Research Article

Injectivity of the Composition Operators of Étale Mappings

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Received 1 May 2014; Revised 10 November 2014; Accepted 19 November 2014; Published 10 December 2014

Academic Editor: Dae San Kim

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Let X be a topological space. The semigroup of all the étale mappings of X (the local homeomorphisms $X \rightarrow X$) is denoted by $\text{et}(X)$. If $G \in \text{et}(X)$, then the G -right (left) composition operator on $\text{et}(X)$ is defined by $R_G (L_G) : \text{et}(X) \rightarrow \text{et}(X)$, $R_G(F) = F \circ G$ ($L_G(F) = G \circ F$). When are the composition operators injective? The Problem originated in a new approach to study étale polynomial mappings $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ and in particular the two-dimensional Jacobian conjecture. This approach constructs a fractal structure on the semigroup of the (normalized) Keller mappings and outlines a new method of a possible attack on this open problem (in preparation). The construction uses the left composition operator and the injectivity problem is essential. In this paper we will completely solve the injectivity problems of the two composition operators for (normalized) Keller mappings. We will also solve the much easier surjectivity problem of these composition operators.

1. Introduction

Let X be a topological space. A mapping $F : X \rightarrow X$ is called a local homeomorphism of X or an étale mapping of X if for any point $x \in X$ there exists a neighborhood U of x such that the restriction of F to U , denoted by $F|_U$, is an homeomorphism. The set of all the étale mappings of X , denoted by $\text{et}(X)$, is a semigroup with a unit with the composition of mappings taken to be the binary operation. If $G \in \text{et}(X)$, then the G -right composition operator on $\text{et}(X)$ is defined by

$$\begin{aligned} R_G : \text{et}(X) &\longrightarrow \text{et}(X) \\ R_G(F) &= F \circ G. \end{aligned} \quad (1)$$

The G -left composition operator on $\text{et}(X)$ is defined by

$$\begin{aligned} L_G : \text{et}(X) &\longrightarrow \text{et}(X) \\ L_G(F) &= G \circ F. \end{aligned} \quad (2)$$

We were interested in the injectivity of these two composition operators in two particular cases. The first is the case of entire functions $\mathbb{C} \rightarrow \mathbb{C}$ that are étale (and normalized). The second case is that of the polynomial mappings $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ with determinant of their Jacobian matrix equal (identically)

to 1 and whose Y -degrees equal their total degrees. For the first case we use the following.

Definition 1. Consider the following

$$\begin{aligned} \text{elh}(\mathbb{C}) &= \{f : \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is entire,} \\ &\quad \forall z \in \mathbb{C} \ f'(z) \neq 0, \ f'(0) = 1\}. \end{aligned} \quad (3)$$

Thus we use in this case the symbol $\text{elh}(\mathbb{C})$ instead of $\text{et}(X)$. Then we have the following.

Proposition 2 (see [1]). *Consider the following $\forall f \in \text{elh}(\mathbb{C})$, R_f is injective.*

Theorem 3 (see [1]). *Let $f \in \text{elh}(\mathbb{C})$. Then L_f is not injective if and only if*

$$f(z) = \frac{1}{b} e^{bz} + a \quad \text{for some } a \in \mathbb{C}, \ b \in \mathbb{C}^\times. \quad (4)$$

This settled the first case. It should be noted (see [1]) that the proof for the left composition operator is much more involved than the proof for the right composition operator (which follows directly from the Picard little theorem). It is in fact the second case that initiated our interest in the injectivity of the composition operators. It results from a new approach

to study étale polynomial mappings $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ and in particular the two-dimensional Jacobian conjecture [2–4]. This approach constructs a fractal structure on the semigroup of the (normalized) Keller mappings and outlines a new method of a possible attack on this open problem (in preparation). The construction uses the left composition operator and the injectivity problem is essential. In this paper we will completely solve the injectivity problems of the two composition operators for (normalized) Keller mappings. We will also solve the much easier surjectivity problem of these composition operators.

2. The Semigroup $\text{et}(\mathbb{C}^2)$ of Normalized Keller Mappings and a Few Facts on Their Asymptotic Variety

Let $F \in \mathbb{C}[X, Y]^2$ be an étale mapping that satisfies the two normalizations

- (1) $\det J_F \equiv 1$;
- (2) $\deg P = \deg_Y P$ and $\deg Q = \deg_Y Q$, where $F(X, Y) = (P(X, Y), Q(X, Y)) \in \mathbb{C}[X, Y]^2$.

The set of all such mappings F will be denoted by $\text{et}(\mathbb{C}^2)$. This semigroup (with respect to composition of mappings) is the parallel of the semigroup $\text{elh}(\mathbb{C})$ for entire functions. The 2-dimensional Jacobian conjecture can be rephrased in each of the following forms:

- (a) $\text{et}(\mathbb{C}^2) \subseteq \text{Aut}(\mathbb{C}^2)$;
- (b) $(\text{et}(\mathbb{C}^2), \circ)$ is a group.

For the next survey of results we refer to the following paper: [5]. We denote by $A(F)$ the asymptotic variety of F , that is, the curve of all the asymptotic values of the mapping F . The canonical geometric basis of F will be denoted by $R_0(F)$. This basis consists of finitely many rational mappings of the following form: $R(X, Y) = (X^{-\alpha}, X^{\beta}Y + X^{-\alpha}\Phi(X))$, where $\alpha \in \mathbb{Z}^+$, $\beta \in \mathbb{Z}^+ \cup \{0\}$, $\Phi(X) \in \mathbb{C}[X]$, and $\deg \Phi < \alpha + \beta$. Also the effective X powers in $X^{\alpha+\beta}Y + \Phi(X)$ have a gcd which equals 1. The cardinality of the geometric basis, $|R_0(F)|$, equals the number of components of the affine algebraic curve $A(F)$. $\forall R \in R_0(F)$ we have the double asymptotic identity $F \circ R = G_R \in \mathbb{C}[X, Y]^2$, where the polynomial mapping G_R is called the R -dual of F . Each $R \in R_0(F)$ generates exactly one component of $A(F)$. This component is normally parametrized by $\{G_R(0, Y) \mid Y \in \mathbb{C}\}$. We will denote by $H_R(X, Y) = 0$ the implicit representation of this component in terms of the irreducible polynomial $H_R \in \mathbb{C}[X, Y]$. There exists a natural number $\gamma(R) \geq 2$ and a polynomial $S_R(X, Y) \in \mathbb{C}[X, Y]$. The affine curve $S_R(X, Y) = 0$ is called the R -phantom curve of F . The R -component of $A(F)$, $H_R(X, Y) = 0$, is a polynomial curve, which is not isomorphic to $\mathbb{A}^1$, and hence in particular must be a singular irreducible curve. We have the relation $H_R(G_R(X, Y)) = X^{\gamma(R)}S_R(X, Y)$. The exponent $\gamma(R)$ satisfies the double inequality $2 \leq \gamma(R) \leq \beta - \alpha$.

In our case of the canonical rational mappings $R \in R_0(F)$, we have $\text{sing}(R) = \{X = 0\}$. The following is true:

$$\begin{aligned} G_R^{-1}(H_R(X, Y) = 0) &= G_R^{-1}(G_R(\text{sing}(R))) \\ &= \text{sing}(R) \cup \{S_R(X, Y) = 0\}. \end{aligned} \quad (5)$$

Thus the G_R -preimage of the R -component of $A(F)$ (which is the G_R -image of $\text{sing}(R)$) is the union of two curves: the first is $\text{sing}(R)$ and the second is the so-called R -phantom curve of F . Even if for a single $R(X, Y)$ the R -phantom curve is empty, then JC(2) follows. Also if $\forall R \in R_0(F)$ $\text{sing}(R) \cap \{S_R(X, Y) = 0\} = \emptyset$, then F is a surjective mapping.

