theorem 2 with Theorem 3, we obtain the following theorem.

Theorem 4. Consider the system of difference equation (1). If relations (7) and (24) are satisfied, then the unique positive equilibrium (x, y) is globally asymptotically stable.

3. Some Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider

FIGURE 1: The dynamics of system (32).

several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations and system of nonlinear difference equations.

Example 1. Consider the following difference equations:

$$\begin{aligned} x_{n+1} &= 0.8 + \frac{x_n}{y_{n-1} + y_{n-2} + y_{n-3}}, \\ y_{n+1} &= 0.6 + \frac{y_n}{x_{n-1} + x_{n-2} + x_{n-3}}, \end{aligned} \quad (32)$$

with the initial values $x_{-i} = y_{-i} = 0.5$ ($i = 1, 2, 3$). Then, the solution (x_n, y_n) of system (32) is bounded and persists and the system has a unique equilibrium $(x, y) = (1.3833, 0.7905)$ which is globally asymptotically stable (see Figure 1).

Example 2. Consider the following difference equations:

$$\begin{aligned} x_{n+1} &= 0.8 + \frac{x_n}{y_{n-1} + y_{n-2} + y_{n-3} + y_{n-4}}, \\ y_{n+1} &= 0.6 + \frac{y_n}{x_{n-1} + x_{n-2} + x_{n-3} + x_{n-4}}, \end{aligned} \quad (33)$$

with the initial values $x_{-i} = y_{-i} = 1.5$ ($i = 1, 2, 3, 4$). Then, the solution (x_n, y_n) of system (33) is bounded and persists and the system has a unique equilibrium $(x, y) = (1.1929, 0.7591)$ which is globally asymptotically stable (see Figure 2).

4. Conclusion

In this paper, we study the dynamics of a system of high order difference equation

$$\begin{aligned} x_{n+1} &= A + \frac{x_n}{\sum_{i=1}^k y_{n-i}}, \quad y_{n+1} = B + \frac{y_n}{\sum_{i=1}^k x_{n-i}}, \\ n &= 0, 1, \dots, \quad k \in \{1, 2, \dots\}. \end{aligned} \quad (34)$$

FIGURE 2: The dynamics of system (33).

It concluded that, under condition $A > 1/k, B > 1/k$, the positive solution (x_n, y_n) of this system is bounded and persists; moreover, if $((k^2 AB - 1)/(kA - 1)) + ((k^2 AB - 1)/(kB - 1)) < 1$, it converges asymptotically the unique equilibrium (x, y) .

We conclude the paper by presenting the following open problem.

Open Problem. Consider the system of difference equation (1) with $A \leq 1/k$ and $B \leq 1/k$. Find the set of all initial conditions that generate bounded solutions. In addition, investigate global behavior of these solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was financially supported by the National Natural Science Foundation of China (Grant no. 11361012), the Scientific Research Foundation of Guizhou Provincial Science and Technology Department ([2013]J2083, [2009]J2061), and the Natural Science Foundation of Guizhou Provincial Educational Department (no. 2008040).

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