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Research Article

On the Spectral Properties of the Weighted Mean Difference Operator $G(u, v; \Delta)$ over the Sequence Space ℓ_1

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In the present work the generalized weighted mean difference operator $G(u, v; \Delta)$ has been introduced by combining the generalized weighted mean and difference operator under certain special cases of sequences $u = (u_k)$ and $v = (v_k)$. For any two sequences u and v of either constant or strictly decreasing real numbers satisfying certain conditions the difference operator $G(u, v; \Delta)$ is defined by $(G(u, v; \Delta)x)_k = \sum_{i=0}^{k} u_k v_i (x_i - x_{i-1})$ with $x_k = 0$ for all $k < 0$. Furthermore, we compute the spectrum and the fine spectrum of the operator $G(u, v; \Delta)$ over the sequence space ℓ_1. In fact, we determine the spectrum, the point spectrum, the residual spectrum, and the continuous spectrum of this operator on the sequence space ℓ_1.

1. Introduction, Preliminaries, and Definitions

Let $u = (u_k)$ and $v = (v_k)$ be two bounded sequences of either constant or strictly decreasing positive real numbers such that $u_k \neq 0$ and $v_k \neq 0$ for all k, and

$$\lim_{k \longrightarrow \infty} u_k = u, \tag{1}$$

$$\lim_{k \longrightarrow \infty} \left| \frac{v_k - v_{k+1}}{v_{k-1} - v_k} \right| = 1 \quad (\text{for strictly decreasing sequence}). \tag{2}$$

By ℓ_1 and bv_p, we denote the spaces of all absolutely summable and p-bounded variation series, respectively. Also, by ℓ_∞, c, and c_0, we denote the spaces of all bounded, convergent, and null sequences, respectively. The main perception of this paper is to introduce the weighted mean difference operator $G(u, v; \Delta)$ as follows.

Let $x = (x_k)$ be any sequence in ℓ_1, and we define the weighted mean difference transform $G(u, v; \Delta)x$ of x by

$$(G(u, v; \Delta)\, x)_k = \sum_{i=0}^{k} u_k v_i (x_i - x_{i-1}) \quad (k \in \mathbb{N}_0), \tag{3}$$

where $\mathbb{N}_0$ denotes the set of nonnegative integers and we assume throughout that any term with negative subscript is zero. Instead of writing (3), the operator $G(u, v; \Delta)$ can be expressed as a lower triangular matrix (g_{nk}), where

$$g_{nk} = \begin{cases} u_k v_k, & (k = n), \\ u_k (v_k - v_{k+1}), & (0 \leq k \leq n-1), \\ 0, & (k > 0). \end{cases} \tag{4}$$

Equivalently, in componentwise the triangle (g_{nk}) can be represented by

$$(g_{nk}) = \begin{pmatrix} u_0v_0 & 0 & 0 & 0 & 0 & \cdots \\ u_1(v_0 - v_1) & u_1v_1 & 0 & 0 & 0 & \cdots \\ u_2(v_0 - v_1) & u_2(v_1 - v_2) & u_2v_2 & 0 & 0 & \cdots \\ u_3(v_0 - v_1) & u_3(v_1 - v_2) & u_3(v_2 - v_3) & u_3v_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

