Hindawi Publishing Corporation
International Journal of Partial Differential Equations
Volume 2014, Article ID 787092, 12 pages
http://dx.doi.org/10.1155/2014/787092

Research Article

Existence of Solution and Approximate Controllability for Neutral Differential Equation with State Dependent Delay

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Received 12 June 2014; Revised 10 September 2014; Accepted 15 September 2014; Published 2 October 2014

Academic Editor: Athanasios N. Yannacopoulos

This paper is divided in two parts. In the first part we study a second order neutral partial differential equation with state dependent delay and noninstantaneous impulses. The conditions for existence and uniqueness of the mild solution are investigated via Hausdorff measure of noncompactness and Darbo Sadovskii fixed point theorem. Thus we remove the need to assume the compactness assumption on the associated family of operators. The conditions for approximate controllability are investigated for the neutral second order system with respect to the approximate controllability of the corresponding linear system in a Hilbert space. A simple range condition is used to prove approximate controllability. Thereby, we remove the need to assume the invertibility of a controllability operator used by authors in (Balachandran and Park, 2003), which fails to exist in infinite dimensional spaces if the associated semigroup is compact. Our approach also removes the need to check the invertibility of the controllability Gramian operator and associated limit condition used by the authors in (Dauer and Mahmudov, 2002), which are practically difficult to verify and apply. Examples are provided to illustrate the presented theory.

1. Introduction

Neutral differential equations appear as mathematical models in electrical networks involving lossless transmission, mechanics, electrical engineering, medicine, biology, ecology, and so forth. Neutral differential equations are functional differential equations in which the highest order derivative of the unknown function appears both with and without derivatives. Second order neutral differential equations model variational problems in calculus of variation and appear in the study of vibrating masses are attached to an electric bar.

Impulsive differential equations are known for their utility in simulating processes and phenomena subject to short term perturbations during their evolution. Discrete perturbations are negligible to the total duration of the process which have been studied in [1–6].

However noninstantaneous impulses are recently studied by Ahmad [7]. Stimulated by their numerous applications in mechanics, electrical engineering, medicine, ecology, and so forth, noninstantaneous impulsive differential equations are recently investigated.

Recently, much attention is paid to partial functional differential equation with state dependent delay. For details see [7–12]. As a matter of fact, in these papers their authors assume severe conditions on the operator family generated by A, which imply that the underlying space X has finite dimension. Thus the equations treated in these works are really ordinary and not partial equations. The literature related to state dependent delay mostly deals with functional differential equations in which the state belongs to a finite dimensional space. As a consequence, the study of partial functional differential equations with state dependent delay is neglected. This is one of the motivations of our paper.

The papers [13, 14] study existence of differential equation via measure of noncompactness. Measure of noncompactness significantly removes the need to assume Lipschitz continuity of nonlinear functions and operators.

In recent years, controllability of infinite dimensional systems has been extensively studied for various applications. In

the papers [15, 16] the authors discuss the exact controllability results by assuming that the semigroup associated with the linear part is compact. However, if the operator B is compact or C_0-semigroup $T(t)$ is compact then the controllability operator is also compact. Hence the inverse of it does not exist if the state space X is infinite dimensional [17].

Another available method in the literature involves the invertibility of operator $(\alpha I+\Gamma_0^T)$, where Γ_0^T is the controllability Gramian and a limit condition which is difficult to check and apply in practical real world problems. See for details [18]. Also it is practically difficult to verify their condition directly. This is one of the motivations of our paper.

However our work is a continuation of coauthor Sukavanam's novel approach in article [19]. We extend our work [20–22] in this paper.

Controllability results are available in overwhelming majority for abstract differential delay systems (see [1, 3–6, 9–12, 14–17, 19–34]), rather than for neutral differential with state dependent delay.

The organization of the paper is as follows. In Section 3 we study the existence and uniqueness of mild solution of the second order equation modelled in the form

$$\begin{aligned}
&\frac{d}{dt}\left(x'(t)+g(t,x_t)\right)=Ax(t)+f\left(t,x_{\rho(t,x_t)}\right),\\
&\qquad t\in\left(s_i,t_{i+1}\right],\quad i=0,\dots,n,\\
&x_0=\phi\in\mathfrak{B},\qquad x'(0)=z\in X,\\
&x(t)=J_i^1\left(t,x_t\right),\quad t\in\left(t_i,s_i\right],\ i=1,2,\dots,n,\\
&x'(t)=J_i^2\left(t,x_t\right),\quad t\in\left(t_i,s_i\right],\ i=1,2,\dots,n,
\end{aligned}\tag{1}$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators on a Banach space X. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically; g, f, J_i^1, J_i^2, $i = 1, \dots, n$ are appropriate functions. $0 = t_0 = s_0 < t_1 \le s_1 \le t_2, < \cdots, < t_n \le s_n \le t_{n+1} = a$ are prefixed numbers. In Section 5 we study the approximate controllability of

$$\begin{aligned}
&\frac{d}{dt}\left(x'(t)+g(t,x_t)\right)=Ax(t)+f\left(t,x_{\rho(t,x_t)}\right)+Bu(t),\\
&\qquad t\in J=[0,a],\\
&x_0=\phi\in\mathfrak{B},\qquad x'(0)=w\in X,
\end{aligned}\tag{2}$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators on a Hilbert space X. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically; g, f are appropriate functions. B is a bounded linear operator on a Hilbert space U.

2. Preliminaries

In this section some definitions, notations, and lemmas that are used throughout this paper are stated. The family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following are satisfied:

(a) $C(0) = I$ (I is the identity operator in X);

(b) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$;

(c) the map $t \to C(t)x$ is strongly continuous for each $x \in X$.

$\{S(t) : t \in \mathbb{R}\}$ is the strongly continuous sine family associated to the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$. It is defined as $S(t)x = \int_0^t C(s)x\,ds$, $x \in X$, $t \in \mathbb{R}$.

The operator A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ and $S(t)$ is the associated sine function. Let N, $\widetilde{N}$ be certain constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \widetilde{N}$ for every $t \in J = [0, a]$. For more details see book by Fattorini [28] and articles [35–37]. In this work we use the axiomatic definition of phase space $\mathfrak{B}$, introduced by Hale and Kato [30].

Definition 1 (see [30]). Let $\mathfrak{B}$ be a linear space of functions mapping $(-\infty, 0]$ into X endowed with seminorm $\|\cdot\|_{\mathfrak{B}}$ and satisfy the following conditions:

(A) If $x : (-\infty, \sigma+b] \to X$, $b > 0$, such that $x_\theta \in \mathfrak{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma+b] : X)$, then for every $t \in [\sigma, \sigma+b)$ the following conditions:

(i) x_t is in $\mathfrak{B}$,

(ii) $\|x(t)\| \le H\|x_t\|_{\mathfrak{B}}$,

(iii) $\|x_t\|_{\mathfrak{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t+\sigma)\|x_\sigma\|_{\mathfrak{B}}$,

where $H > 0$ is a constant $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathfrak{B}$ is complete.

Definition 2 (see [31]). Hausdorff's measure of noncompactness χ_Y for a bounded set B in any Banach space Y is defined by $\chi_Y(B) = \inf\{r > 0,\ B$ can be covered by finite number of balls with radii $r.\}$

Lemma 3 (see [31]). *Let Y be a Banach space and $B, C \subset Y$ be bounded, then the following properties hold:*

(1) *B is precompact if and only if $\chi_Y(B) = 0$;*

(2) *$\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(\operatorname{conv} B)$, where $\overline{B}$ and $\operatorname{conv} B$ are closure and convex hull of B, respectively;*

(3) *$\chi_Y(B) \le \chi_Y(C)$ when $B \subset C$;*

(4) *$\chi_Y(B+C) \le \chi_Y(B) + \chi_Y(C)$ where $B+C = \{x+y;\ x \in B,\ y \in C\}$;*

(5) *$\chi_Y(B \cup C) = \max\{\chi_Y(B), \chi_Y(C)\}$;*

(6) $\chi_Y(\lambda B) = \|\lambda\|\chi_Y(B)$ *for any* $\lambda \in \mathbb{R}$;

(7) *if the map* $Q : D(Q) \subset Y \to Z$ *is Lipschitz continuous with constant* k *then* $\chi_Z(QB) \le k\chi_Y(B)$ *for any bounded subset* $B \subset D(Q)$, *where* Z *is a Banach space;*

(8) *if* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subset of* Y *and* $\lim_{n\to\infty}\chi_Y(W_n) = 0$, *then* $\bigcap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y.