Proposition 4. *If $F, G \in \text{et}(\mathbb{C}^2)$, then $R_0(G) \subseteq R_0(F \circ G)$, $F(A(G)) \subseteq A(F \circ G)$.*

Proof. $R \in R_0(G) \Rightarrow G \circ R \in \mathbb{C}[X, Y]^2 \Rightarrow F \circ (G \circ R) \in \mathbb{C}[X, Y]^2 \Rightarrow (F \circ G) \circ R \in \mathbb{C}[X, Y]^2 \Rightarrow R \in R_0(F \circ G)$. Next we have $(a, b) \in F(A(G)) \Rightarrow \exists R \in R_0(G) \exists Y \in \mathbb{C}$ such that $(a, b) = F((G \circ R)(0, Y)) \Rightarrow \exists R \in R_0(F \circ G) \exists Y \in \mathbb{C}$ such that $(a, b) = ((F \circ G) \circ R)(0, Y) \Rightarrow (a, b) \in A(F \circ G)$. $\square$

The proposition tells us that compositions of étale mappings do not decrease the geometric basis of the right factor and consequently do not decrease the left image of its asymptotic variety. We naturally ask under what conditions the geometric basis of $F \circ G$ is actually larger than that of G ? In other words, we would like to know when is it true that $R_0(G) \subset R_0(F \circ G)$? This happens exactly when $\exists R \in R_0(F \circ G) - R_0(G)$. This means that $(F \circ G) \circ R \in \mathbb{C}[X, Y]^2$, $G \circ R \notin \mathbb{C}[X, Y]^2$. Let $R(X, Y) = (X^{-\alpha}, X^{\beta}Y + X^{-\alpha}\Phi(X))$, $G(X, Y) = (P(X, Y), Q(X, Y))$. Then

$$\begin{aligned} (G \circ R)(X, Y) &= (P(X^{-\alpha}, X^{\beta}Y + X^{-\alpha}\Phi(X)), \\ &\quad Q(X^{-\alpha}, X^{\beta}Y + X^{-\alpha}\Phi(X))) \\ &\in \mathbb{C}(X, Y)^2 - \mathbb{C}[X, Y]^2. \end{aligned} \quad (6)$$

We clearly have $\text{sing}(G \circ R) \subseteq \text{sing}(R)$ and so $\text{sing}(G \circ R) = \{X = 0\}$. By $F \circ (G \circ R) = (F \circ G) \circ R \in \mathbb{C}[X, Y]^2$ we have $G \circ R \in R(F)$. This is not necessarily a member of the geometric basis of F . The canonical geometric basis of F , $R_0(F)$ contains finitely many rational mappings of the form $S(X, Y) = (X^{-a}, X^bY + X^{-a}\Psi(X))$. Since $G \in \text{et}(\mathbb{C}^2)$, it follows that $|\mathbb{C}^2 - G(\mathbb{C}^2)| < \infty$ (a similar phenomenon as the Picard little theorem). If L is an asymptotic tract of F , then $G^{-1}(L)$ cannot be a bounded subset of $\mathbb{C}^2$. The reason is that if $G^{-1}(L)$ is compact, then $G(G^{-1}(L))$ is compact and since $L \subseteq G(G^{-1}(L)) \subseteq G(G^{-1}(L))$ this would imply the contradiction that L is bounded (and hence cannot be an asymptotic tract). Hence $G^{-1}(L)$ has at least one component, say L_1 , that goes to infinity. This is because the number of components of $G^{-1}(L)$ is finite and $G^{-1}(L)$ is not bounded. So $F \circ G$ has a limit along L_1 which equals the above asymptotic value of F . This proves the following generalization of the second part of Proposition 4, namely, the following proposition.

Proposition 5. *If $F, G \in \text{et}(\mathbb{C}^2)$, then $A(F) \cup F(A(G)) = A(F \circ G)$.*

This proposition implies that if $A(F) \subset F(A(G))$, then necessarily $R_0(G) \subset R_0(F \circ G)$ because, as shown in the proof of Proposition 4 $\forall R \in R_0(G)$, $((F \circ G) \circ R)(\text{sing}(R)) \subseteq F(A(G))$.

Proposition 6. *Let $F \in \text{et}(\mathbb{C}^2)$. If $\exists G \in \text{et}(\mathbb{C}^2)$ such that $R_0(G) = R_0(F \circ G)$, then $F(\mathbb{C}^2) = \mathbb{C}^2$; that is, F is a surjective mapping.*

Proof. Since $F \in \text{et}(\mathbb{C}^2)$ we have $\mathbb{C}^2 - F(\mathbb{C}^2) \subseteq A(F)$ because in this case the only points in the complement of the image of F are the finitely many Picard exceptional values of F which are asymptotic values of F . If, as the assumption says, $R_0(G) = R_0(F \circ G)$, then by Proposition 5 we must have $A(F) \subseteq F(A(G)) \subseteq F(\mathbb{C}^2)$ and so there are no Picard exceptional values of the mapping F . $\square$

If $F \in \text{et}(\mathbb{C}^2)$ is not a surjective mapping, then the last proposition implies that $\forall G \in \text{et}(\mathbb{C}^2)$; we must have $R_0(G) \subset R_0(F \circ G)$. In particular $R_0(F) \subset R_0(F \circ F)$. This is the choice $G = F$. If we choose $G = F \circ F$, we get $R_0(F \circ F) \subset R_0(F \circ (F \circ F))$. Now it is clear that by induction we get the infinite chain of strict inequalities:

$$\begin{aligned} R_0(F) &\subset R_0(F \circ F) \subset R_0(F \circ F \circ F) \\ &\subset \dots \subset R_0(F^{\circ n}) \subset \dots, \end{aligned} \quad (7)$$

where $F^{\circ n} = F \circ \dots \circ F$ n times fold composition. Since the cardinality of the geometric basis $R_0(G)$ is the number of components of the asymptotic variety $A(G)$, it follows in this case that the asymptotic varieties of iterates of F are of increasing complexity in the sense that the number of components of the curve $A(F^{\circ(n+1)})$ is strictly larger than the number of components of the curve $A(F^{\circ n})$. By Proposition 5 we always have $A(F) \cup F(A(F)) = A(F \circ F)$. Hence $A(F) \cup F(A(F)) \cup (F \circ F)(A(F)) = A(F) \cup F(A(F) \cup F(A(F))) = A(F) \cup F(A(F \circ F)) = A(F \circ F \circ F)$. By induction we get in general:

$$A(F) \cup \bigcup_{k=1}^n F^{\circ k}(A(F)) = A(F^{\circ(n+1)}). \quad (8)$$

3. The Composition Operators on $\text{et}(\mathbb{C}^2)$ Are Not Surjective, but the Right Composition Operator Is Injective

Proposition 7. *The mappings R_F, L_F are not surjective if and only if $F \notin \text{Aut}(\mathbb{C}^2)$. In fact in this case we have $R_F(\text{et}(\mathbb{C}^2)) \subset \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$, $L_F(\text{et}(\mathbb{C}^2)) \subset \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$.*

Proof. Consider the following:

$$\begin{aligned} R_0(R_F(G)) &= R_0(G \circ F) \supseteq R_0(F) \neq \emptyset, \\ A(L_F(G)) &= A(F \circ G) \supseteq A(F) \neq \emptyset. \end{aligned} \quad (9)$$

$\square$

Proposition 8. *R_F is injective.*

Proof. $R_F(G) = R_F(H) \Rightarrow G \circ F = H \circ F$. Since $F \in \text{et}(\mathbb{C}^2)$ we have $|\mathbb{C}^2 - F(\mathbb{C}^2)| < \infty$ and by the assumption $G|_{F(\mathbb{C}^2)} = H|_{F(\mathbb{C}^2)}$. Hence $G \equiv H$. $\square$

We naturally inquire if also L_F is injective. So let us assume that $L_F(G) = L_F(H)$. Then $F \circ G = F \circ H$. If we denote $T = F \circ G$, then $R_0(G), R_0(H) \subseteq R_0(T)$ and also $A(F) \cup F(A(G)) = A(F) \cup F(A(H))$. If $A(F) = \emptyset$, then $F \in \text{Aut}(\mathbb{C}^2)$ and so $G = F^{-1}(F \circ G) = F^{-1}(F \circ H) = H$. If, on the other hand, $G \neq H$, then there are points $(X, Y) \in \mathbb{C}^2$ for which $G(X, Y) \neq H(X, Y)$. By our assumption, $F(G(X, Y)) = F(H(X, Y))$, so F is not in $\text{Aut}(\mathbb{C}^2)$ and it identifies different images of G and H of the same (X, Y) . We ask the following question: suppose that $F \circ G = F \circ H$, $G \neq H$, is there a point $(X, Y) \in \mathbb{C}^2$ for which $G(X, Y) = H(X, Y)$?

Based on our experience with entire functions we tend to prove that the answer to the question is negative. Indeed this is the case and the proof is almost identical to the entire case; see [1]. Namely, if the answer is affirmative, then we have two types of points in $\mathbb{C}^2$: $(U, V) \in \mathbb{C}^2$ for which $G(U, V) \neq H(U, V)$ and the complimentary set, where both sets are nonempty. Let us denote by N the first subset of $\mathbb{C}^2$; that is,

$$N = \{(U, V) \in \mathbb{C}^2 \mid G(U, V) \neq H(U, V)\}. \quad (10)$$

The subset N of $\mathbb{C}^2$ is open in the strong topology because G, H are étale mappings and if $G(U, V) \neq H(U, V)$, then $\exists O$ an open neighborhood of (U, V) in the strong topology such that $G(O) \cap H(O) = \emptyset$. So the complimentary subset of N is a closed nonempty subset of $\mathbb{C}^2$. Let $(X, Y) \in \partial N^c$ be a boundary point of N^c . Let $(U_n, V_n) \in N$ satisfy $\lim(U_n, V_n) = (X, Y)$. Then $\forall n \in \mathbb{Z}^+$, $G(U_n, V_n) \neq H(U_n, V_n)$, $F(G(U_n, V_n)) = F(H(U_n, V_n))$, and $G(X, Y) = H(X, Y)$. This implies that in any strong neighborhood of $G(X, Y) = H(X, Y)$ there are different points, say $G(U_n, V_n) \neq H(U_n, V_n)$ for $n \in \mathbb{Z}^+$ large enough, so that $F(G(U_n, V_n)) = F(H(U_n, V_n))$. Hence F is not injective in any strong neighborhood of $G(X, Y) = H(X, Y)$. Thus $F \notin \text{et}(\mathbb{C}^2)$. This contradiction proves the following.