The main objective of this paper is to determine the spectrum of the operator $G(u, v; \Delta)$ over the basic sequence space ℓ_1. The operator $G(u, v; \Delta)$ has been studied by Polat et al. [1] in detail by introducing the difference sequence spaces $\ell_\infty(u, v; \Delta)$, $c_0(u, v; \Delta)$, and $c(u, v; \Delta)$. In the existing literature several researchers have been actively engaged in finding the spectrum and fine spectrum of different bounded linear operators over various sequence spaces. The spectrum of weighted mean operator has been studied by Rhoades [2], whereas that of the difference operator Δ over the sequence spaces ℓ_p for $0 < p < 1$ and c, c_0 has been studied by Altay and Başar [3, 4]. Kayaduman and Furkan [5] have determined the fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv_p and on generalizing these results, Srivastava and Kumar [6, 7] have determined the fine spectrum of the operator Δ_v over the sequence spaces ℓ_1 and c_0, where (v_k) is a sequence of either constant or strictly deceasing sequence of reals satisfying certain conditions. Dutta and Baliarsingh [8–10] have computed the spectrum of the operator Δ_v^r $(r \in \mathbb{N}_0)$ and Δ^2 over the sequence spaces ℓ_1, c_0, and c_0, respectively. The fine spectrum of the generalized difference operators $B(r, s)$ and $B(r, s, t)$ over the sequence spaces ℓ_p, bv_p and ℓ_1, bv has been studied by Bilgiç and Furkan [11, 12], respectively. Recently, the spectrum of some particular limitation matrices in certain sequence spaces has been studied by [13–15] and many others.

Let X and Y be Banach spaces and $T : X \longrightarrow Y$ be a bounded linear operator. By $\mathscr{R}(T)$, we denote the range of T; that is,

$$\mathscr{R}(T) = \{y \in Y : y = Tx; x \in X\}. \tag{6}$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\mathbf{0}\}$ be a normed linear space over the complex field and $T : D(T) \longrightarrow X$ be a linear operator, where $D(T)$ denotes the domain of T. With T, for a complex number λ, we associate an operator $T_\lambda = (T - \lambda I)$, where I is called identity operator on $D(T)$ and if T_λ has an inverse, we denote it by T_λ^{-1}, that is,

$$T_\lambda^{-1} = (T - \lambda I)^{-1} \tag{7}$$

and is called the *resolvent* operator of T. Many properties of T_λ and T_λ^{-1} depend on λ and the spectral theory is concerned with those properties. We are interested in the set of all λ's in the complex plane such that T_λ^{-1} exists/T_λ^{-1} is bounded/domain of T_λ^{-1} is dense in X. Now, we state the following results which are essential for our investigation.

Definition 1 (see [16, page 371]). Let $X \neq \{\mathbf{0}\}$ be normed linear space over the complex field and $T : D(T) \longrightarrow X$ be a linear operator with domain $D(T) \subseteq X$. A regular value of T is a complex number λ such that

(R1) T_λ^{-1} exists;

(R2) T_λ^{-1} is bounded;

(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent* set $\rho(T, X)$ of T is the set of all *regular* values of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\rho(T, X)$ is partitioned into three disjoint sets as follows.

(I) *Point Spectrum* $\sigma_p(T, X)$. It is the set of all $\lambda \in \mathbb{C}$ such that (R1) does not hold. The elements of $\sigma_p(T, X)$ are called eigen values of T.

(II) *Continuous Spectrum* $\sigma_c(T, X)$. It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds and satisfies (R3) but does not satisfy (R2).

(III) *Residual Spectrum* $\sigma_r(T, X)$. It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds but does not satisfy (R3). Condition (R2) may or may not hold.

Goldberg's Classification of the Operator T_λ (see [17], page 58–71). Let X be a Banach space and $T_\lambda = (T - \lambda I) \in B(X)$, where λ is a complex number. Again, let $R(T_\lambda)$ and T_λ^{-1} denote the range and inverse of the operator T_λ, respectively. Then the following possibilities may occur:

(A) $R(T_\lambda) = X$;

(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$;

(C) $\overline{R(T_\lambda)} \neq X$

and

(1) T_λ is injective and T_λ^{-1} is continuous;

(2) T_λ is injective and T_λ^{-1} is discontinuous;

(3) T_λ is not injective.

Taking permutations (A), (B), (C) and (1), (2), (3), we get nine different states. These are labelled by A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, and C_3. If λ is a complex number such that $T_\lambda \in A_1$ or $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T, X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda \in B_2\sigma(T, X)$ which means the operator $T_\lambda \in B_2$, that is, $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but T_λ^{-1} is discontinuous; similarly are the others.