Definition 4 (see [31]). The map $Q : W \subset Y \to Y$ is said to be a χ-contraction if there exists a positive constant $k < 1$ such that $\chi_Y Q(C) \le k\chi_Y(C)$ for any bounded close subset $C \subset W$ where Y is a Banach space.

Lemma 5 (Darbo-Sadovskii [31]). *If* $W \subset Y$ *is closed and convex and* $0 \in W$, *the continuous map* $Q : W \to W$ *is* χ*-contraction, then the map* Q *has at least one fixed point.*

$PC([0,a], X)$ is the space formed by normalized piecewise continuous function from $[0,b]$ into X. In particular it is the space PC formed by all functions $u : [0,b] \to X$ such that u is continuous at $t \ne t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, 2, \ldots, n$. It is clear that PC endowed with the norm $\|x\|_{PC} = \sup_{t\in J}\|x(t)\|$ is a Banach space. For any $x \in PC$

$$\tilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i,\ i = 1, 2, \ldots, n. \end{cases} \tag{3}$$

So, $\tilde{x} \in C([t_i, t_{i+1}], X)$.

Lemma 6 (see [31]). (1) *If* $W \subset PC([a,b]; X)$ *is bounded, then* $\chi(W(t)) \le \chi_{PC}(W)$ *for any* $t \in [a,b]$ *where* $W(t) = \{u(t) : u \in W\} \subset X$.

(2) *If* W *is piecewise equicontinuous on* $[a,b]$, *then* $\chi(W(t))$ *is piecewise continuous for* $t \in [a,b]$, *and*

$$\chi_{PC}(W) = \sup\{\chi(W(t)),\ t \in [a,b]\}. \tag{4}$$

(3) *If* $W \subset PC([a,b]; X)$ *is bounded and piecewise equicontinuous, then* $\chi(W(t))$ *is piecewise continuous for* $t \in [a,b]$ *and*

$$\chi\left(\int_a^t W(s)\,ds\right) \le \int_a^t \chi(W(s))\,ds \quad t \in [a,b]. \tag{5}$$

Lemma 7 (see [35]). *If the semigroup* $S(t)$ *is equicontinuous and* $\eta \in L([0,a]; R^+)$, *then the set* $\{\int_0^t S(t-s)u(s)ds : \|u(s)\| \le \eta(s)$ *for a.e* $s \in [0,a]\}$ *is equicontinuous for* $t \in [0,b]$.

3. Existence and Uniqueness of Mild Solution

We define mild solution of problem (1) as follows.

Definition 8. A function $x : (-\infty, a] \to X$ is a mild solution of the problem (1) if $x_0 = \phi$; $x(\cdot)|_{[0,a]} \in PC(X)$, $x(t) = J_i^1(t, x_t)$ $\forall t \in (t_i, s_i]$ $i = 1, \ldots, n$, and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)[z + g(0,\phi)] \\ &\quad - \int_0^t C(t-s)\,g(s, x_s)\,ds \\ &\quad + \int_0^t S(t-s)\,f\left(s, x_{\rho(s,x_s)}\right)ds, \\ &\qquad t \in [0, t_1], \\ x(t) &= C(t-s_i)\,J_i^1\left(s_i, x_{s_i}\right) \\ &\quad + S(t-s_i)\left(J_i^2\left(s_i, x_{s_i}\right) + g\left(s_i, x_{s_i}\right)\right) \\ &\quad - \int_{s_i}^t C(t-s)\,g(s, x_s)\,ds \\ &\quad + \int_{s_i}^t S(t-s)\,f\left(s, x_{\rho(s,x_s)}\right)ds \\ &\qquad \text{for } t \in [s_i, t_{i+1}] \quad i = 1, \ldots, n. \end{aligned} \tag{6}$$

To prove our result we always assume $\rho : J \times \mathfrak{B} \to (-\infty, a]$ is a continuous function. The following hypotheses are used.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s,\psi) : \rho(s,\psi) \le 0\}$ into $\mathfrak{B}$ and there exists a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \le J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(Hf) $f : J \times \mathfrak{B} \to X$ satisfies the following.

(1) For every $x : (-\infty, a] \to X$, $x_0 \in \mathfrak{B}$ and $x|_J \in PC$, the function $f(\cdot,\psi) : J \to X$ is strongly measurable for every $\psi \in \mathfrak{B}$ and $f(\cdot, t)$ is continuous for a.e. $t \in J$.

(2) There exists an integrable function $\alpha : J \to [0,+\infty)$ and a monotone continuous nondecreasing function $\Omega : [0,+\infty) \to (0,+\infty)$ such that $\|f(t,v)\| \le \alpha(t)\Omega(\|v\|_{\mathfrak{B}})$ $\forall t \in J$ and $v \in \mathfrak{B}$.

(3) There exists an integrable function $\eta : J \to [0,\infty)$ such that $\chi(S(s)f(t,D)) \le \eta(t)\sup_{-\infty<\theta<0}\chi(D(\theta))$ for a.e. $s, t, \in J$, where $D(\theta) = \{v(\theta) : v \in D\}$.

(Hg) The function $g(\cdot)$ is continuous $\forall t, v \in J\times\mathfrak{B}$ and $g(t,\cdot)$ is Lipschitz continuous such that there exists positive constant L_g such that

$$\begin{aligned} \|g(t,v_1) - g(t,v_2)\| &\le L_g\|v_1 - v_2\|_{\mathfrak{B}}, \\ (t, v_i) &\in J \times \mathfrak{B}, \quad i = 1, 2. \end{aligned} \tag{7}$$

(HJ)

(1) There exist positive constants c_i^1, c_i^2, d_i^1, d_i^2 such that $\|J_i^1(t,v)\| \le c_i^1\|v\|_{\mathfrak{B}}+c_i^2$ and $\|J_i^2(t,v)\| \le d_i^1\|v\| + d_i^2$.

(2) $\|J_i^j(t,u)-J_i^j(t,v)\| \le L_{J_i^j}\|u-v\|_{\mathfrak{B}}$ for all $u,v \in \mathfrak{B}$ $i=1,\dots,n$, $j=1,2$.

(1) $K_a(NaL_g + \widetilde{N}\int_0^a \alpha(s)ds \lim_{\tau\to\infty} \sup(\Omega(\tau)/\tau) + \sum_{i=1}^n(Nc_i^1 + \widetilde{N}(d_i^1 + L_g))) + \sum_{i=1}^n c_i^1 K_a < 1$;

(2) $(K_aNL_ga + \sum_{i=1}^n\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}K_a + \sum_{i=0}^n\int_{s_i}^t \eta(s)ds) + \sum_{i=1}^n\{L_{J_i^1}\}K_a < 1$.

(H1)

(1) $K_a(NaL_g + \widetilde{N}\int_0^a \alpha(s)ds \lim_{\tau\to\infty} \sup(\Omega(\tau)/\tau) + \sum_{i=1}^n(Nc_i^1+\widetilde{N}(d_i^1 + L_g))) + \sum_{i=1}^n c_i^1 K_a < 1$;

(2) $(K_aNL_ga + \sum_{i=1}^n\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}K_a + \sum_{i=0}^n\int_{s_i}^t \eta(s)ds) + \sum_{i=1}^n\{L_{J_i^1}\}K_a < 1$.