Proposition 9. *Suppose that L_F is not injective for some $F \in \text{et}(\mathbb{C}^2)$. Then $\forall G \neq H$, $G, H \in \text{et}(\mathbb{C}^2)$ such that $L_F(G) = L_F(H)$ and $\forall (X, Y) \in \mathbb{C}^2$ we have $G(X, Y) \neq H(X, Y)$.*

Remark 10. Proposition 7 asserts that $\forall F \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \exists G, H \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $\forall M \in \text{et}(\mathbb{C}^2)$ we have $M \circ F \neq G$ and $F \circ M \neq H$.

4. The Size, d_F , of the Generic Fiber of a Keller Mapping $F \in \text{et}(\mathbb{C}^2)$

We will need the generic size of a fiber of a mapping $F = (P, Q) \in \text{et}(\mathbb{C}^2)$. If we denote $\deg P(X, Y) = n$ and $\deg Q(X, Y) = m$, then $\forall (a, b) \in \mathbb{C}^2$ the F fiber over (a, b) is $F^{-1}(a, b) = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = F(a, b)\}$. It is well known that this set is a finite subset of $\mathbb{C}^2$ and, by the Bezout theorem,

we have $|\{(x, y) \in \mathbb{C}^2 \mid F(x, y) = (a, b)\}| = |F^{-1}(a, b)| \leq n \cdot m$. Moreover, there is a number that we will denote by d_F such that generically in (a, b) we have $|F^{-1}(a, b)| = d_F$. This means that $\{(a, b) \in \mathbb{C}^2 \mid |F^{-1}(a, b)| \neq d_F\}$ is a closed and proper Zariski subset of $\mathbb{C}^2$. In fact $\forall (a, b) \in \mathbb{C}^2, |F^{-1}(a, b)| \neq d_F \Rightarrow |F^{-1}(a, b)| < d_F$. Thus we have $d_F = \max\{|F^{-1}(a, b)| \mid (a, b) \in \mathbb{C}^2\}$.

Definition 11. Let $F \in \text{et}(\mathbb{C}^2)$. We will denote $d_F = \max\{|F^{-1}(a, b)| \mid (a, b) \in \mathbb{C}^2\}$. We will call d_F the geometrical degree of the étale mapping F .

Proposition 12. Consider the following $\forall F, G \in \text{et}(\mathbb{C}^2)$, $d_{F \circ G} = d_F \cdot d_G$.

This is a well-known result. We include one of its proofs for convenience.

Proof. $\forall (a, b) \in \mathbb{C}^2, (F \circ G)^{-1}(a, b) = G^{-1}(F^{-1}(a, b))$. But generically in (a, b) , $|F^{-1}(a, b)| = d_F$ and generically in (c, d) , $|G^{-1}(c, d)| = d_G$. $\square$

Definition 13. An étale mapping $F \in \text{et}(\mathbb{C}^2)$ is composite if $\exists G, H \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $F = G \circ H$. An étale mapping $A \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ is prime if it is not composite. This is equivalent to $A = B \circ C$ for some $B, C \in \text{et}(\mathbb{C}^2) \Rightarrow B \in \text{Aut}(\mathbb{C}^2) \vee C \in \text{Aut}(\mathbb{C}^2)$. The subset of $\text{et}(\mathbb{C}^2)$ of all the prime mappings will be denoted by $\text{et}_p(\mathbb{C}^2)$. Thus the set of all the composite étale mappings is $\text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2)$.

Proposition 14. $\forall F \in \text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2)$, d_F is not a prime number. Equivalently, $\forall F \in \text{et}(\mathbb{C}^2)$, d_F is a prime number $\Rightarrow F \in \text{et}_p(\mathbb{C}^2)$.

Proof. $F \in \text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2) \Rightarrow \exists G, H \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $F = G \circ H$ (by the definition) $\Rightarrow d_F = d_G \cdot d_H$, $d_G, d_H > 1$ (by Proposition 12 and the fact that $d_M = 1 \Leftrightarrow M \in \text{Aut}(\mathbb{C}^2)$) $\Rightarrow d_F$ is a composite integer. $\square$

Theorem 15. The following hold true.

- (1) If $\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \neq \emptyset$, then $\text{et}_p(\mathbb{C}^2) \neq \emptyset$.
- (2) $\forall F \in \text{et}(\mathbb{C}^2) \exists k \in \mathbb{Z}^+ \cup \{0\} \exists A_0 \in \text{Aut}(\mathbb{C}^2) \exists P_1, \dots, P_k \in \text{et}_p(\mathbb{C}^2)$ such that $F = A_0 \circ P_1 \circ \dots \circ P_k$.

Proof. If $\text{et}_p(\mathbb{C}^2) = \emptyset$, then $\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ are all composite étale mappings. Let $F \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$, then $\exists G_1, G_2' \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $F = G_1 \circ G_2'$. So $\exists G_2, G_3' \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $G_2' = G_2 \circ G_3'$. Hence $F = G_1 \circ G_2 \circ G_3'$. Continuing this we get for any $k \in \mathbb{Z}^+ \exists G_1, \dots, G_k \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $F = G_1 \circ \dots \circ G_k$ and by Proposition 12 $d_F = \prod_{j=1}^k d_{G_j}$. But $\forall 1 \leq j \leq k, d_{G_j} \geq 2$ and so $\forall k \in \mathbb{Z}^+, d_F \geq 2^k$, a contradiction to $d_F < \infty$. Thus $\text{et}_p(\mathbb{C}^2) \neq \emptyset$.

Now part 2 is standard, for if $F \in \text{Aut}(\mathbb{C}^2)$ we take $A_0 = F$ and $k = 0$. If $F \in \text{et}_p(\mathbb{C}^2)$ we take $A_0 = \text{id}$, $k = 1$, and $P_1 = F$. If $F \in \text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2)$, then $F = G \circ H$ for some $G, H \in$

$\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$. So by Proposition 12 $d_F = d_G \cdot d_H$ and since $d_G, d_H \geq 2$ it follows that $d_G, d_H < d_F$ and we conclude the proof of part 2 using induction on the geometrical degree. Namely, $G = P_1 \circ \dots \circ P_m, H = P_{m+1} \circ \dots \circ P_k$ for $m \geq 1, k \geq m + 1$, and some primes $P_1, \dots, P_k \in \text{et}_p(\mathbb{C}^2)$. $\square$

5. The Metric Spaces $(\text{et}(\mathbb{C}^2), \rho_D)$

We will need a special kind of four (real) dimensional subsets of $\mathbb{R}^4$. These will serve us to construct suitable metric structures on $\text{et}(\mathbb{C}^2)$.

Definition 16. Let D be an open subset of $\mathbb{C}^2$ with respect to the strong topology that satisfies the following conditions.

- (1) $\text{int}(\overline{D}) = D$ (D has no “slits”).
- (2) $\overline{D}$ is a compact subset of $\mathbb{C}^2$ (in the strong topology).
- (3) $\forall G_1, G_2 \in \text{et}(\mathbb{C}^2), G_1(D) = G_2(D) \Leftrightarrow G_1 = G_2$.

We define the following real valued function:

$$\begin{aligned} \rho_D : \text{et}(\mathbb{C}^2) \times \text{et}(\mathbb{C}^2) &\longrightarrow \mathbb{R}^+ \cup \{0\}, \\ \rho_D(G_1, G_2) &= \text{the volume of } G_1(D) \Delta G_2(D). \end{aligned} \quad (11)$$

Here we use the standard set-theoretic notation of the symmetric difference between two sets A and B ; that is, $A \Delta B = (A - B) \cup (B - A)$.

Remark 17. It is not clear how to construct an open subset D of $\mathbb{C}^2$ that will satisfy the three properties that are required in Definition 16. We will postpone for a while the demonstration that such open sets exist.

Proposition 18. ρ_D is a metric on $\text{et}(\mathbb{C}^2)$.

Proof. (1) $\rho_D(G_1, G_2) = 0 \Leftrightarrow$ the volume of $G_1(D) \Delta G_2(D) = 0 \Leftrightarrow G_1(D) = G_2(D)$ (where the last equivalence follows by the fact that G_1 and G_2 are local homeomorphisms in the strong topology and because of condition (1) in Definition 16) $\Leftrightarrow G_1 = G_2$ (by condition (3) in Definition 16).

(2) By $G_1(D) \Delta G_2(D) = G_2(D) \Delta G_1(D)$, it follows that $\rho_D(G_1, G_2) = \rho_D(G_2, G_1)$.