Lemma 2 (see [17, page 59]). *A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 3 (see [17, page 60]). *The adjoint operator T^* is onto if and and only if T has a bounded inverse.*

Let P, Q be two nonempty subsets of the space w of all real or complex sequences and let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}_0$. For every $x = (x_k) \in P$, we write

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{8}$$

The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (P, Q)$ if and only if $Ax \in Q$ whenever $x \in P$.

Lemma 4 (see [18, page 126]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum ℓ_1 norms of the columns of A is bounded.*

Lemma 5 (see [19], Theorem 2). *If $a_k(0) \neq 0$ for all $k \in \mathbb{N}_0$, then the inverse of the difference operator $B(a[m])$ is given by a lower triangular Toeplitz matrix $C = (c_{nk})$ as follows:*

$$c_{nk} = \begin{cases} \dfrac{1}{a_n(0)}, & (k = n), \\ \dfrac{(-1)^{n-k}}{\prod_{j=k}^{n} a_j(0)} \\ \quad \times D_{n-k}^{(k)}(a[m]), & (0 \le k \le n-1), \\ 0, & (k > n), \end{cases} \qquad (n, k \in \mathbb{N}_0), \tag{9}$$

where $(B(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + \cdots + a_{k-m}(m)x_{k-m}$, $(k \in \mathbb{N}_0)$ and

$$D_n^{(k)}(a[m]) = \begin{vmatrix} a_k(1) & a_{k+1}(0) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_k(2) & a_{k+1}(1) & a_{k+2}(0) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_k(m) & a_{k+1}(m-1) & a_{k+2}(m-2) & \cdots & a_{m-1}(1) & a_m(0) & \cdots & 0 \\ 0 & a_{k+1}(m) & a_{k+2}(m-1) & \cdots & a_{m-1}(2) & a_m(1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-m}(m) & \cdots & \cdots & \cdots & a_{n-1}(1) \end{vmatrix}. \tag{10}$$

2. The Spectrum of the Operator $G(u, v; \Delta)$ over the Sequence ℓ_1

In this section, we compute the spectrum, the point spectrum, the continuous spectrum, and the residual spectrum of the difference matrix $G(u, v; \Delta)$ on the sequence space ℓ_1.

Theorem 6. *The operator $G(u, v; \Delta) : \ell_1 \longrightarrow \ell_1$ is a linear operator and*

$$\|G(u, v; \Delta)\|_{(\ell_1:\ell_1)} = \sup_n |u_n| \left(\sum_{k=0}^{n-1} |v_k - v_{k+1}| + |v_n| \right). \tag{11}$$

Proof. Proof of this theorem follows from Lemma 4 and in particular cases.

(i) If (u_k) is a strictly decreasing sequence and (v_k) is a constant sequence of positive reals (say $v_k = v$ for all k), then

$$\|G(u, v; \Delta)\|_{(\ell_1:\ell_1)} = u_0 v. \tag{12}$$

(ii) If (u_k) and (v_k) are strictly decreasing sequences of positive reals, then

$$\|G(u, v; \Delta)\|_{(\ell_1:\ell_1)} = u_0 v_0. \tag{13}$$

(iii) If (u_k) and (v_k) are constant sequences of positive reals, then

$$\|G(u, v; \Delta)\|_{(\ell_1:\ell_1)} = uv. \tag{14}$$

(iv) If (u_k) is a constant sequence and (v_k) is a strictly decreasing sequence of positive reals, then

$$\|G(u, v; \Delta)\|_{(\ell_1:\ell_1)} = uv_0. \tag{15}$$

□

Theorem 7. *The spectrum of $G(u,v;\Delta)$ over the sequence space ℓ_1 is given by*

$$\sigma\left(G\left(u,v;\Delta\right),\ell_1\right)=\left\{\lambda\in\mathbb{C}:\left|\frac{\lambda-u_kv_{k+1}}{\lambda-u_kv_k}\right|\geq 1;\left(k\in\mathbb{N}_0\right)\right\}. \tag{16}$$

Proof. The proof is divided into four parts as follows.