Lemma 9 (see [9]). *If $y : (-\infty,a] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in PC(X)$ then*

$$\left\|y_{\rho(s,y_s)}\right\|_{\mathfrak{B}} \le \left(M_a + \widetilde{J^\phi}\right)\|\phi\|_{\mathfrak{B}} + K_a \sup\left\{\|y(\theta)\|; \theta \in [0,\max\{0,s\}]\right\}, \quad s \in \mathbb{R}(\rho^-) \cup [0,a], \tag{8}$$

where $\widetilde{J^\phi} = \sup_{t\in\mathbb{R}(\rho^-)} J^\phi(t)$, $M_a = \sup_{t\in J} M(t)$ and $K_a = \max_{t\in J} K(t)$.

In this section : $(y-\infty,a] \to X$ is the function defined by $y_0 = \phi$ and $y(t) = C(t)\phi(0) + S(t)(z + g(0,\phi))$ on $J_1 = [0,t_1]$. Clearly $\|y_t\|_{\mathfrak{B}} \le K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_b = \sup_{0\le t\le b}\|y(t)\|$.

Theorem 10. *If the hypotheses (Hf), (Hg), (HI), (H1) are satisfied, then the initial value problem* (1) *has at least one mild solution.*

Proof. Let $S(a)$ be the space $S(a) = \{x : (-\infty,a] \to X \mid x_0 = 0,\ x|_J \in \mathrm{PC}\}$ endowed with supremum norm $\|\cdot\|_a$.

Let $\Gamma : S(a) \to S(a)$ be the map defined by $(\Gamma x)_0 = 0$ and $\Gamma = \sum_{i=1}^n \Gamma_i^1 + \sum_{i=1}^n \Gamma_i^2$:

$$\left(\Gamma_i^1 x\right)(t) = \begin{cases} J_i^1\left(t,\overline{x}_t\right), & t \in (t_i,s_i];\ i=1,\dots,n \\ C(t-s_i)J_i^1\left(s_i,\overline{x}_{s_i}\right) \\ \quad +S(t-s_i)\left(J_i^2\left(s_i,\overline{x}_{s_i}\right)\right. \\ \qquad \left.+g\left(s_i,x_{s_i}\right)\right), & t \in (s_i,t_{i+1}];\ i=1,\dots,n; \end{cases} \tag{9}$$

$$\left(\Gamma_i^2 x\right)(t) = \begin{cases} \int_{s_i}^t C(t-s)g\left(s,\overline{x}_s\right)ds \\ \quad +\int_{s_i}^t S(t-s) \\ \qquad \times f\left(s,\overline{x}_{\rho(s,\overline{x}_s)}\right)ds, & t \in (s_i,t_{i+1}];\ i=0,\dots,n \\ 0, & t \notin (s_i,t_{i+1}],\ i=0,\dots,n, \end{cases} \tag{10}$$

where $\overline{x}_0 = \phi$ and $\overline{x} = x + y$ on J. It is easy to see that

$$\|\overline{x}_t\|_{\mathfrak{B}} \le K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a\|x\|_t, \tag{11}$$

where $\|x\|_t = \sup_{0\le s\le t}\|x(s)\|$.

$$\left\|\overline{x}_{\rho(s,\overline{x}_s)}\right\|_{\mathfrak{B}} \le k^* := \left(M_a + \widetilde{J^\phi}\right)\|\phi\|_{\mathfrak{B}} + K_a\|y\|_a + K_a\|x\|_a. \tag{12}$$

Thus Γ is well defined and has values in $S(a)$. Also by axioms of phase space, the Lebesgue dominated convergence theorem, and the conditions (Hf), (Hg) it can be shown that Γ is continuous.

Step 1. There exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(a) : \|x\|_a \le k\}$. In fact, if we assume that the assertion is false, then for $k > 0$ there exist $x_k \in B_k$ and $t_k \in (s_i,t_{i+1}]$ such that $k < \|\Gamma x_k(t_k)\|$:

$$\begin{aligned} k &\le \sum_{i=0}^n\left\|\Gamma_i^2 x_k(t_k)\right\| + \sum_{i=1}^n\left\|\Gamma_i^1 x_k(t_k)\right\| \\ &\le \sum_{i=0}^n N\int_{s_i}^{t_k} L_g\left(\|\overline{x_k}_s\|_{\mathfrak{B}} + \|g(s,0)\|\right)ds \\ &\quad + \sum_{i=1}^n \widetilde{N}\int_{s_i}^{t_k}\alpha(s)\,\Omega\left(\left\|\overline{x_k}_{\rho(s,\overline{x_k}_s)}\right\|_{\mathfrak{B}}\right) \\ &\quad + \sum_{i=1}^n N\left(c_i^1\|\overline{x_k}_s\| + c_i^2\right) \\ &\quad + \sum_{i=1}^n \widetilde{N}\left(d_i^1\|\overline{x_k}_s\| + d_i^2\right. \\ &\qquad \left.+L_g\left\|\overline{x_k}_{s_i} - 0\right\| + \|g(s,0)\|\right) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n} N \int_{s_i}^{t_k} L_g \left(K_a \|y\|_a + M_a \|\phi\|_{\mathfrak{B}} \right. \\
&\qquad \left. + K_a k + \|g(s,0)\|\right) ds \\
&+ \sum_{i=0}^{n} \widetilde{N} \int_{s_i}^{t_k} \alpha(s)\, ds \\
&\times \Omega\left(K_a \|y\|_a + \left(M_a + \widetilde{J^{\phi}}\right) \|\phi\|_{\mathfrak{B}} + K_a k\right) \\
&+ \sum_{i=1}^{n} N \left(c_i^1 \left(K_a \|y\|_a + M_a \|\phi\|_{\mathfrak{B}} + K_a k\right) + c_i^2\right) \\
&+ \widetilde{N} \left(d_i^1 \left(K_a \|y\|_a + M_a \|\phi\|_{\mathfrak{B}} + K_a k\right) + d_i^2 \right. \\
&\qquad \left. + L_g \left(K_a \|y\|_a + M_a \|\phi\|_{\mathfrak{B}} + K_a k\right) + \|g(s,0)\|\right).
\end{aligned} \tag{13}$$

Hence

$$\begin{aligned}
1 &< \left(\widetilde{N} \int_0^a \alpha(s)\, ds \right. \\
&\qquad \times \lim_{k\to\infty} \sup \frac{\Omega\left(K_a \|y\|_a + \left(M_a + \widetilde{J^{\phi}}\right) \|\phi\|_{\mathfrak{B}} + K_a k\right)}{k} \\
&\qquad \left. + N a K_a L_g \right) + K_a \sum_{i=1}^{n} \left(N c_i^1 + \widetilde{N}\left(d_i^1 + L_g\right)\right) \\
&\leq K_a \left(N a L_g + \widetilde{N} \int_0^a \alpha(s)\, ds \right. \\
&\qquad \left. \times \lim_{\tau\to\infty} \sup \frac{\Omega(\tau)}{\tau} + \sum_{i=1}^{n} \left(N c_i^1 + \widetilde{N}\left(d_i^1 + L_g\right)\right) \right)
\end{aligned} \tag{14}$$

which is a contradiction to the hypothesis (H1). Similarly $(\Gamma x)(t) < k$, for $t_k \in (t_i, s_i]$ $\forall i = 1, 2, \ldots, n$. Suppose on the contrary,

$$\begin{aligned}
k &< \sum_{i=1}^{n} \left(\Gamma_i^1 x_k\right)(t_k) \\
&= \sum_{i=1}^{n} \left\| J_i^1 \left(t_k, \overline{x_{k t_k}}\right) \right\| \\
&\leq \sum_{i=1}^{n} \left\{ c_i^1 \left\| \overline{x_{k t_k}} \right\|_{\mathfrak{B}} + c_i^2 \right\} \\
&\leq \sum_{i=1}^{n} \left\{ c_i^1 \left(K_a \|y\|_a + M_a \|\phi\|_{\mathfrak{B}} + K_a k\right) + c_i^2 \right\}.
\end{aligned} \tag{15}$$

Hence,

$$1 < \sum_{i=1}^{n} c_i^1 K_a, \tag{16}$$

which is a contradiction.