(3) Here we use a little technical set-theoretic containment. Namely, for any three sets A, B , and C we have

$$A \Delta C \subseteq (A \Delta B) \cup (B \Delta C). \quad (12)$$

This implies that $G_1(D) \Delta G_3(D) \subseteq (G_1(D) \Delta G_2(D)) \cup (G_2(D) \Delta G_3(D))$ from which it follows that

$$\begin{aligned} &(\text{the volume of } G_1(D) \Delta G_3(D)) \\ &\leq (\text{the volume of } G_1(D) \Delta G_2(D)) \\ &\quad + (\text{the volume of } G_2(D) \Delta G_3(D)). \end{aligned} \quad (13)$$

Hence the triangle inequality $\rho_D(G_1, G_3) \leq \rho_D(G_1, G_2) + \rho_D(G_2, G_3)$ holds. $\square$

So far we thought of the volume of $G_1(D)\Delta G_2(D)$ as the volume of the open set which is the symmetric difference between the G_1 image and the G_2 image of the open set D . However, the mappings G_1 and G_2 are étale and in particular need not be injective. We will take into the volume computation the multiplicities of G_1 and of G_2 . By Theorem 3 on page 39 of [6] we have the following: given that $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ we define $\tilde{F} = (\operatorname{Re} F_1, \operatorname{Im} F_1, \dots, \operatorname{Re} F_n, \operatorname{Im} F_n) ; \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. Then $\det J_{\tilde{F}} = |\det J_F|^2$. Thus the Jacobian condition $\det J_F \equiv 1$ implies that $\det J_{\tilde{F}} \equiv 1$. So the real mapping $\tilde{F}$ preserves the usual volume form. In order to take into account the multiplicities of the étale mappings G_1 and G_2 when computing the volume of the symmetric difference $G_1(D)\Delta G_2(D)$ we had to do the following. For any $G \in \operatorname{et}(\mathbb{C}^2)$ instead of computing,

$$\iiint_D (\det J_{\tilde{G}} \cdot dV) = \iiint_D dV, \quad (14)$$

we compute

$$\iiint_{\tilde{G}(D)} dX_1 dX_2 dY_1 dY_2 \quad (15)$$

where $X = X_1 + iX_2$, $Y = Y_1 + iY_2$.

For every $j = 1, 2, \dots, d_G$ we denote by D_j the subset of D such that for each point of D_j there are exactly j points of D that are mapped by G to the same image of that point. In other words, $D_j = \{\alpha \in D \mid |\tilde{G}^{-1}(\tilde{G}(\alpha)) \cap D| = j\}$. We assume that D is large enough so that $\forall j = 1, \dots, d_G$ we have $D_j \neq \emptyset$. For our étale mappings it is well known that if $j < d_G$, then $\dim D_j < \dim D$, so the volume of these D_j 's contribution equals 0. The dimension claim follows by the well-known fact that the size of a generic fiber $|G^{-1}(x)|$ equals d_G and that d_G is also the maximal size of any of the fibers of G . However, for the sake of treating more general families of mappings, we denote by $\operatorname{vol}(D_j)$ the volume of the set D_j . Then D has a partition into exactly j subsets of equal volume. The volume of each such a set is $\operatorname{vol}(D_j)/j$ and each such a set has exactly one of the j points in $\tilde{G}^{-1}(\tilde{G}(\alpha)) \cap D$ for each $\alpha \in D_j$. We note that $\operatorname{vol}(\tilde{G}(D_j)) = \operatorname{vol}(D_j)/j$ by the Jacobian Condition. Thus the volume with the multiplicity of $\tilde{G}$ taken into account is given by

$$\begin{aligned} \operatorname{vol}(\tilde{G}(D)) + \sum_{j=2}^{d_G} (j-1) \cdot \frac{\operatorname{vol}(D_j)}{j} \\ = \operatorname{vol}(\tilde{G}(D)) + \sum_{j=2}^{d_G} (j-1) \cdot \operatorname{vol}(\tilde{G}(D_j)). \end{aligned} \quad (16)$$

We note that $\tilde{G}(D) = \bigcup_{j=1}^{d_G} \tilde{G}(D_j)$ is a partition, so $\operatorname{vol}(\tilde{G}(D)) = \sum_{j=1}^{d_G} \operatorname{vol}(\tilde{G}(D_j))$. Hence we can express the desired volume by

$$\begin{aligned} \operatorname{vol}(\tilde{G}(D)) + \sum_{j=2}^{d_G} (j-1) \cdot \operatorname{vol}(\tilde{G}(D_j)) \\ = \sum_{j=1}^{d_G} j \cdot \operatorname{vol}(\tilde{G}(D_j)). \end{aligned} \quad (17)$$

We note that this equals $\sum_{j=1}^{d_G} \operatorname{vol}(D_j)$ and since $D = \bigcup_{j=1}^{d_G} D_j$ is a partition we have $\operatorname{vol}(D) = \sum_{j=1}^{d_G} \operatorname{vol}(D_j)$. As expected, the volume computation that takes into account the multiplicity of G is in general larger than the geometric volume $\operatorname{vol}(\tilde{G}(D))$. The access can be expressed in several forms:

$$\begin{aligned} \operatorname{vol}(D) - \operatorname{vol}(\tilde{G}(D)) &= \sum_{j=2}^{d_G} (j-1) \cdot \operatorname{vol}(\tilde{G}(D_j)) \\ &= \sum_{j=2}^{d_G} \left(1 - \frac{1}{j}\right) \operatorname{vol}(D_j). \end{aligned} \quad (18)$$

Coming back to the computation of the metric distance $\rho_D(G_1, G_2)$ = the volume of $G_1(D)\Delta G_2(D)$ we compute the volume of $G_1(D) - G_2(D)$ with the multiplicity of G_1 while the volume of $G_2(D) - G_1(D)$ is computed with the multiplicity of G_2 .

6. Characteristic Sets of Families of Holomorphic Local Homeomorphisms $\mathbb{C}^2 \rightarrow \mathbb{C}^2$

In this section we prove the existence of sets D that satisfy the three properties that are required in Definition 16. The third property will turn out to be the tricky one.

Definition 19. Let Γ be a family of holomorphic local homeomorphisms $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. A subset $D \subseteq \mathbb{C}^2$ is called a characteristic set of Γ if it satisfies the following condition: $\forall F_1, F_2 \in \Gamma, F_1(D) = F_2(D) \Leftrightarrow F_1 = F_2$.

We start by recalling the well-known rigidity property of holomorphic functions in one complex variable. Also known as the permanence principle, or the identity theorem. The identity theorem for analytic functions of one complex variable says that if $D \subseteq \mathbb{C}$ is a domain (an open and a connected set) and if E is a subset of D that has a nonisolated point and if $f(z)$ is an analytic function defined on D and vanishing on E , then $f(z) = 0$ for all $z \in D$.

There is an identity theorem for analytic functions in several complex variables, but for more than one variable the above statement is false. One correct statement is as follows: "Let $f(z)$ and $g(z)$ be holomorphic functions in a domain D in $\mathbb{C}^n$. If $f(z) = g(z)$ for all z in a non-empty set δ in D , then $f(z) = g(z)$ in D . Hence, analytic continuation of holomorphic functions in several complex variables can be performed as in the case of one complex variable. Contrary to the case of one complex variable, the zero set of a holomorphic function in a domain $D \subseteq \mathbb{C}^n, n \geq 2$, contains no isolated points. Thus even if $f(z) = g(z)$ in a set

with accumulation points in D , it does not necessarily follow that $f(z) = g(z)$ in D . For example, in $\mathbb{C}^2$ with variables z and w we can take $f(z, w) = z$ and $g(z, w) = z^2$ [7, Chapter 1, page 16].

In spite of the above standard identity theorem for $n \geq 2$ complex variables that requires a thick set E (i.e., an open set) on which $f(z) = g(z)$ one can do much better. Let us start with the following. Let $F(Z, W)$ be an entire function of two complex variables Z and W . Let us define a subset E of $\mathbb{C}^2$ as follows. We take a convergent sequence $\{Z_k\}_{k=1}^\infty$ of different numbers. Thus $\lim Z_k = a$ and $j \neq k \Rightarrow Z_j \neq Z_k$. For each k , let $\{W_j^{(k)}\}_{j=1}^\infty$ be a convergent sequence of different numbers, such that their limit is $\lim_{j \rightarrow \infty} W_j^{(k)} = Z'_k$. We define $E = \{(Z_k, W_j^{(k)}) \mid j, k = 1, 2, 3, \dots\}$. Now we have the following.

Proposition 20. *If $F(Z, W)$ vanishes on E , that is, $F(Z_k, W_j^{(k)}) = 0$ for $j, k = 1, 2, 3, \dots$, then $F(Z, W) \equiv 0$ is the zero function.*

Remark 21. We note that E is a thin set, in fact a countable set. Even the closure $\bar{E}$ is thin.