Let (u_k) and (v_k) be two bounded sequences of positive reals satisfying (1) and (2) and $\lambda\in\mathbb{C}$ such that $|(\lambda-u_kv_{k+1})/(\lambda-u_kv_k)|<1$ for all $k\in\mathbb{N}_0$; then $(G(u,v;\Delta)-\lambda I)$ is a triangle and hence has an inverse, that is,

$$\left(G\left(u,v;\Delta\right)-\lambda I\right)^{-1}=\begin{pmatrix}\frac{1}{u_0v_0-\lambda} & 0 & 0 & 0 & \cdots\\ -\frac{u_1\left(v_0-v_1\right)}{\prod_{j=0}^{1}\left(u_jv_j-\lambda\right)} & \frac{1}{u_1v_1-\lambda} & 0 & 0 & \cdots\\ \frac{u_2\left(v_0-v_1\right)\left(\lambda-u_1v_2\right)}{\prod_{j=0}^{2}\left(u_jv_j-\lambda\right)} & -\frac{u_2\left(v_1-v_2\right)}{\prod_{j=1}^{2}\left(u_jv_j-\lambda\right)} & \frac{1}{u_2v_2-\lambda} & 0 & \cdots\\ -\frac{u_3\left(v_0-v_1\right)\left(\lambda-u_1v_2\right)\left(\lambda-u_2v_3\right)}{\prod_{j=0}^{3}\left(u_jv_j-\lambda\right)} & \frac{u_3\left(v_1-v_2\right)\left(\lambda-u_2v_3\right)}{\prod_{j=1}^{3}\left(u_jv_j-\lambda\right)} & -\frac{u_3\left(v_2-v_3\right)}{\prod_{j=2}^{3}\left(u_jv_j-\lambda\right)} & \frac{1}{u_3v_3-\lambda} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}. \tag{17}$$

In general, by using Lemma 5, we observe that

$$\begin{aligned}D_n^{(k)}\left(u,v,\lambda\right)&=\begin{vmatrix}u_{k+1}\left(v_k-v_{k+1}\right) & \left(u_{k+1}v_{k+1}-\lambda\right) & 0 & \cdots & 0\\ u_{k+2}\left(v_k-v_{k+1}\right) & u_{k+2}\left(v_{k+1}-v_{k+2}\right) & \left(u_{k+2}v_{k+2}-\lambda\right) & \cdots & 0\\ u_{k+3}\left(v_k-v_{k+1}\right) & u_{k+3}\left(v_{k+1}-v_{k+2}\right) & u_{k+3}\left(v_{k+2}-v_{k+3}\right) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ u_n\left(v_k-v_{k+1}\right) & u_n\left(v_{k+1}-v_{k+2}\right) & u_n\left(v_{k+2}-v_{k+3}\right) & \cdots & u_n\left(v_{n-1}-v_n\right)\end{vmatrix}\\ &=\left(v_k-v_{k+1}\right)\left(v_{k+1}-v_{k+2}\right)\cdots u_n\left(v_{n-1}-v_n\right)\times\left(u_{k+1}-\frac{\left(u_{k+1}v_{k+1}-\lambda\right)}{\left(v_{k+1}-v_{k+2}\right)}\right)\\ &\quad\times\left(u_{k+2}-\frac{\left(u_{k+2}v_{k+2}-\lambda\right)}{\left(v_{k+2}-v_{k+3}\right)}\right)\times\cdots\times\left(u_{n-1}-\frac{\left(u_{n-1}v_{n-1}-\lambda\right)}{\left(v_{n-1}-v_n\right)}\right).\end{aligned} \tag{18}$$