Step 2. To prove that Γ is a χ-contraction. Let $\Gamma = \sum_{i=1}^{n} \Gamma_i^1 + \sum_{i=0}^{n} \Gamma_i^2$ be split into $\Gamma = \sum_{i=1}^{n} \Gamma_i^1 + \sum_{i=0}^{n} \{\Gamma_{i_1}^2 + \Gamma_{i_2}^2\}$ for $t > 0$

$$\begin{aligned}
\Gamma_{i_1}^2 x(t) &= \int_{s_i}^{t} C(t-s)\, g\left(s, \overline{x_s}\right) ds, \\
\Gamma_{i_2}^2 x(t) &= \int_{s_i}^{t} S(t-s)\, f\left(s, \overline{x}_{\rho(s,\overline{x}_s)}\right) ds.
\end{aligned} \tag{17}$$

For arbitrary $x_1, x_2 \in \mathrm{B}_k$, and $t \in (s_i, t_{i+1}]$

$$\begin{aligned}
&\sum_{i=0}^{n} \left\| \Gamma_{i_1}^2 x_1(t) - \sum_{i=0}^{n} \Gamma_{i_2}^1 x_2(t) \right\| \\
&\leq \sum_{i=0}^{n} \left\| \int_{s_i}^{t} C(t-s) \left(g\left(s, x_{1_s} + y_s\right) - g\left(s, x_{2_s} + y_s\right)\right) ds \right\| \\
&\leq \sum_{i=0}^{n} N L_g a \left\| x_{1_t} - x_{2_t} \right\|_{\mathfrak{B}} \\
&\leq K_a N L_g a \|x_1 - x_2\|_a.
\end{aligned} \tag{18}$$

So, $\Gamma_{i_1}^2$ $\forall i = 0, \ldots, n$ is Lipschitz continuous with Lipschitz constant $N L_g a K_a$.

For any $W \subset \Gamma_{i_1}^2(B_k)$, W is piecewise equicontinuous since $S(t)$ is equicontinuous. Hence from the fact that $\rho(s, \overline{x_s}) \leq s$, $s \in [0, a]$ and Lemma 6 and $\chi_{\mathrm{PC}}(W) = \sup\{\chi(W(t)),\ t \in J\}$ we have

$$\begin{aligned}
&\chi\left(\sum_{i=0}^{n} \Gamma_{i_1}^2 W(t)\right) \\
&= \sum_{i=0}^{n} \chi\left(\int_{s_i}^{t} S(t-s)\, f\left(s, W_{\rho(s,\overline{x}_s)} + y_s\right) ds\right) \\
&\leq \sum_{i=0}^{n} \int_{s_i}^{t} \eta(s) \sup_{-\infty<\theta\leq 0} \chi\left(W\left(\rho\left(s, \overline{x}_s\right) + \theta\right) + y(s+\theta)\right) ds \\
&\leq \sum_{i=0}^{n} \int_{s_i}^{t} \eta(s) \sup_{-\infty<\theta\leq 0} \chi\left(W(s+\theta) + y(s+\theta)\right) ds \\
&\leq \sum_{i=0}^{n} \int_{s_i}^{t} \eta(s) \sup_{-\infty<\tau\leq 0} \chi W(\tau)\, ds \\
&\leq \chi_{\mathrm{PC}}(W) \sum_{i=0}^{n} \int_{s_i}^{t} \eta(s)\, ds.
\end{aligned} \tag{19}$$

For arbitrary $x_1, x_2 \in B_k$ and $t \in (s_i, t_{i+1}]$

$$\begin{aligned}
&\sum_{i=1}^{n}\left\|\left(\Gamma_i^1 x_1\right)(t)-\sum_{i=1}^{n}\left(\Gamma_i^1 x_2\right)(t)\right\| \\
&\quad\leq \sum_{i=1}^{n}\left\{NL_{J_i^1}\left\|\overline{x_{1_{s_i}}}-\overline{x_{2_{s_i}}}\right\|\right. \\
&\qquad\left.+\widetilde{N}\left(L_{J_i^2}\left\|\overline{x_{1_{s_i}}}-\overline{x_{2_{s_i}}}\right\|+L_g\left\|x_{2_{s_i}}-x_{1_{s_i}}\right\|\right)\right\} \qquad (20)\\
&\quad\leq \sum\left\{NL_{J_i^1}+\widetilde{N}\left(L_{J_i^2}+L_g\right)\right\}\left\|x_{1_{s_i}}+y_s-x_{2_{s_i}}-y_s\right\| \\
&\quad\leq \sum\left\{NL_{J_i^1}+\widetilde{N}\left(L_{J_i^2}+L_g\right)\right\}\left\|x_{1_{s_i}}-x_{2_{s_i}}\right\|_{\mathfrak{B}} \\
&\quad\leq \sum\left\{NL_{J_i^1}+\widetilde{N}\left(L_{J_i^2}+L_g\right)\right\}K_a\|x_1-x_2\|_a.
\end{aligned}$$

So, Γ_i^1 $\forall i = 1, \ldots, n$ is Lipschitz continuous with Lipschitz constant $(NL_{J_i^1} + \widetilde{N}L_{J_i^2})K_a$.

For arbitrary $x_1, x_2 \in B_k$ and $t \in (t_i, s_i]$,

$$\begin{aligned}
\sum_{i=1}^{n}\left\|\left(\Gamma_i^1 x_1\right)(t)-\sum_{i=1}^{n}\left(\Gamma_i^1 x_2\right)(t)\right\| &\leq \sum_{i=1}^{n}L_{J_i^1}\left\|x_{1_t}-x_{2_t}\right\|_{\mathfrak{B}} \\
&\leq \sum_{i=1}^{n}K_aL_{J_i^1}\|x_1-x_2\|_a.
\end{aligned} \qquad (21)$$

For each bounded set $W \in \mathrm{PC}(J; X)$ and $t \in (s_i, t_{i+1}]$, $\forall i = 0, \ldots, n$ we have

$$\begin{aligned}
\chi_{\mathrm{PC}}(\Gamma W) &\leq \sum_{i=1}^{n}\chi_{\mathrm{PC}}\left(\Gamma_i^1 W\right)+\sum_{i=0}^{n}\chi_{\mathrm{PC}}\left(\Gamma_{i_1}^2 W+\Gamma_{i_2}^2 W\right) \\
&\leq \left(K_aNL_ga+\sum\left\{NL_{J_i^1}+\widetilde{N}\left(L_{J_i^2}+L_g\right)\right\}K_a\right. \\
&\qquad\left.+\sum_{i=0}^{n}\int_{s_i}^{t}\eta(s)\,ds\right)\chi_{\mathrm{PC}}(W).
\end{aligned} \qquad (22)$$

For each bounded set $W \in \mathrm{PC}(J; X)$ and $t \in (t_i, s_i]$ $\forall i = 1, 2, \ldots, n$ we have

$$\begin{aligned}
\chi_{\mathrm{PC}}(\Gamma W) &\leq \sum_{i=1}^{n}\chi_{\mathrm{PC}}\left(\Gamma_i^1 W\right)+\sum_{i=0}^{n}\chi_{\mathrm{PC}}\left(\Gamma_{i_1}^2 W+\Gamma_{i_2}^2 W\right) \\
&\leq \left(\sum_{i=1}^{n}\left\{L_{J_i^1}\right\}K_a+0+0\right)\chi_{\mathrm{PC}}(W).
\end{aligned} \qquad (23)$$

Therefore, Γ is a χ-contraction. So, by Darbo-Sadovskii fixed point theorem we conclude that Γ has a fixed point in $S(a)$. hence, $z = x + y$ is a mild solution of (1). □

4. Approximate Controllability

In this section the approximate controllability of the control system (1) without the impulsive conditions is studied. We consider

$$\begin{aligned}
&\frac{d}{dt}\left(x'(t)+g(t,x_t)\right)=Ax(t)+f\left(t,x_{\rho(t,x_t)}\right)+Bu(t), \\
&\qquad t\in J=[0,a], \\
&x_0=\phi\in\mathfrak{B},\qquad x'(0)=w\in X,
\end{aligned} \qquad (24)$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators on a Hilbert space X. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically; g, f are appropriate functions. B is a bounded linear operator on a Hilbert space U. We define mild solution of problem (24) as follows.