Proof of Proposition 20. Since $F(Z, W)$ is an entire function, it can be represented as a convergent power series centered at $(0, 0)$ with an infinite radius of convergence. We can sum the terms in the order we please. Let us write $F(Z, W)$ as a power series in W with coefficients that are entire functions in Z . Thus we have $F(Z, W) = \sum_{k=0}^\infty a_k(Z)W^k$, where for each $k = 0, 1, 2, \dots$, $a_k(Z)$ is an entire function in the variable Z . For a fixed $l \in \{1, 2, 3, \dots\}$ we have by our assumptions the following: $F(Z_l, W_j^{(l)}) = 0$ for $j = 1, 2, 3, \dots$. But $\lim_{j \rightarrow \infty} W_j^{(l)} = Z'_l$ so that $g_l(W) = F(Z_l, W)$ is an entire function of the single variable W , which vanishes on a convergent sequence $\{W_j^{(l)}\}_{j=1}^\infty$. By the identity theorem of one complex variable we deduce that $g_l(W) \equiv 0$, the zero function. Since $g_l(W) = \sum_{k=0}^\infty a_k(Z_l)W^k$ it follows that the Maclaurin coefficients $a_k(Z_l)$, $k = 0, 1, 2, \dots$ vanish. Now, this is valid for each l , and $\lim Z_l = a$ converges. Since each $a_k(Z)$ is an entire function which vanishes on a convergent sequence $\{Z_l\}_{l=1}^\infty$ it follows, once again, by the identity theorem in one complex variable, that is, $a_k(Z) \equiv 0$, the zero function $k = 0, 1, 2, \dots$. Hence we conclude that $F(Z, W) = \sum_{k=0}^\infty a_k(Z)W^k \equiv 0$. $\square$

This type of elementary arguments that was used to construct a thin set E for identity purpose is not new. For example, see the following.

Theorem 22 (see [8]). *Let $D \subseteq \mathbb{C}$ be a domain, and let E be a subset of D that has a nonisolated point. Let $F(Z, W)$ be a function defined for $Z, W \in D$ such that $F(Z, W)$ is analytic in Z for each fixed $W \in D$ and analytic in W for each fixed $Z \in D$. If $F(Z, W) = 0$ whenever Z and W both belong to E , then $F(Z, W) = 0$ for all $Z, W \in D$.*

Advancing along the lines of the construction of the thin set in Proposition 20 we note that if $\{Z_k\}_{k=1}^\infty$ is a sequence of

different numbers that converges to $\lim Z_k = a$, and if for each $k = 1, 2, 3, \dots$ there is a straight line segment $[\alpha_k, \beta_k]$ of W 's such that two entire functions $F(Z, W)$ and $G(Z, W)$ agree on the union (a countable union) of the segments $\{Z_k\} \times [\alpha_k, \beta_k]$, that is, $F(Z_k, W) = G(Z_k, W)$, $\forall W \in [\alpha_k, \beta_k]$, then $F(Z, W) \equiv G(Z, W)$, $\forall (Z, W) \in \mathbb{C}^2$.

We now will construct characteristic sets of families Γ of holomorphic local homeomorphisms $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$.

Definition 23. Let m be a natural number and $\alpha \in \mathbb{C}^2$. An m -star at α is the union of m line segments, so that any pair intersects in α .

Definition 24. Let l be a line segment and let $\{\alpha_k\}$ be a countable dense subset of l . Let $\{n_k\}$ be a sequence of different natural numbers and $\forall k$, let S_{n_k} be an n_k -star at α_k such that one of the star's segments lies on l , and such that $\forall k_1 \neq k_2$, $\tilde{S}_{n_{k_1}} \cap \tilde{S}_{n_{k_2}} = \emptyset$. Here we denoted $\tilde{S} = S - l$. Moreover, we group the stars in bundles of, say 5, thus getting the sequence of star bundles:

$$\{S_{n_1}, S_{n_2}, S_{n_3}, S_{n_4}, S_{n_5}\},$$

$$\{S_{n_6}, \dots, S_{n_{10}}\}, \dots, \{S_{n_{5j+1}}, \dots, S_{n_{5j+5}}\}, \dots \quad (19)$$

and for each bundle of five we take the maximal length of its rays to be at most $1/10$ the length of the maximal length of the previous bundle. We define

$$l_0^{\{n_k\}} = l \cup \bigcup_{k=1}^\infty S_{n_k}. \quad (20)$$

Let $\{Z_k\}_{k=1}^\infty$ be a sequence of different complex numbers that converges to $\lim Z_k = a$. Let $\{\{n_j^{(k)}\}_{j=1}^\infty\}_{k=1}^\infty$ be a partition of the natural numbers, $\mathbb{Z}^+$. In fact all we need is the disjointness, that is, $k_1 \neq k_2 \Rightarrow \{n_j^{(k_1)}\}_{j=1}^\infty \cap \{n_j^{(k_2)}\}_{j=1}^\infty = \emptyset$. Let us consider the starred segments

$$\{l_0^{\{n_j^{(k)}\}_{j=1}^\infty} \mid k = 1, 2, 3, \dots\} \quad (21)$$

and define the following countable union of starred segments in $\mathbb{C}^2$:

$$\bigcup_{k=1}^\infty \{Z_k\} \times l_0^{\{n_j^{(k)}\}_{j=1}^\infty}, \quad (22)$$

where we assume that the lengths of the star rays were chosen to satisfy disjointness in $\mathbb{C}^2$; namely,

$$k_1 \neq k_2 \Rightarrow \{Z_{k_1}\} \times l_0^{\{n_j^{(k_1)}\}_{j=1}^\infty} \cap \{Z_{k_2}\} \times l_0^{\{n_j^{(k_2)}\}_{j=1}^\infty} = \emptyset. \quad (23)$$

We let

$$E = \bigcup_{k=1}^\infty \{Z_k\} \times l_0^{\{n_j^{(k)}\}_{j=1}^\infty}, \quad (24)$$

or if we need a closed (compact) set, the closure of this union.

Proposition 25. *Let Γ be any family of entire holomorphic local homeomorphisms $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Then E is a characteristic set of Γ .*

Proof. Let $F_1, F_2 \in \Gamma$ satisfy $F_1(E) = F_2(E)$. Then each starred line segment

$$\{Z_k\} \times l_0^{\{n_j^{(k)}\}_{j=1}^\infty} \quad (25)$$

must be mapped onto a curve

$$F_1 \left(\{Z_k\} \times l_0^{\{n_j^{(k)}\}_{j=1}^\infty} \right) = F_2 \left(\{Z_k\} \times l_0^{\{n_j^{(k)}\}_{j=1}^\infty} \right) \quad (26)$$

and each $n_j^{(k)}$ -star on l , $S_{n_j^{(k)}}$ is mapped onto a holomorphic $n_j^{(k)}$ -star

$$F_1 \left(\{Z_k\} \times S_{n_j^{(k)}} \right) = F_2 \left(\{Z_k\} \times S_{n_j^{(k)}} \right). \quad (27)$$

This is because the valence sequences of the stars

$$\left\{ \left\{ n_j^{(k)} \right\}_{j=1}^\infty \right\}_{k=1}^\infty \quad (28)$$

are pairwise disjoint natural numbers, and F_1, F_2 are local homeomorphisms and hence preserve the star valencies $n_j^{(k)}$. The centers of the holomorphic stars

$$\left\{ F_1 \left(\alpha_{n_j^{(k)}} \right) \right\} = \left\{ F_2 \left(\alpha_{n_j^{(k)}} \right) \right\} \quad (29)$$

form a countable and a dense subset of the curves $F_1(\{Z_k\} \times l) = F_2(\{Z_k\} \times l)$. By continuity this implies that the restrictions

$$F_1|_{\{Z_k\} \times l}, \quad F_2|_{\{Z_k\} \times l} \quad (30)$$

coincide. Since F_1 and F_2 are holomorphic, this implies by Proposition 20 (which is a variant of the identity theorem for entire functions $\mathbb{C}^2 \rightarrow \mathbb{C}^2$) that $F_1 \equiv F_2$. $\square$

Remark 26. Proposition 25 holds true for any rigid family of local homeomorphisms. Rigidity here means that

$$F_1|_{\{Z_k\} \times l} = F_2|_{\{Z_k\} \times l} \iff F_1 \equiv F_2. \quad (31)$$

So the proposition holds true for holomorphic mappings, for harmonic mappings, and in particular for $\text{et}(\mathbb{C}^2)$.

We recall that Definition 16 required also two additional topological properties; namely, the open set D should satisfy $\text{int}(\overline{D}) = D$; $\overline{D}$ is compact (all in the strong topology). These automatically exclude the set E that was constructed in Definition 24. However, we can modify this construction to get at least an open set.

Proposition 27. *Let Γ be any family of holomorphic local homeomorphisms $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Let U be any open subset of $\mathbb{C}^2$ with a smooth boundary that contains the compact E . Then the open set $U - E$ is a characteristic set of Γ .*

Proof. Since E can not be mapped in the smooth ∂U by an holomorphic local homeomorphism, we have for any $F_1, F_2 \in \Gamma$ for which $F_1(U - E) = F_2(U - E)$ that also $F_1(E) = F_2(E)$. Now the result follows by Proposition 25. $\square$

Remark 28. We note that if $\overline{U}$ is a compact then $U - E$ satisfies, at least, the requirement that $\overline{U - E}$ is compact. However, the “no-slit” condition $\text{int}(\overline{U - E}) = \text{int}(\overline{U}) \neq U - E$ fails.