Therefore, for all $n=k$, $((G(u,v;\Delta)-\lambda I)^{-1})_{n,k}=1/(u_nv_n-\lambda)$, for all $k>n$, $((G(u,v;\Delta)-\lambda I)^{-1})_{n,k}=0$, and for all $0\leq k<n$ one can deduce that

$$\left(\left(G\left(u,v;\Delta\right)-\lambda I\right)^{-1}\right)_{n,k}=\frac{\left(v_k-v_{k+1}\right)u_n}{\left(u_kv_k-\lambda\right)\left(u_nv_n-\lambda\right)}\prod_{j=k+1}^{n-1}\frac{\left(\lambda-u_jv_{j+1}\right)}{\left(u_jv_j-\lambda\right)}. \tag{19}$$

By using Lemma 4, we obtain that

$$\begin{aligned}&\left\|\left(G\left(u,v;\Delta\right)-\lambda I\right)^{-1}\right\|_{(\ell_1:\ell_1)}\\ &=\sup_n\sum_{k=0}^{n}\left|\left(\left(G\left(u,v;\Delta\right)-\lambda I\right)^{-1}\right)_{n,k}\right|\\ &\leq\sup_n\left(\left|\frac{1}{u_nv_n-\lambda}\right|+\sum_{k=0}^{n-1}\left|\frac{\left(v_k-v_{k+1}\right)u_n}{\left(u_kv_k-\lambda\right)\left(u_nv_n-\lambda\right)}\times\prod_{j=k+1}^{n-1}\frac{\left(\lambda-u_jv_{j+1}\right)}{\left(u_jv_j-\lambda\right)}\right|\right)\end{aligned}$$

$$
\begin{aligned}
&= \sup_n \left|\frac{1}{u_n v_n - \lambda}\right| \left(1 + \sum_{k=0}^{n-1} \left|\frac{(v_k - v_{k+1})\, u_n}{(u_k v_k - \lambda)}\right| \right. \\
&\qquad \left. \times \prod_{j=k+1}^{n-1} \left|\frac{(\lambda - u_j v_{j+1})}{(u_j v_j - \lambda)}\right|\right) \\
&\le \sup_n \left|\frac{u_n}{u_n v_n - \lambda}\right| \left(\frac{1}{|u_n|} + \sum_{k=0}^{n-1} \left|\frac{(v_k - v_{k+1})}{(u_k v_k - \lambda)}\right| \prod_{j=k+1}^{n-1} |r_\lambda|\right) \\
&\le \sup_n \left|\frac{Q_\lambda u_n}{u_n v_n - \lambda}\right| \left(\frac{1}{|Q_\lambda u_n|} + \frac{1}{|r_\lambda|} \sum_{k=0}^{n-1} |r_\lambda|^{n-k-1}\right) \\
&= \sup_n \left|\frac{Q_\lambda u_n}{u_n v_n - \lambda}\right| \left[\frac{1}{|Q_\lambda u_n|} + \frac{1}{|r_\lambda|} \left(\frac{1 - |r_\lambda|^{n-1}}{1 - |r_\lambda|}\right)\right] < \infty.
\end{aligned}
\tag{20}
$$

For simplicity, we write $Q_\lambda = \sup_k((v_k - v_{k+1})/(u_k v_k - \lambda))$ and $r_\lambda = \sup_j((\lambda - u_j v_{j+1})/(u_j v_j - \lambda))$. Since (u_k) and (v_k) are two bounded sequences of positive reals, the quantity Q_λ is finite. As per the assumption, $\lambda \in \mathbb{C}$ and $|(\lambda - u_k v_{k+1})/(\lambda - u_k v_k)| < 1$ for all $k \in \mathbb{N}_0$. Therefore, $|r_\lambda| < 1$. Now, we consider the four possible cases of the sequences (u_k) and (v_k).