Definition 11. A function $x : (-\infty, a] \to X$ is a mild solution of the problem (24) if $x_0 = \phi$; $x(\cdot)|_{[0,a]} \in C(J, X)$, the functions $f(s, x_{\rho(s,x_s)})$ and $g(s, x_s)$ are integrable and the integral equation is satisfied:

$$\begin{aligned}
x(t) &= C(t)\phi(0)+S(t)\left[w+g(0,\phi)\right] \\
&\quad-\int_0^t C(t-s)\,g(s,x_s)\,ds \\
&\quad+\int_0^t S(t-s)\left[f\left(s,x_\rho(s,x_s)\right)+Bu(s)\right]ds, \\
&\qquad t\in[0,a].
\end{aligned} \qquad (25)$$

Lemma 12 (see [11]). *Under the assumption that $h : [0, a] \to X$ is an integrable function, such that*

$$\begin{aligned}
x''(t) &= Ax(t)+h(t), \quad t\in J, \\
x(0) &= x^0, \\
x'(0) &= x^1
\end{aligned} \qquad (26)$$

and h is a function continuously differentiable, then

$$\int_0^t C(t-s)\,h(s)\,ds = S(t)\,h(0)+\int_0^t S(t-s)\,h'(s)\,ds. \qquad (27)$$

Set $a := T$.

Definition 13. The set given by $\mathscr{R}_T(f) = \{x(T) \in X : x$ is the mild solution of (24)$\}$ is called reachable set of the system (24). $\mathscr{R}_T(0)$ is the reachable set of the corresponding linear control system (31).

Definition 14. The system (24) is said to be approximately controllable on $[0, T]$ if $\mathscr{R}_T(f)$ is dense in X. The corresponding linear system is approximately controllable if $\mathscr{R}(0)$ is dense in X.

Lemma 15. *Let X be Hilbert space and X_1, X_2 closed subspaces such that $X = X_1 + X_2$. Then there exists a bounded linear operator $P : X \to X_2$ such that for each $x \in X$, $x = x - Px \in X_1$ and $\|x_1\| = \min\{\|y\| : y \in X_1, (1-Q)(y) = (1-Q)(x)\}$ where Q denotes the orthogonal projection on X_2.*

Let us define a continuous linear operator $\mathfrak{L} : \mathscr{L}_2([0,T];X) \to C([0,T];X)$ as

$$\mathfrak{L}p = \int_0^T S(T-s)\,p(s)\,ds, \quad p \in \mathscr{L}_2([0,T];X). \tag{28}$$

Let us denote the kernel of the operator $\mathfrak{L}$ by N which is a closed subspace of $\mathscr{L}_2([0,T];X)$. Let $N_0^{\perp}$ denote the corresponding orthogonal subspace of $\mathscr{L}_2([0,T];X)$. Let $\mathfrak{P}$ be a projection on $\mathscr{L}_2([0,T];X)$ with range $N_0^{\perp}$. Let $\overline{R(B)}$ denote the closure of the range of operator B. The following hypothesis is required to prove the approximate controllability

(HR) $\forall \epsilon > 0$ and $p(\cdot) \in \mathscr{L}_2([0,T];X)$, $\exists u(\cdot) \in U$ such that $\|\mathfrak{L}p - \mathfrak{L}Bu\|_X < \epsilon$.

It is easily seen that hypothesis (HR) is equivalent to the $\mathscr{L}_2([0,T];X) = \overline{R(B)} + N_0$ or $\mathfrak{P}\overline{R(B)} = N_0^{\perp}$. Theorem 16 shows that (HR) implies approximate controllability of the system (29). It is also known that approximate controllability of (31) implies $\mathscr{L}_2([0,T];X) = \overline{R(B) + N_0}$. Hence the closeness of the product space implies that (HR) is equivalent to approximate controllability of (29).

Theorem 16. *If the assumptions (Hg) and (HR) hold then the corresponding neutral system*

$$\begin{gathered} \frac{d\left(x'(t) + g(t,x_t)\right)}{dt} = Ax(t) + Bu(t), \quad t \in J, \\ x(0) = \phi(0), \\ x'(0) = w \end{gathered} \tag{29}$$

with $f \equiv 0$ is approximately controllable.

Proof. It is sufficient to prove that $D(A) \subset \overline{\mathscr{R}_T(0)}$ since $D(A)$ is dense in X. Let $h(T,\phi) = C(t)\phi(0) + S(t)[w + g(0,\phi(0))] - \int_0^T C(t-s)g(s,x_s)ds$ for any chosen $\xi \in D(A)$, then $\xi - h(T,\phi) \in D(A)$. It can be easily seen from Lemma 12 and [28] that there exists some $p \in C^1([0,T];X)$ such that

$$\eta = \xi - h(T,\phi) = \int_0^T S(T-s)\,p(s)\,ds. \tag{30}$$

By hypothesis (HR) there exists a control function $u(\cdot) \in \mathscr{L}_2([0,T];U)$ such that $\|\eta - \mathfrak{L}Bu\| < \epsilon$. As ϵ is arbitrary it implies that $K_T(0) \subset D(A)$. Since the $D(A)$ is dense in X, $K_T(0)$ is dense in X. Hence the neutral system with $f \equiv 0$ is approximately controllable. □

We state the corresponding linear control system

$$\begin{gathered} x''(t) = Ax(t) + Bu(t), \quad t \in J, \\ x(0) = x^0, \\ x'(0) = x^1. \end{gathered} \tag{31}$$

Both exact and approximate controllability of the above system are studied extensively in [33, 38] and so forth.

Assume that f, g satisfy the following conditions with $\mu_f, \mu_g, \nu_f, \nu_g \in L^2(J)$. For a fixed $\phi \in \mathfrak{B}$ and $x \in C(J,X)$ such that $x(0) = \phi(0)$, we define maps $F, G : C_0(J,X) \to L^2(J,X)$ by $F(z)(t) = f(t, z_t + x_t)$ and $G(z)(t) = g(t, z_t + x_t)$. Here $x_t(\theta) = x(t+\theta)$, for $t+\theta \geq 0$ and $x_t(\theta) = \phi(t+\theta)$ for $t+\theta \leq 0$ and $z_t(\theta) = z(t+\theta)$ for $t+\theta \geq 0$ and $z_t(\theta) = 0$ for $t+\theta \leq 0$. Clearly, F, G are continuous maps.

(C1) The function $F(t,\cdot) : \mathfrak{B} \to X$ is continuous for almost all $t \in I$ and $F(\cdot,z) : J \to X$ is strongly measurable, $\forall z \in \mathfrak{B}$.

(C2) There exists integrable functions $\mu_F, \nu_F : I \to [0,\infty)$ and a continuous nondecreasing function $W_F : [0,\infty) \to (0,\infty)$ such that $\|F(t,z)\|_2 \leq \mu_F(t)W_F(\|z\|_1) + \nu_F(t)$, $(t,z) \in J \times \mathfrak{B}$.