Now that we gained some experience with the topological construction of E we are going to make one more step and fix its shortcomings that were mentioned above. We need to construct a domain D of $\mathbb{C}^2$ which has the following three properties.

- (1) $\text{int}(\overline{D}) = D$ relative to the complex topology.
- (2) $\overline{D}$ is a compact subset of $\mathbb{C}^2$ relative to the strong topology.
- (3) $\forall G_1, G_2 \in \text{et}(\mathbb{C}^2), G_1(D) = G_2(D) \iff G_1 \equiv G_2$.

The complex topology and the strong topology are the same. Our construction will be a modification of the construction of the domain that was constructed in Proposition 27. We start by modifying the notion of an m -star that was introduced in Definition 23.

Definition 29. Let m be a natural number and $\alpha \in \mathbb{C}^n$. A thick m -star at α is a union of $2m$ triangles, so that any pair intersects exactly at one vertex, and this vertex (i.e., common to all the $2m$ triangles) is α .

Definition 30. Let E be the construction of Definition 24 that uses thick m -stars.

Proposition 31. *Let Γ be any family of holomorphic local homeomorphisms $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Then E is a characteristic set of Γ .*

Proof. The proof is the same word by word as that of Proposition 25 where we replace k -star S_k by thick k -star S_k . $\square$

We finally obtain our construction.

Proposition 32. *Let Γ be any family of holomorphic local homeomorphisms $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Let $B(0, R)$ be an open ball centered at 0 with a radius R large enough so that $E \subset B(0, R)$ (where E is the set in Proposition 31). Then the domain $D = B(0, R) - E$ is a characteristic set of Γ .*

Proof. The proof is the same as that of Proposition 27 where we replace k -star S_k by thick k -star S_k . $\square$

7. Injectivity of the Left Composition Operator L_F

We would like our natural mappings, the right mapping R_F and the left mapping L_F , to be say bi-Lipschitz with respect to the metric ρ_D (that reflects the fact that our mappings $\text{et}(\mathbb{C}^2)$

satisfy the Jacobian condition). Considering first the right mapping R_F , it would mean that given three étale mappings $G_1, G_2, F \in \text{et}(\mathbb{C}^2)$ and a characteristic set D of $\text{et}(\mathbb{C}^2)$ we need to compare the volume of $G_1(D)\Delta G_2(D)$ (multiplicities of G_1 and of G_2 are taken into account) with the volume of the R_F deformed set $(G_1 \circ F)(D)\Delta(G_2 \circ F)(D)$. A short reflection shows that the two volumes are not comparable (in the sense of bi-Lipschitz). The situation is completely different when we replace the right mapping, R_F by the left mapping, L_F . For example, we have the following.

Proposition 33. $\forall F \in \text{Aut}(\mathbb{C}^2)$ the mapping L_F is an isometry of the metric space $(\text{et}(\mathbb{C}^2), \rho_D)$.

Proof. For any two mappings G_1 and G_2 in $\text{et}(\mathbb{C}^2)$ we need to compare $\rho_D(G_1, G_2)$ with $\rho_D(F \circ G_1, F \circ G_2)$. We have (using our assumption on F)

$$(F \circ G_1)(D) \Delta (F \circ G_2)(D) = F(G_1(D) \Delta G_2(D)). \quad (32)$$

Since F is also (globally) volume preserving we have

$$\begin{aligned} & \text{the volume of } F(G_1(D) \Delta G_2(D)) \\ &= \text{the volume of } (G_1(D) \Delta G_2(D)). \end{aligned} \quad (33)$$

This proves that $\rho_D(G_1, G_2) = \rho_D(F \circ G_1, F \circ G_2)$. $\square$

We now drop the restrictive assumption that $F \in \text{Aut}(\mathbb{C}^2)$. Thus we merely have $F \in \text{et}(\mathbb{C}^2)$ and we still want to compare $\rho_D(G_1, G_2)$ with $\rho_D(F \circ G_1, F \circ G_2)$, for any pair $G_1, G_2 \in \text{et}(\mathbb{C}^2)$. We only know that F is a local diffeomorphism of $\mathbb{C}^2$ and (by the Jacobian condition) that it preserves (locally) the volume. In this case the geometrical degree of F , d_F can be larger than 1. We have the identity $d_F = |F^{-1}(\{(a, b)\})|$ which holds generically (in the Zariski sense) in $(a, b) \in \mathbb{C}^2$. Hence the (complex) dimension of the set $\{(a, b) \in \mathbb{C}^2 \mid |F^{-1}(a, b)| < d_F\}$ is at most 1. The Jacobian condition $\det J_F \equiv 1$ implies (as we noticed before) that F preserves volume taking into account the multiplicity. The multiplicity is a result of the possibility that F is not injective and hence the deformation of the characteristic set D by F convolves (i.e., might overlap at certain locations). However, this overlapping is bounded above by d_F . So if $A \subseteq \mathbb{C}^2$ is a measurable subset of $\mathbb{C}^2$ and we compare the volume of A with the volume of its image $F(A)$, then

$$\begin{aligned} & \text{the volume of } F(A) \leq \text{the volume of } A \\ & \leq d_F \cdot \{\text{the volume of } F(A)\}. \end{aligned} \quad (34)$$

This can be rewritten as follows:

$$\begin{aligned} & \frac{1}{d_F} \cdot \{\text{the volume of } A\} \leq \text{the volume of } F(A) \\ & \leq \text{the volume of } A. \end{aligned} \quad (35)$$

This is the place to emphasize also the following conclusion (that follows by the generic identity $d_F = |F^{-1}(\{(a, b)\})|$); namely,

$$\lim_{A \rightarrow \mathbb{C}^2} \frac{\text{the volume of } F(A)}{\text{the volume of } A} = \frac{1}{d_F}, \quad (36)$$

provided that the set A tends to cover the whole of the complex space $\mathbb{C}^2$ in an appropriate manner. To better understand why the quotient tends to the lower limit $1/d_F$ rather than to any number in the interval $[1/d_F, 1]$ (if at all) we recall that our mapping belongs to $\text{et}(\mathbb{C}^2)$ and so is a polynomial étale mapping. So any point $(a, b) \in \mathbb{C}^2$ for which $|F^{-1}(a, b)| < d_F$ is an asymptotic value of F and hence belongs to the curve A_F which is the asymptotic variety of F . In other words the identity $d_F = |F^{-1}(a, b)|$ is satisfied exactly on the semialgebraic set $\mathbb{C}^2 - A_F$ which is the complement of an algebraic curve. We now state and prove the main result of this paper.

Theorem 34. Let $F, G_1, G_2 \in \text{et}(\mathbb{C}^2)$. Then one has the following.

$$(i) \rho_D(F \circ G_1, F \circ G_2) \leq \rho_D(G_1, G_2).$$

(ii) Suppose that D is a family of characteristic sets of $\text{et}(\mathbb{C}^2)$ such that $D \rightarrow \mathbb{C}^2$, then $\forall \epsilon > 0$ one has

$$\left(\frac{1}{d_F} - \epsilon\right) \cdot \rho_D(G_1, G_2) \leq \rho_D(F \circ G_1, F \circ G_2) \quad (37)$$

for D being large enough.

(iii) Under the assumptions in (ii) one has

$$\lim_{D \rightarrow \mathbb{C}^2} \frac{\rho_D(F \circ G_1, F \circ G_2)}{\rho_D(G_1, G_2)} = \frac{1}{d_F}. \quad (38)$$

In particular, the left mapping $L_F : \text{et}(\mathbb{C}^2) \rightarrow \text{et}(\mathbb{C}^2)$, $L_F(G) = F \circ G$, is a bi-Lipschitz self-mapping of the metric space $(\text{et}(\mathbb{C}^2), \rho_D)$ with the constants $1/d_F \leq 1$.

Proof. (i) $x \in (F \circ G_1)(D) \Delta (F \circ G_2)(D) \Rightarrow \exists y \in G_j(D)$, $j = 1$ or 2 such that $x = F(y)$ and $x \notin (F \circ G_{3-j})(D)$. By $x \notin (F \circ G_{3-j})(D)$ it follows that $y \notin G_{3-j}(D)$ and so $y \in G_1(D) \Delta G_2(D)$ and $x = F(y) \in F(G_1(D) \Delta G_2(D))$. Hence $(F \circ G_1)(D) \Delta (F \circ G_2)(D) \subseteq F(G_1(D) \Delta G_2(D))$, so $\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D)) \leq \text{vol}(F(G_1(D) \Delta G_2(D)))$, and finally $\rho_D(F \circ G_1, F \circ G_2) \leq \rho_D(G_1, G_2)$.