Case 1. If (u_k) and (v_k) are strictly decreasing sequences of positive reals, then from the above relation we have $(G(u, v; \Delta) - \lambda I)^{-1} \in (\ell_1 : \ell_1)$. Thus,

$$
\sigma(G(u, v; \Delta), \ell_1) \subseteq \left\{\lambda \in \mathbb{C} : \left|\frac{\lambda - u_k v_{k+1}}{\lambda - u_k v_k}\right| \ge 1; (k \in \mathbb{N}_0)\right\}. \tag{21}
$$

Conversely, consider $\lambda \ne u_k v_k$ and $|(\lambda - u_k v_{k+1})/(\lambda - u_k v_k)| \ge 1$ for all $k \in \mathbb{N}_0$, and clearly $(G(u, v; \Delta) - \lambda I)$ is a triangle and hence $(G(u, v; \Delta) - \lambda I)^{-1}$ exists, but $\| (G(u, v; \Delta) - \lambda I)^{-1}\|_{(\ell_1:\ell_1)}$ is not finite. Again, for $\lambda = u_k v_k$, the matrix $(G(u, v; \Delta) - u_k v_k I)(k \in \mathbb{N}_0)$ is not invertible. Thus,

$$
\left\{\lambda \in \mathbb{C} : \left|\frac{\lambda - u_k v_{k+1}}{\lambda - u_k v_k}\right| \ge 1; (k \in \mathbb{N}_0)\right\} \subseteq \sigma(G(u, v; \Delta), \ell_1). \tag{22}
$$

Combining (21) and (22) we complete the proof for this case.

Case 2. If (u_k) is a strictly decreasing sequence and (v_k) is a constant sequence of positive reals, then

$$
(G(u, v; \Delta) - \lambda I)^{-1} = \begin{pmatrix}
\frac{1}{u_0 v - \lambda} & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{u_1 v - \lambda} & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{u_2 v - \lambda} & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{u_3 v - \lambda} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{23}
$$

Since $Q = 0$, $\|(G(u, v; \Delta) - \lambda I)^{-1}\|_{(\ell_1:\ell_1)} = \sup_n |1/(u_n v - \lambda)| < \infty$ and the rest part of the proof is similar to that of Case 1.

Case 3. If (u_k) and (v_k) are constant sequences of positive reals, then the proof is similar to that of Case 2 and only the difference is $\|(G(u, v; \Delta) - \lambda I)^{-1}\|_{(\ell_1:\ell_1)} = |1/uv - \lambda| < \infty$.

Case 4. If (u_k) is constant sequence and (v_k) is strictly decreasing sequence of positive reals, then the proof is similar to that of Case 1. □

Theorem 8. *The point spectrum of the operator $G(u, v; \Delta)$ over ℓ_1 is given by*

$$
\sigma_p(G(u, v; \Delta), \ell_1) = \begin{cases} u_k v, & (\textit{if } (v_k) \textit{ is a constant sequence}) \\ \emptyset, & (\textit{otherwise}). \end{cases} \tag{24}
$$

Proof. Suppose $x \in \ell_1$ and consider the system of linear equations $G(u, v; \Delta)x = \lambda x$ for $x \ne \mathbf{0} = \{0, 0, 0, \ldots\}$ in ℓ_1,

$$
\begin{aligned}
u_0 v_0 x_0 &= \lambda x_0 \\
u_1 (v_0 - v_1) x_0 + u_1 v_1 x_1 &= \lambda x_1 \\
u_2 (v_0 - v_1) x_0 + u_2 (v_1 - v_2) x_1 + u_2 v_2 x_2 &= \lambda x_2 \\
&\vdots \\
u_n (v_0 - v_1) x_0 + u_n (v_1 - v_2) x_1 + \cdots + u_n v_n x_n &= \lambda x_n \\
&\vdots
\end{aligned}
\tag{25}
$$

On solving the system of (25), we obtain that $\lambda = u_0 v_0$ whenever $x_0 \ne 0$ and

$$
\begin{aligned}
x_1 &= \frac{u_1 (v_0 - v_1)}{u_0 v_0 - u_1 v_1} x_0 \\
x_2 &= \frac{u_2 (v_0 - v_1)(u_0 v_0 - u_1 v_2)}{(u_0 v_0 - u_1 v_1)(u_0 v_0 - u_2 v_2)} x_0 \text{ and in general} \\
x_n &= \frac{u_n (v_0 - v_1)}{u_0 v_0 - u_n v_n} \prod_{j=1}^{n-1} \frac{u_0 v_0 - u_j v_{j+1}}{u_0 v_0 - u_j v_j} x_0.
\end{aligned}
\tag{26}
$$

Now, we have the following cases.