(C3) The function $f(\cdot)$ is continuous $\forall t, v \in J \times \mathfrak{B}$ and $f(t,\cdot)$ is Lipschitz continuous such that there exists positive constant L_f such that

$$\begin{gathered} \|f(t,v_1) - f(t,v_2)\| \leq L_f \|v_1 - v_2\|_{\mathfrak{B}}, \\ (t,v_i) \in J \times \mathfrak{B}, \quad i = 1,2. \end{gathered} \tag{32}$$

The above same conditions also hold for G.

Also, $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi$ and $y(t) = C(t)\phi(0) + S(t)(z + g(0,\phi))$ on J. Clearly $\|y_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_b = \sup_{0\leq t\leq b}\|y(t)\|$.

The operators $\Lambda_i : L^2(J,X) \to X$ $i = 1,2$ are defined as

$$\begin{gathered} \Lambda_1 x(t) = \int_0^a S(t-s)\,x(s)\,ds, \\ \Lambda_2 x(t) = \int_0^a C(t-s)\,x(s)\,ds. \end{gathered} \tag{33}$$

Clearly Λ_i are bounded linear operators. We set $\mathfrak{N}_i = \ker(\Lambda_i)$, $\Lambda = (\Lambda_1, \Lambda_2)$ and $\mathfrak{N} = \ker(\Lambda)$. Let $C_0(J,X)$ denote the space consisting of continuous functions $x : J \to X$ such that $x(0) = 0$, endowed with the norm of uniform convergence. Let $J_i : L^2(J,X) \to C_0(J,X)$, $i = 1,2$ be maps defined as follows:

$$\begin{gathered} J_1 x(t) = \int_0^t S(t-s)\,x(s)\,ds, \\ J_2 x(t) = \int_0^t C(t-s)\,x(s)\,ds. \end{gathered} \tag{34}$$

So, $J_i x(a) = \Lambda_i(x)$, $i = 1,2$.

As a continuation of coauthor Sukavanam's work [19] and from hypothesis $(B1)$ in [39] we assume that $L^2(J,X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1,2$.

By using Lemma 15 we denote P_i the map associated to this decomposition and construct $X_2 = \mathfrak{N}_i$ and $X_1 = \overline{R(B)}$. Also set $c_i = \|P_i\|$.

We introduce the space

$$Z = \{z \in C_0(J,X) : z = J_1(n_1) + J_2(n_2),\ n_i \in \mathfrak{N}_i,\ i = 1,2\} \tag{35}$$

and we define the map $\Gamma : \overline{Z} \to C_0(J,X)$ by

$$\Gamma = J_1 \circ P_1 \circ F - J_2 \circ P_2 \circ G. \tag{36}$$

Lemma 17. *If the hypothesis (H_ϕ)–(Hg) and conditions (C1)-(C2) hold for f, g and $aK_a(c_1\widetilde{N}L_f + c_2NL_g) < \sqrt{2}$ then Γ has a fixed point.*

Proof. For $z^1, z^2 \in \overline{Z}$ let $\Delta f(s) = f(s, z^2_{\rho(s,z^2(s))} + x_{\rho(s,x(s))}) - f(s, z^1_{\rho(s,z^1(s))} + x_{\rho(s,x(s))})$ and $\Delta g(s) = g(s, z^2_s + x_s) - f(s, z^1_s + x_s)$. $\forall 0 \le t \le a$

$$\begin{aligned}
&\left\|\left(\Gamma z^2 - \Gamma z^2\right)(t)\right\| \\
&\le \left\|\int_0^t S(t-s)\left[P_1(\Delta f)\right](s)\,ds\right\| \\
&\quad + \left\|\int_0^t C(t-s)\left[P_2(\Delta g)\right](s)\,ds\right\| \\
&\le \widetilde{N}\int_0^t \left\|\left[P_1(\Delta f)\right](s)\right\| ds + N\int_0^t \left\|\left[P_2(\Delta g)\right](s)\right\| ds \\
&\le \widetilde{N}t^{1/2}c_1\|\Delta f\|_2 + Nt^{1/2}c_2\|\Delta g\|_2.
\end{aligned} \tag{37}$$

Now

$$\begin{aligned}
\|\Delta f\|_2^2 &= \int_0^a \Big\| f\left(s, z^2_{\rho(s,z^2(s))} + x_{\rho(s,x(s))}\right) \\
&\qquad - f\left(s, z^1_{\rho(s,z^1(s))} + x_{\rho(s,x(s))}\right)\Big\|^2 ds \\
&\le L_f^2 \int_0^a \left\|z^2_{\rho(s,z^2(s))} - z^1_{\rho(s,z^1(s))}\right\|^2_{\mathfrak{B}} ds \\
&\le L_f^2 \int_0^a \left\|z^2_s - z^1_s\right\|^2_{\mathfrak{B}} ds \\
&\le aL_f^2K_a^2\left\|z^2 - z^1\right\|^2_\infty ds.
\end{aligned} \tag{38}$$

Similarly we find for g. So,

$$\left\|\left(\Gamma z^2 - \Gamma z^1\right)(t)\right\| \le bt^{1/2}\left\|z^2 - z^1\right\|_\infty, \tag{39}$$

where $b = a^{1/2}K_a(c_1\widetilde{N}L_f + c_2NL_g)$. Repeating this we get

$$\left\|\left(\Gamma^n z^2 - \Gamma^n z^1\right)(t)\right\|_\infty \le \frac{\left(bt^{1/2}\right)^n}{2^{(n-1)/(2n)}}\left\|z^2 - z^1\right\|_\infty. \tag{40}$$

As $b = aK_a(c_1\widetilde{N}L_f + c_2NL_g) < \sqrt{2}$ and $2^{(n-1)/2n} \to \sqrt{2}$ as $n \to \infty$, the map Γ^n is a contraction for n sufficiently large and therefore Γ has a fixed point. □

Theorem 18. *If the associated linear control system (31) is approximately controllable on J, the space $L^2([0,a],X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1,2$ and condition of the preceding Lemma 17 hold then the semilinear control system (24) with state dependent delay is approximately controllable on J.*

Proof. Assume $x(\cdot)$ to be the mild solution and $u(\cdot)$ to be an admissible control function of system (31) with initial conditions $x(0) = \phi(0)$ and $x'(0) = w + g(0,\phi)$. Let z be the fixed point of Γ. So, $z(0) = 0$ and $z(a) = \Lambda_1(P_1(F(z))) - \Lambda_2(P_2(G(z))) = 0$. By Lemma 12 we can split the functions $F(z)$, $G(z)$ with respect to the decomposition $L^2(J,X) = \mathfrak{N}_i + \overline{R(B)}$ $i = 1,2$, respectively, by setting $q_1 = F(z) - P_1(F(z))$ and $q_2 = G(z) - P_2(G(z))$. We define the function $y(t) = z(t) + x(t)$ for $t \in J$ and $y_0 = \phi$. So, $x(a) = y(a)$. Thus by the properties of x and z

$$\begin{aligned}
&y(t) \\
&\quad = \int_0^t S(t-s)\left(f\left(s, y_{\rho(s,y(s))}\right) - q_1(s) + Bu(s)\right) ds \\
&\qquad - \int_0^t C(t-s)\left(g(s, y_s) - q_2(s)\right) ds \\
&\qquad + C(t)x(0) + S(t)x'(0).
\end{aligned} \tag{41}$$