(ii) and (iii). Here the proof is not just set theoretic. We will elaborate more in the remark that follows this proof. We recall that $F, G_1, G_2 \in \text{et}(\mathbb{C}^2)$. This implies that $\forall (\alpha, \beta) \in \mathbb{C}^2$ we have $|F^{-1}(\alpha, \beta)| \leq [C(X, Y) : C(F)]$, the extension degree of F ; see [3]. This is the so-called Fiber theorem for étale mappings. Moreover the image is cofinite; $|\mathbb{C}^2 - F(\mathbb{C}^2)| < \infty$ [3]. Also F has a finite set of exactly d_F maximal domains $\{\Omega_1, \dots, \Omega_{d_F}\}$. This means that F is injective on each maximal domain Ω_j , and $i \neq j \Rightarrow \Omega_i \cap \Omega_j = \emptyset$, and $\mathbb{C}^2 = \bigcup_{j=1}^{d_F} \overline{\Omega_j}$ and the boundaries $\partial \Omega_j$ are piecewise smooth (even piecewise analytic). For the theory of maximal domains of entire functions in one complex variable see [9], and for that theory for meromorphic functions in one complex variable see [10, 11]. Here we use only basic facts of the theory which are valid also for more than complex variable. If D is a family of characteristic sets of $\text{et}(\mathbb{C}^2)$ such that $D \rightarrow \mathbb{C}^2$, then by the

above $G_1(D), G_2(D) \rightarrow \mathbb{C}^2 - A$, where A is a finite set, and if $G_1 \neq G_2$, then we have the identity

$$\begin{aligned} & F(G_1(D) \Delta G_2(D)) - (F \circ G_1)(D) \Delta (F \circ G_2(D)) \\ &= \{x = F(y) = F(z) \mid y \in G_1(D) - G_2(D) \\ &\quad \wedge z \in G_2(D) - G_1(D)\}. \end{aligned} \quad (39)$$

Recalling that $(F \circ G_1)(D) \Delta (F \circ G_2(D)) \subseteq F(G_1(D) \Delta G_2(D))$ we write the last identity as follows:

$$\begin{aligned} & F(G_1(D) \Delta G_2(D)) \\ &= (F \circ G_1)(D) \Delta (F \circ G_2(D)) \\ &\cup \{x = F(y) = F(z) \mid y \in G_1(D) - G_2(D) \\ &\quad \wedge z \in G_2(D) - G_1(D)\}. \end{aligned} \quad (40)$$

Taking any two points $y \in G_1(D) - G_2(D)$ and $z \in G_2(D) - G_1(D)$ (as in the defining equation of the set on the right-hand side in the last identity), we note that there are $i \neq j$, $1 \leq i, j \leq d_F$ such that $y \in \Omega_i \wedge z \in \Omega_j$ (for $F(y) = F(z)$!). For $\tilde{D}$ a large enough characteristic set of $\text{et}(\mathbb{C}^2)$, we will have $z \in G_1(\tilde{D})$ and $y \in G_2(\tilde{D})$ and so $y, z \in G_1(\tilde{D}) \cap G_2(\tilde{D})$ (since $G_1(D), G_2(D) \rightarrow \mathbb{C}^2 - \{\text{a finite set}\}$). Hence $F(G_1(\tilde{D}) \Delta G_2(\tilde{D})) - (F \circ G_1)(\tilde{D}) \Delta (F \circ G_2)(\tilde{D})$ will not include the point x . We conclude that if y and z are F -equivalent ($F(y) = F(z)$), then $x = F(y) = F(z)$ will not belong to $F(G_1(D) \Delta G_2(D)) - (F \circ G_1)(D) \Delta (F \circ G_2)(D)$ for large enough D . We obtain the following crude estimate:

$$\begin{aligned} & \text{vol}(\{x = F(y) = F(z) \mid y \in G_1(D) - G_2(D) \\ &\quad \wedge z \in G_2(D) - G_1(D)\}) \\ &= o(\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D))). \end{aligned} \quad (41)$$

One can think of D as a large open ball centered at the origin of $\mathbb{R}^4$, $D \approx B(R)$ and with the radius R and look at the images of the two polynomial étale mappings $(F \circ G_1)(B(R))$ and $(F \circ G_2)(B(R))$ and compare the volume of $(F \circ G_1)(B(R)) \Delta (F \circ G_2)(B(R))$ which is of the order of magnitude R^{4d} , where d depends on the algebraic degrees of $F \circ G_1$ and $F \circ G_2$, with the volume of the set in the left-hand side of the last equation. Similar estimates are used in the theory of covering surfaces by Ahlfors, see [12, chapter 5]. We conclude that

$$\lim_{D \rightarrow \mathbb{C}^2} \frac{\text{vol}(F(G_1(D) \Delta G_2(D)))}{\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D))} = 1. \quad (42)$$

Hence

$$\begin{aligned} & \lim_{D \rightarrow \mathbb{C}^2} \frac{\rho_D(F \circ G_1, F \circ G_2)}{\rho_D(G_1, G_2)} \\ &= \lim_{D \rightarrow \mathbb{C}^2} \frac{\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D))}{\text{vol}(G_1(D) \Delta G_2(D))} \\ &= \lim_{D \rightarrow \mathbb{C}^2} \frac{\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D))}{\text{vol}(F(G_1(D) \Delta G_2(D)))} \quad (43) \\ &\quad \cdot \frac{\text{vol}(F(G_1(D) \Delta G_2(D)))}{\text{vol}(G_1(D) \Delta G_2(D))} \\ &= 1 \cdot \frac{1}{d_F} = \frac{1}{d_F}. \end{aligned}$$

□

Remark 35. The facts we used in proving (ii) and (iii) for étale mappings are in fact true in any dimension n , that is, in $\mathbb{C}^n$. In dimension $n = 2$ it turns out that the codimension of the image of the mapping is 0 and in fact the coimage is a finite set. Also the fibers are finite and have a uniform bound on their cardinality (one can get a less tight uniform bound by the Bezout theorem). Here are few well-known facts (which one can find in Hartshorne's book on Algebraic Geometry, [13]).

(1) The following two conditions are equivalent.

- (a) The Jacobian condition: the determinant $\det J_F$ is a nonzero constant.
- (b) The map F^* is étale (in standard sense of algebraic geometry). In particular it is flat.

Let $F^* : Y \rightarrow X$ be étale. Let $X^{\text{im}} := F^*(Y) \subseteq X$.

- (2) For every prime ideal $\mathfrak{p} \subseteq A$ ($X = \text{spec}(A)$), with residue field $k(\mathfrak{p})$ the ring $B \otimes_A k(\mathfrak{p})$ is finite over $k(\mathfrak{p})$ ($Y = \text{spec}(B)$).
- (3) F^* is a quasifinite mapping.
- (4) The set X^{im} is open in X .
- (5) For every point $x \in X(\mathbb{C})$ the fiber $(F^*)^{-1}(x)$ is a finite subset of $Y(\mathbb{C})$.
- (6) The ring homomorphism $A \rightarrow B$ is injective, and the induced field extension $K \rightarrow L$ is finite.
- (7) There is a nonempty open subset $X^{\text{fin}} \subseteq X^{\text{im}}$ such that on letting $Y^{\text{fin}} := (F^*)^{-1}(X^{\text{fin}}) \subseteq Y$, the map of schemes $F^*|_{Y^{\text{fin}}} : Y^{\text{fin}} \rightarrow X^{\text{fin}}$ is finite. For any point $x \in X^{\text{fin}}(\mathbb{C})$ we have the equality $d_x = d_{F^*}$ the geometrical degree of F^* .
- (8) The dimension of the set $Z := X - X^{\text{im}}$ is at most $n - 2$.
- (9) If $X^{\text{im}} = X - Z$ is affine, then $Z = \emptyset$ and $X^{\text{im}} = X$.

Let X_{cl} be the topological space which is the set $X(\mathbb{C}) \cong \mathbb{C}^n$ given the classical topology. Similarly for Y_{cl} , the map of

schemes $F^* : Y \rightarrow X$ induces a map of topological spaces $F_{cl} : Y_{cl} \rightarrow X_{cl}$ ($F_{cl} = f^*|_{Y(\mathbb{C})}$).

(10) The map $F_{cl} : Y_{cl} \rightarrow X_{cl}$ is a local homeomorphism. An immediate conclusion from Theorem 34 is the following,

Corollary 36. $\forall F \in \text{et}(\mathbb{C}^2)$ the left mapping $L_F : \text{et}(\mathbb{C}^2) \rightarrow L_F(\text{et}(\mathbb{C}^2))$, $L_F(G) = F \circ G$ is an injective mapping.

8. Extending the Notion of Geometrical Degree

In this section we will outline the fact that some of the notions and results that are related to geometrical degree of an étale mapping originate, in fact, in the more basic topological spaces (no algebraic or holomorphic structure is needed). We will skip most of the proofs (that are elementary).

Definition 37. Let X be a topological space. The semigroup of all the continuous mappings, $F : X \rightarrow X$, will be denoted by $C(X)$. Here, as usual, the binary operation is composition of mappings.

Proposition 38. (1) Let X be a topological space, $F \in C(X)$, and $\overline{F(X)} = X$. Then $R_F : C(X) \rightarrow C(X)$ is injective.

(2) Let X be a topological space, $F \in C(X)$, and $R_F : C(X) \rightarrow C(X)$ is injective. Then for any $G, H \in C(X)$, the property $G|_{F(X)} = H|_{F(X)}$ implies that $G \equiv H$; that is, any $G \in C(X)$ is determined by its restriction $G|_{F(X)}$.