Case 1. If (u_k) and (v_k) are strictly decreasing sequences of positive reals, then we need to show that $|(u_0 v_0 - u_j v_{j+1})/(u_0 v_0 - u_j v_j)| \ge 1$. Suppose, for the contrary, if $|(u_0 v_0 - u_j v_{j+1})/(u_0 v_0 - u_j v_j)| < 1$, then $v_{j+1} > v_j$ which contradicts the fact that (v_k) is strictly decreasing sequence. Therefore, $x \notin \ell_1$ and hence $\sigma_p(G(u, v; \Delta), \ell_1) = \emptyset$.

Case 2. If (u_k) is a strictly decreasing and (v_k) is a constant sequence of positive reals, then for all $k \in \mathbb{N}_0$, $\lambda = u_k v$ is an eigen value corresponding to the eigen

vector $x = \{0, 0, \ldots, 0, 1, 0, \ldots\}$ whose kth entry is 1. Thus, $\sigma_p(G(u, v; \Delta), \ell_1) = u_k v$.

Case 3. If (u_k) and (v_k) are constant sequences of positive reals, then the proof is similar to that of Case 2 and $\sigma_p(G(u, v; \Delta), \ell_1) = uv$.

Case 4. If (u_k) is a constant and (v_k) is a strictly decreasing sequence of positive reals, then the proof is similar to that of Case 1. □

Theorem 9. *The point spectrum of the dual operator $(G(u, v; \Delta))^*$ over the sequence space $\ell_1^* \cong \ell_\infty$ is given by*

$$\sigma_p\left((G(u,v;\Delta))^*, \ell_\infty\right) = \begin{cases} u_k v, & (\text{if } (v_k) \text{ is a constant sequence}) \\ \left\{\lambda \in \mathbb{C} : \left|\dfrac{\lambda - u_k v_{k+1}}{\lambda - u_k v_k}\right| > 1; (k \in \mathbb{N}_0)\right\}, & (\text{otherwise}). \end{cases} \tag{27}$$

Proof. Consider $(G(u, v; \Delta))^* f = \lambda f$ and $\mathbf{0} \neq f \in \ell_1$, and then the system of equations

$$\begin{aligned} &u_0 v_0 f_0 + (v_0 - v_1) u_1 f_1 + (v_0 - v_1) u_2 f_2 + \cdots = \lambda f_0 \\ &u_1 v_1 f_1 + (v_1 - v_2) u_2 f_2 + (v_1 - v_2) u_3 f_3 + \cdots = \lambda f_1 \\ &u_2 v_2 f_2 + (v_2 - v_3) u_3 f_3 + (v_2 - v_3) u_4 f_4 + \cdots = \lambda f_2 \\ &\qquad \vdots \\ &u_k v_k f_k + (v_k - v_{k+1}) u_{k+1} f_{k+1} + (v_k - v_{k+1}) u_{k+1} f_{k+1} \\ &\quad + \cdots = \lambda f_k \\ &\qquad \vdots \end{aligned} \tag{28}$$

From the system of linear equations (28), we obtain that

$$f_{k-1} = \frac{v_{k-1} - v_k}{\lambda - u_{k-1} v_{k-1}} \left(u_k f_k + u_{k+1} f_{k+1} + u_{k+2} f_{k+2} + \cdots\right), \tag{29}$$

$$f_k = \frac{v_k - v_{k+1}}{\lambda - u_k v_k} \left(u_{k+1} f_{k+1} + u_{k+2} f_{k+2} + u_{k+3} f_{k+3} + \cdots\right) \quad (k \geq 1). \tag{30}$$