As $C_0^1(J,U)$ is dense in $L^2(J,U)$ we can choose a sequence $v_n^1 \in L^2(J,U)$ and a sequence $v_n^2 \in L^2(J,X)$ such that $Bv_n^1 \to q_1$ and $Bv_n^2 \to q_2$ as $n \to \infty$. By Lemma 15 we get

$$\begin{aligned}
&y^n(t) \\
&\quad = \int_0^t S(t-s)\left(f\left(s, y^n_{\rho(s,y(s))}\right) - Bv_n^1(s) + Bu(s)\right) ds \\
&\qquad - \int_0^t C(t-s)\left(g(s, y^n_s) - Bv_n^2(s)\right) ds \\
&\qquad + C(t)\phi(0) + S(t)\left(w - g(0,\phi)\right) \\
&\quad = \int_0^t S(t-s)\Big(f\left(s, y^n_{\rho(s,y(s))}\right) - Bv_n^1(s) \\
&\qquad\qquad + B\frac{d}{ds}v_n^2(s) + Bu(s)\Big) ds \\
&\qquad - \int_0^t C(t-s)\,g(s, y^n_s)\,ds + C(t)\phi(0) \\
&\qquad + S(t)\left(w + g(0,\phi)\right).
\end{aligned} \tag{42}$$

Hence by Definition 11 and the last expression we conclude that y^n is the mild solution of the following equation:

$$\begin{aligned}
&\frac{d}{dt}\left(y'(t) + g(t, x_t)\right) \\
&\quad = Ay(t) + f\left(t, y_{\rho(t,y(t))}\right) + B\left(-v_n^1(t) + \frac{d}{dt}v_n^2(t) + u(t)\right) \\
&\qquad x(0) = \phi \in \mathfrak{B} \qquad x'(0) = w.
\end{aligned} \tag{43}$$

Hence $y^n(a) \in \mathscr{R}_T(a, f, g, \phi, w)$. Since the solution map is generally continuous, $y^n \to y$ as $n \to \infty$. Thus $y(a) \in \overline{\mathscr{R}_T(a, f, g, \phi, w)}$. Therefore $\mathscr{R}_T(0)(a, \phi(0), w + g(0, \phi)) \subset \overline{\mathscr{R}_T(a, f, g, \phi, w)}$, which means $\mathscr{R}_T(a, f, g, \phi, w)$ is dense in X. Thus the system (1) is controllable. □

5. Examples

Example 1. In this section we discuss a partial differential equation applying the abstract results of this paper. In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$ (see [10]).

Consider the second order neutral differential equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial u(t,\xi)}{\partial t} + \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\xi)\, u(s,\eta)\, d\eta\, ds\right) \\
&\quad = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} \\
&\qquad + \int_{-\infty}^{t} a(t-s)\, u(s-\rho_1(t)\rho_2(\|u(t)\|), \xi)\, ds, \\
&\qquad\qquad t \in (s_i, t_{i+1}], \quad i = 0, \ldots, n, \ \xi \in [0, \pi], \\
&\qquad u(t,0) = u(t,\pi) = 0, \quad t \in [0, a], \\
&\qquad u(\tau,\xi) = \phi(\tau,\xi) \quad \tau \le 0, \ 0 \le \xi \le \pi, \\
&\qquad u'(\tau,\xi) = \omega(t,\xi) \quad \tau \le 0, \ 0 \le \xi \le \pi, \\
&\qquad u(t)(\xi) = \int_{\infty}^{t_i} a_i^1(t_i - s)\, u(s,\xi)\, ds \\
&\qquad\qquad t \in (t_i, s_i], \quad i = 1, 2, \ldots, n, \\
&\qquad u'(t)(\xi) = \int_{\infty}^{t_i} a_i^2(t_i - s)\, u(s,\xi)\, ds \\
&\qquad\qquad t \in (t_i, s_i], \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{44}$$

where $\phi \in C_0 \times L^2(h, X)$, $0 < t_1 <, \ldots, t_n < a$. For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$, where $\phi_n(x) = \sqrt{2/\pi}\sin nx$, $0 \le x \le \pi$, $n = 1, 2, 3, \ldots$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate the operators $S(t)$, $C(t)$ such that $S(t)y = \sum_{n=1}^{\infty}((\sin(nt))/n) < y, \phi_n > \phi_n$, $n = 1, 2, \ldots$ $\forall y \in X$, and the operator $C(t)y = \sum_{n=1}^{\infty}\cos(nt) < y, \phi_n > \phi_n$, $n = 1, 2, \ldots$ $\forall y \in X$. To find a solution to this problem we will assume that $h(\cdot)$ satisfies the conditions (g-5)–(g-7) in [34]. From Theorems 1.37 and 7.1.1 in [34] we conclude that $C_a((-\infty, 0], X)$ is continuously included in $\mathfrak{B}$. Let us suppose that the functions $\rho_i : \mathbb{R} \to [0, \infty)$, $a : \mathbb{R} \to \mathbb{R}$ are piecewise continuous. By defining maps $\rho, G, F : [0, a] \times \mathfrak{B} \to X$ by

$$\begin{aligned}
\rho(t,\psi) &:= \rho_1(t)\rho_2(\|\psi(0)\|), \\
g(t,\psi)(\xi) &:= \int_{-\infty}^{0}\int_0^{\pi} b(s,v,\xi)\,\psi(s,v)\, dv\, ds, \\
f(t,\psi)(\xi) &:= \int_{-\infty}^{0} a(s)\,\psi(s,\xi)\, ds, \\
J_i^j(\psi)(\xi) &:= \int_{-\infty}^{0} a_i^j(s)\,\psi(s,\xi)\, ds \quad i = 1, \ldots, n \ j = 1, 2
\end{aligned} \tag{45}$$

the system (51) can be transformed into system (1). Assume that the following conditions hold:

(a) the functions $b(s,\eta,\xi)$, $\partial b(s,\eta,\xi)/\partial\xi$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and

$$\begin{aligned}
&L_g \\
&:= \max\left\{\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{h(s)}\left(\frac{\partial^i b(s,\eta,\xi)}{\partial\xi^i}\right)^2 d\eta\, ds\, d\xi\right)^{1/2} : \right. \\
&\qquad \left. i = 0, 1\right\} \\
&< \infty
\end{aligned} \tag{46}$$

such that $\|g\|_{\mathscr{L}(X)} \le L_g$.

(b) The function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $L_f = \int_{-\infty}^{0}(a(s)^2/h(s))ds < \infty$ and $\|F\|_{\mathscr{L}(X)} \le L_f$.

(c) The functions $a_i^j \in C([0,\infty);\mathbb{R})$ and $L_i^j := (\int_{-\infty}^{0}((a_i^j(s))^2/h(s))ds)^{1/2} < \infty$ for all $i = 1, 2, \ldots, n$ $j = 1, 2$.

Moreover $g(t,\cdot)$, J_i^j, $i = 1, \ldots, n$, $j = 1, 2$ are bounded linear operators.

Hence by assumptions (a)–(c) and Theorem 10 it is ensured that mild solution to the problem (51) exists.

Now let us consider a particular example from the point of view of an application:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial u(t,\xi)}{\partial t}+\int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)\,u(s,\eta)\,d\eta\,ds\right)\\
&\quad=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}+a(t)\,b\left(u\left(t-\mu\left(u(t,0)\right),\xi\right)\right),\\
&\qquad t\in\left(s_i,t_{i+1}\right],\quad i=0,\ldots,n,\ \xi\in[0,\pi],\\
&\qquad u(t,0)=u(t,\pi)=0,\quad t\in[0,a],\\
&\qquad u(\tau,\xi)=\phi(\tau,\xi)\quad \tau\le 0,\ 0\le\xi\le\pi,\\
&\qquad u'(\tau,\xi)=\omega(\tau,\xi)\quad \tau\le 0,\ 0\le\xi\le\pi,\\
&\qquad u(t)(\xi)=d_i^1\sin\left|u(t,\xi)\right|,\quad t\in\left(t_i,s_i\right],\ i=1,2,\ldots,n,\\
&\qquad u'(t)(\xi)=d_i^2\cos\left|u(t,\xi)\right|,\quad t\in\left(t_i,s_i\right],\ i=1,2,\ldots,n,
\end{aligned}\tag{47}$$

where $\phi\in\mathfrak{B}=C_h^0(X)$. The functions $a:J\to\mathbb{R}$, $b:\mathbb{R}\times J\to\mathbb{R}$, $\mu:\mathbb{R}\to\mathbb{R}^+$ are piecewise continuous. We assume the existence of positive constants b_1, b_2 such that

$$|b(t)|\le b_1|t|+b_2,\quad \forall t\in\mathbb{R}.\tag{48}$$

If we define maps

$$\begin{aligned}
f(t,\psi)(\xi)&=a(t)\,b\left(\psi(0,\xi)\right),\\
\rho(t,\psi)&=t-\mu\left(\psi(0,0)\right),
\end{aligned}\tag{49}$$

and $g(t,\psi)(\xi)$ as in the problem (51) we can transform (47) into (1). Also a simple estimate shows that $\|f(t,\psi)\|\le a(t)[b_1\|\psi\|_{\mathfrak{B}}+b_2\pi^{1/2}]$ $\forall(t,\psi)\in J\times\mathfrak{B}$.