(3) Let X be a topological space that has the following property: for any closed $C \subseteq X$ and any point $x \in X - C$ there exist two continuous mappings $F, G : X \rightarrow X$ such that $F|_C = G|_C$, but $F(x) \neq G(x)$.

Let $F \in C(X)$ be such that $R_F : C(X) \rightarrow C(X)$ is injective. Then $\overline{F(X)} = X$.

Remark 39. Proposition 8 follows.

We are ready to discuss the notion of the geometrical degree, d_F , of appropriate mappings in $C(X)$.

Lemma 40. Let X be a topological space, $F \in C(X)$ and $n \in \mathbb{Z}^+ \cup \{0\}$. If F is open, then the set $B_n = \{x \in X \mid |F^{-1}(x)| \leq n\}$ is closed.

Corollary 41. Let X be a topological space, $F \in C(X)$, F is open, and $n \in \mathbb{Z}^+$. Then one has the following.

- (1) $\{x \in X \mid |F^{-1}(x)| \neq n\} = B_{n-1} \cup (X - B_n)$, the union of a closed set and an open set.
- (2) $\{x \in X \mid |F^{-1}(x)| = n\} = B_n - B_{n-1} = B_n \cap (X - B_{n-1})$, the intersection of a closed set and an open set.
- (3) If $d_F = \max\{|F^{-1}(x)| \mid x \in X\} < \infty$ exists, then $\{x \in X \mid |F^{-1}(x)| \neq d_F\}$ is a closed set.

Remark 42. $B_n \subseteq B_{n+1}$.

Definition 43. Let X be a topological space, $F \in C(X)$, F is open, and the maximum $d_F = \max\{|F^{-1}(x)| \mid x \in X\} < \infty$ exists. Then we call d_F the geometrical degree of F .

Example 44. If $X = \mathbb{C}^2$ with the complex topology and $F \in \text{et}(\mathbb{C}^2)$, then we know that d_F exists. We also know that the set $A_F = \{x \in \mathbb{C}^2 \mid |F^{-1}(x)| < d_F\}$ is a plane algebraic curve (possibly empty). Thus it is closed in $\mathbb{C}^2$. Moreover it is also small because $\dim A_F < 2 = \dim \mathbb{C}^2$.

We need one more property to hold for our mappings, namely, that the fiber size will generically be d_F , that is, that the set of all $x \in X$ for which $d_F = |F^{-1}(x)|$ will be a large set measured in the topology of X . This leads us to the following.

Definition 45. Let X be a topological space. We will denote by $E(X)$ the set of all the mappings $F : X \rightarrow X$ that have the following properties.

- (1) $F \in C(X)$.
- (2) F is open.
- (3) The maximum $d_F = \max\{|F^{-1}(x)| \mid x \in X\} < \infty$ exists.
- (4) $\overline{X - B_{d_F-1}} = X$.

Proposition 46. Let X be an Hausdorff space. Then one has the following.

- (1) $E(X)$ is a semigroup with an identity (where the binary operation is composition of mappings). In fact $\text{Aut}(X) \subseteq E(X)$.
- (2) $\forall F, G \in E(X)$, $d_{F \circ G} = d_F \cdot d_G$.

Proof. Checking that $E(X)$ is closed for composition: $F, G \in C(X) \Rightarrow F \circ G \in C(X)$. Also F, G open $\Rightarrow F \circ G$ is open. If $x \in X$, then $|F^{-1}(x)| \leq d_F$ and $\forall y \in F^{-1}(x)$; we have $|G^{-1}(y)| \leq d_G$. Since $(F \circ G)^{-1}(x) = G^{-1}(F^{-1}(x))$, it follows that $d_{F \circ G}$ exists and, in fact, that $d_{F \circ G} \leq d_F \cdot d_G$. This gives the first three properties in the last definition. We need to check that $\overline{X - B_{d_{F \circ G}-1}} = X$. For that matter it will be convenient to denote $S_F = \{x \in X \mid |F^{-1}(x)| < d_F\}$. Let $x \in X - S_G = X - B_{d_G-1}$. This set is open and dense in X . By our definitions $G^{-1}(x) = \{a_1, a_2, \dots, a_{d_G}\}$, a finite set of exactly d_G points. Since X is Hausdorff we can find d_G open neighborhoods $V_{a_1}, V_{a_2}, \dots, V_{a_{d_G}}$ of $a_1, a_2, \dots, a_{d_G}$, respectively which are pairwise disjoint. The images $G(V_{a_1}), G(V_{a_2}), \dots, G(V_{a_{d_G}})$ are open (since G is open) neighborhoods of the point $x = G(a_1) = G(a_2) = \dots = G(a_{d_G})$. Let us take the intersection

$$V = \bigcap_{j=1}^{d_G} G(V_{a_j}). \quad (44)$$

Then V is an open neighborhood of x and $G^{-1}(V)$ consists of d_G neighborhoods $V_1, V_2, \dots, V_{d_G}$ of $a_1, a_2, \dots, a_{d_G}$. We define $U_j = V_j - S_G$, $j = 1, 2, \dots, d_G$. Then since S_G is closed and S_G^c is open and dense in X , the U_j 's are open and dense subsets of the V_j 's. Since G is continuous and $G(V_j) = V$, it follows that $G(U_j)$ are open and dense in V . We note that each point $y \in U_j$ is such that $|F^{-1}(y)| = d_F$ because S_F is disjoint of U_j .

Thus the point $x \in X - S_G$ has a neighborhood V and d_G open and dense subsets $G(U_j)$ of V such that each point

$$x' \in \bigcap_{j=1}^{d_G} G(U_j) \quad (45)$$

(this set is still open and dense in V) is such that $|G^{-1}(x')| = d_G$ and each $a \in G^{-1}(x')$ is such that $|F^{-1}(a)| = d_F$. Hence $|(F \circ G)^{-1}(x')| = d_F \cdot d_G$. This proves both that $\overline{X - B_{d_{F \circ G}}} = X$ and that $d_{F \circ G} = d_F \cdot d_G$. $\square$

Corollary 47. *Let X be Hausdorff and $\mathcal{F}$ a semigroup of mappings $F : X \rightarrow X$ which are continuous and open and suppose that there is an absolute constant $c \in \mathbb{Z}^+$ such that $\forall F \in \mathcal{F}, d_F \leq c$. Then the sets $X - B_{d_F-1}, F \in \mathcal{F}$ cannot be dense in X , unless $\forall F \in \mathcal{F}, d_F = 1$.*

Corollary 48. *Let X be Hausdorff and $\mathcal{F}$ a family of mappings $F : X \rightarrow X$ which are continuous and open and satisfy $\overline{X - B_{d_{F \circ G}}} = X, \forall F \in \mathcal{F}$ (in particular $\forall F \in \mathcal{F}, d_F < \infty$). Let $S = \langle \mathcal{F} \rangle$ be the semigroup generated by $\mathcal{F}$ (composition of mappings is the binary operation). Then $\forall F, G \in S, d_{F \circ G} = d_F \cdot d_G$.*

Definition 49. Let X be an Hausdorff space. A mapping $F \in E(X)$ is called a composite mapping if $\exists G, H \in E(X) - \text{Aut}(X)$ such that $F = G \circ H$. A mapping $A \in E(X) - \text{Aut}(X)$ is a prime mapping if it is not composite. This is equivalent to the following: if $A = B \circ C$ for some $B, C \in E(X)$, then $B \in \text{Aut}(X) \vee C \in \text{Aut}(X)$. The subset of $E(X)$ of all the prime mappings will be denoted by $E_p(X)$. Thus the set of all the composite mappings is $E(X) - E_p(X)$.

Proposition 50. $\forall F \in E(X) - E_p(X), d_F$ is not a prime integer. Equivalently, $\forall F \in E(X), d_F$ is a prime integer $\Rightarrow F \in E_p(X)$.

Theorem 51. (1) If $E(X) - \text{Aut}(X) \neq \emptyset$, then $E_p(X) \neq \emptyset$.

(2) $\forall F \in E(X), \exists k \in \mathbb{Z}^+ \cup \{0\}, \exists A_0 \in \text{Aut}(X), \exists P_1, \dots, P_k \in E_p(X)$ such that $F = A_0 \circ P_1 \circ \dots \circ P_k$.

Maybe few elementary examples are in place.

Example 52. In Definition 45 we take $X = \mathbb{C}$ with the complex topology and (with an abuse of notation) take $\mathbb{C}[T] - \mathbb{C}$ for $E(X)$. Then Proposition 46(2), $d_{F \circ G} = d_F \cdot d_G$, is the elementary fact from algebra that $\forall P, Q \in \mathbb{C}[T]$ we have $\deg P \circ Q = \deg P \cdot \deg Q$.

A second example is given in Section 4 of this paper.

Example 53. In Definition 45 we take $X = \mathbb{C}^2$ with the complex topology, and $E(X) = \text{et}(\mathbb{C}^2)$. Then the theory that was outlined in Propositions 12 and 14 and Theorem 15 is a special case of the above more general topological theory.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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