Combining (29) and (30), we have

$$f_k = \frac{v_k - v_{k+1}}{\lambda - u_k v_k} \left(\frac{(\lambda - u_{k-1} v_{k-1})}{(v_{k-1} - v_k)} f_{k-1} - u_k f_k\right). \tag{31}$$

Therefore, by using ratio text, it is observed that

$$\lim_{k \longrightarrow \infty} \left|\frac{f_k}{f_{k-1}}\right| = \lim_{k \longrightarrow \infty} \left|\frac{(v_k - v_{k+1})(\lambda - u_{k-1} v_{k-1})}{(v_{k-1} - v_k)(\lambda - u_k v_{k+1})}\right|. \tag{32}$$

Now, we have the following cases.

Case 1. If (u_k) and (v_k) are strictly decreasing sequences of positive reals, then by using (2), we conclude that $\lim_{k \longrightarrow \infty} |f_k / f_{k-1}| < 1$ if and only if $|(\lambda - u_{k-1} v_{k-1})/(\lambda - u_k v_{k+1})| < 1$ for all $k \geq 1$. Therefore, $\sigma_p((G(u, v; \Delta))^*, \ell_\infty) = \{\lambda \in \mathbb{C} : |(\lambda - u_k v_{k+1})/(\lambda - u_k v_k)| > 1; (k \in \mathbb{N}_0)\}$.

Case 2. If (u_k) is a strictly decreasing and (v_k) is a constant sequence of positive reals, then from the system of (28), we observe that for all $k \in \mathbb{N}_0$, $\lambda = u_k v$ is an eigen value corresponding to the eigen vector $f = \{0, 0, \ldots, 0, 1, 0, \ldots\}$ whose kth entry is 1. Thus, $\sigma_p((G(u, v; \Delta))^*, \ell_\infty) = u_k v$.

Case 3. If (u_k) and (v_k) are constant sequences of positive reals, then the proof is similar to that of Case 2.

Case 4. If (u_k) is a constant and (v_k) is a strictly decreasing sequence of positive reals, then the proof is similar to that of Case 1. □

Theorem 10. *The residual spectrum of the operator $G(u, v; \Delta)$ over the sequence space ℓ_1 is given by*

$$\sigma_r\left(G(u,v;\Delta), \ell_1\right) = \begin{cases} \emptyset, & (\text{if } (v_k) \text{ is a constant sequence}) \\ \left\{\lambda \in \mathbb{C} : \left|\dfrac{\lambda - u_k v_{k+1}}{\lambda - u_k v_k}\right| > 1; (k \in \mathbb{N}_0)\right\}, & (\text{otherwise}). \end{cases} \tag{33}$$

Proof. Proof follows from Lemma 2 and Theorems 8 and 9. □

Theorem 11. *The continuous spectrum of the operator $G(u, v; \Delta)$ over the sequence space ℓ_1 is given by*

$$\sigma_c\left(G\left(u,v;\Delta\right),\ell_1\right)=\begin{cases}\emptyset, & \left(if\ \left(v_k\right)\ is\ a\ constant\ sequence\right)\\ \left\{\lambda\in\mathbb{C}:\left|\dfrac{\lambda-u_kv_{k+1}}{\lambda-u_kv_k}\right|=1;\left(k\in\mathbb{N}_0\right)\right\}, & \left(otherwise\right).\end{cases}\tag{34}$$

Proof. The proof of this theorem follows from Theorems 7, 8, and 10 and along with the fact that

$$\begin{aligned}&\sigma\left(G\left(u,v;\Delta\right),\ell_1\right)\\&\quad=\sigma_p\left(G\left(u,v;\Delta\right),\ell_1\right)\cup\sigma_r\left(G\left(u,v;\Delta\right),\ell_1\right)\\&\qquad\cup\sigma_c\left(G\left(u,v;\Delta\right),\ell_1\right).\end{aligned}\tag{35}$$

□

3. Conclusion

In this work, the authors have determined the spectrum of the generalized weighted mean difference operator $G(u,v;\Delta)$ over the Banach space ℓ_1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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