Also if we define $J_i^1(t,u(t))=d_i^1\sin|u(t)|$ and $J_i^2=d_i^2\cos|u(t)|$ for all $i=1,\ldots,n$ then the hypotheses (HJ) can be easily proved. For instance,

$$\begin{aligned}
&\left\|J_i^1(t,u(t))\right\|=\left\|d_i^1\sin|u(t)|\right\|\le d_i^1\|u(t)\|,\\
&\left\|J_i^1(t,u_1(t))-J_i^1(t,u_2(t))\right\|\\
&\quad=\left\|d_i^1\sin|u_1(t)|-d_i^1\sin|u_2(t)|\right\|\\
&\quad\le\left\|d_i^1\left[|u_1(t)|-|u_2(t)|\right]\right\|.
\end{aligned}\tag{50}$$

Similarly it is easily seen for J_i^2. Now, if ϕ satisfies the hypothesis (H_ϕ) then $\exists$ a mild solution of (47).

Example 2. Consider the second order neutral differential equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial u(t,\xi)}{\partial t}+\int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)\,u(s,\eta)\,d\eta\,ds\right)\\
&\quad=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}\\
&\qquad+\int_{-\infty}^{t} a(t-s)\,u\left(s-\rho_1(t)\,\rho_2(\|u(t)\|),\xi\right)ds+Bv(t)\\
&\qquad t\in[0,a],\quad \xi\in[0,\pi],\\
&\qquad u(t,0)=u(t,\pi)=0,\quad t\in[0,a],\\
&\qquad u(t,\xi)=\phi(t,\xi)\quad \tau\le 0,\ 0\le\xi\le\pi,
\end{aligned}\tag{51}$$

where $\phi\in C_0\times L^2(h,X)$, $0<t_1<,\ldots,t_n<a$. For $y\in D(A)$, $y=\sum_{n=1}^{\infty}<y,\phi_n>\phi_n$, and $Ay=-\sum_{n=1}^{\infty}n^2<y,n>\phi_n$, where $\phi_n(x)=\sqrt{2/\pi}\sin nx$, $0\le x\le\pi$, $n=1,2,3,\ldots$ is the eigenfunction corresponding to the eigenvalue $\lambda_n=-n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate the operators $S(t)$, $C(t)$ such that $S(t)y=\sum_{n=1}^{\infty}((\sin(nt))/n)<y,\phi_n>\phi_n$, $n=1,2,\ldots$ $\forall y\in X$, and the operator $C(t)y=\sum_{n=1}^{\infty}\cos(nt)<y,\phi_n>\phi_n$, $n=1,2,\ldots$ $\forall y\in X$. Let the infinite dimensional control space be defined as $U=\{u:u=\sum_{n=2}^{\infty}u_n\phi_n,\ \sum_{n=2}^{\infty}u_n^2<\infty\}$ with norm $\|u\|_U=(\sum_{n=2}^{\infty}u_n^2)^{1/2}$. Thus U is a Hilbert space. By defining maps $\rho,G,F:[0,a]\times\mathfrak{B}\to X$ by

$$\begin{aligned}
\rho(t,\psi)&:=\rho_1(t)\,\rho_2\left(\|\psi(0)\|\right),\\
G(\psi)(\xi)&:=\int_{-\infty}^{0}\int_{0}^{\pi} b(s,v,\xi)\,\psi(s,v)\,dv\,ds,\\
F(\psi)(\xi)&:=\int_{-\infty}^{0} a(s)\,\psi(s,\xi)\,ds,
\end{aligned}\tag{52}$$

the system (51) can be transformed into system (1). Assume that the functions $\rho_i:\mathbb{R}\to[0,\infty)$, $a:\mathbb{R}\to\mathbb{R}$ are continuous and satisfy the following conditions.

(a) The functions $b(s,\eta,\xi)$, $\partial b(s,\eta,\xi)/\partial\xi$ are measurable, $b(s,\eta,\pi)=b(s,\eta,0)=0$ and

$$\begin{aligned}
&L_g\\
&:=\max\left\{\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{h(s)}\left(\frac{\partial^i b(s,\eta,\xi)}{\partial\xi^i}\right)^2 d\eta\,ds\,d\xi\right)^{1/2}:\right.\\
&\qquad\left. i=0,1\right\}\\
&<\infty
\end{aligned}\tag{53}$$

such that $\|g\|_{\mathscr{L}(X)}\le L_g$.

(b) The function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $L_f = \int_{-\infty}^{0}(a(s)^2/h(s))ds < \infty$ and $\|F\|_{\mathscr{L}(X)} \le L_f$.

(c) The functions $a_i^j \in C([0,\infty);\mathbb{R})$ and $L_i^j := (\int_{-\infty}^{0}((a_i^j(s))^2/h(s))ds)^{1/2} < \infty$ for all $i = 1,2,\ldots,n$ $j = 1,2$.

Moreover $g(t,\cdot)$ is a bounded linear operator.

Here we examine the conditions (HR) for this control system. Then by using Theorem 18 we show its approximate controllability. Let $\widetilde{B} : U \to X : \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n$ for $u = \sum_{n=2}^{\infty} u_n\phi_n \in U$. The bounded linear operator $B : L_2([0,T];U) \to L_2([0,T];X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$.

Let $\alpha \in N \subset L_2(0,T:X)$, N is the null space of Γ. Also $\alpha = \sum_1^{\infty} \alpha_n(s)\phi_n$. Therefore

$$\int_0^T S(T-s)\,\alpha(s)\,ds = 0. \tag{54}$$

This implies that

$$\int_0^T \frac{\sin n(T-s)}{n}\alpha_n(s)\,ds = 0, \quad n \in \mathscr{N}. \tag{55}$$

The Hilbert space $L_2(0,T)$ can be written as

$$L_2(0,T) = \mathrm{Sp}\{\sin s\}^{\perp} + \mathrm{Sp}\{\sin 4s\}^{\perp}. \tag{56}$$

Thus for $h_1, h_2 \in L_2(0,T)$ there exists $\alpha_1 \in \{\sin s\}\perp$, $\alpha_2 \in \{\sin 4s\}^{\perp}$ such that $h_1 - 2h_2 = \alpha_1 - 2\alpha_2$. So let $u_2 = h_2 - \alpha_2$. Then $h_1 = \alpha_1 + 2u_2$, $h_2 = \alpha_2 + u_2$ also let $u_n = h_n$, $n = 3,4,\ldots$ and $\alpha_n = 0$, $n = 3,4,\ldots$. Thus we see that hypothesis (HR) is satisfied as $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n,\ \sum_{n=2}^{\infty} u_n^2 < \infty\}$ and $\widetilde{B} : U \to X : \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n$.

Hence by assumptions (a)–(c) and Theorem 18 it is ensured that the problem (51) is approximately controllable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express sincere gratitude to the reviewer for his valuable suggestions. The first author would like to thank Ministry of Human Resource and Development with Grant no. MHR-02-23-200-429/304 for their funding.

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