Research Article

On the Spectrum and Spectral Norms of \( r \)-Circulant Matrices with Generalized \( k \)-Horadam Numbers Entries

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This work is concerned with the spectrum and spectral norms of \( r \)-circulant matrices with generalized \( k \)-Horadam numbers entries. By using Abel transformation and some identities we obtain an explicit formula for the eigenvalues of them. In addition, a sufficient condition for an \( r \)-circulant matrix to be normal is presented. Based on the results we obtain the precise value for spectral norms of normal \( r \)-circulant matrix with generalized \( k \)-Horadam numbers, which generalize and improve the known results.

1. Introduction

There is no doubt that the \( r \)-circulant matrices have been one of the most interesting research areas in computation mathematics. It is well known that these matrices have a wide range of applications in signal processing, digital image disposal, coding theory, linear forecast, and design of self-regress.

There are many works concerning estimates for spectral norms of \( r \)-circulant matrices with special entries. For example, Solak [1] established lower and upper bounds for the spectral norms of circulant matrices with Fibonacci and Lucas numbers entries. Subsequently, Ipek [2] investigated some improved estimations for spectral norms of these matrices. Bani-Domi and Kittaneh [3] established two general norm equalities for circulant and skew circulant operator matrices. Shen and Cen [4] gave the bounds of the spectral norms of \( r \)-circulant matrices whose entries are Fibonacci and Lucas numbers. In [5] they defined \( r \)-circulant matrices involving \( k \)-Lucas and \( k \)-Fibonacci numbers and also investigated the upper and lower bounds for the spectral norms of these matrices.

Recently, Yazlik and Taskara [6] define a generalization \( \{ H_{kn} \} \) of the special second-order sequences such as Fibonacci, Lucas, \( k \)-Fibonacci, \( k \)-Lucas, generalized \( k \)-Fibonacci and \( k \)-Lucas, Horadam, Pell, Jacobsthal, and Jacobsthal-Lucas sequences. For any integer number \( k \geq 1 \), the generalized \( k \)-Horadam sequence \( \{ H_{kn} \} \) is defined by the following recursive relation:

\[
H_{k, n+2} = f(k) H_{k, n+1} + g(k) H_{k, n},
H_{k, 0} = a, \quad H_{k, 1} = b,
\]

where \( f(k) \) and \( g(k) \) are scaler-value polynomials, \( f^2(k) + 4g(k) > 0 \). The following are some particular cases.

(i) If \( f(k) = k, g(k) = 1 \) and \( a = 0, b = 1 \), the \( k \)-Fibonacci sequence is obtained:

\[
F_{k,n+2} = kF_{k,n+1} + F_{k,n}, \quad F_{k,0} = 0, \quad F_{k,1} = 1.
\]

(ii) If \( f(k) = k, g(k) = 1 \) and \( a = 2, b = k \), the \( k \)-Lucas sequence is obtained:

\[
L_{k,n+2} = kL_{k,n+1} + L_{k,n}, \quad F_{k,0} = 0, \quad F_{k,1} = k.
\]

(iii) If \( f(k) = 1, g(k) = 1 \) and \( a = 0, b = 1 \), the Fibonacci sequence is obtained:

\[
F_{n+2} = F_{n+1} + F_{n}, \quad F_{0} = 0, \quad F_{1} = 1.
\]

(iv) If \( f(k) = 1, g(k) = 1 \) and \( a = 2, b = 1 \), the Lucas sequence is obtained:

\[
L_{n+2} = L_{n+1} + L_{n}, \quad L_{0} = 2, \quad L_{1} = 1.
\]
(v) If \( f(k) = 1, g(k) = 2 \) and \( a = 0, b = 1 \), the Jacobsthal sequence is obtained:

\[
J_{n+2} = J_{n+1} + 2J_n, \quad J_0 = 0, \quad J_1 = 1.
\]

In [7], the authors present new upper and lower bounds for the spectral norm of an \( r \)-circulant matrix \( C_r(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1}) \), and they study the spectral norm of circulant matrix with generalized \( k \)-Horadam numbers in [8]. In this paper, we first give an explicit formula for the eigenvalues of \( r \)-circulant matrix with generalized \( k \)-Horadam numbers entries using different methods in [7]. Afterwards, we present a sufficient condition for an \( r \)-circulant matrix to be normal. Based on the results, the precise value for spectral norms of normal \( r \)-circulant matrix whose entries are generalized \( k \)-Horadam numbers is obtained, which generalize and improve the main results in [1, 2, 4, 5].

2. Preliminaries

In this section, we present some known lemmas and results that will be used in the following study.

Definition 1. For any given \( c_0, c_1, \ldots, c_{n-1} \in \mathbb{C} \), the \( r \)-circulant matrix \( C_r \), denoted by \( C_r(c_0, c_1, \ldots, c_{n-1}) \), is of the form

\[
\begin{pmatrix}
  c_0 & c_1 & \cdots & c_{n-1} \\
r_{c_{n-1}} & c_0 & \cdots & c_{n-2} \\
r_{c_{n-2}} & r_{c_{n-1}} & \cdots & c_{n-3} \\
\vdots & \vdots & \ddots & \vdots \\
r_{c_1} & r_{c_2} & \cdots & c_0
\end{pmatrix}.
\]

(7)

It is obvious that the matrix \( C_r \) turns into a classical circulant matrix for \( r = 1 \).

Lemma 2 (see [9]). Let \( C = C_r(c_0, c_1, \ldots, c_{n-1}) \) be an \( r \)-circulant matrix; then the eigenvalues of \( C \) are given by

\[
\lambda_i = \sum_{j=0}^{n-1} c_j x_j, \quad \mu_i = r^{j/n} \omega^j, \quad i = 0, 1, \ldots, n-1,
\]

where \( \omega = e^{-2\pi i/n} \) is the \( n \)th root of unity.

Let us take any matrix \( A = [a_{ij}] \) of order \( n \); it is well known that the spectral norm of matrix \( A \) is

\[
\|A\|_2 = \sqrt{\max_{0 \leq g < n} \lambda_1(A^H A)},
\]

(9)

where \( A^H \) is the conjugate transpose of \( A \) and \( \lambda_1(A^H A) \) is the eigenvalue of \( A^H A \).

For a normal matrix \( A \) (i.e., \( AA^H = A^H A \)), we have the following lemma.

Lemma 3 (see [10]). Let \( A \) be a normal matrix with eigenvalues \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \). Then the spectral norm of \( A \) is

\[
\|A\|_2 = \max_{0 \leq g < n} |\lambda_i|.
\]

The following lemma can be found in [11].

Lemma 4 (see [11], Abel transformation). Suppose that \( \{a_i\} \) and \( \{b_i\} \) are two sequences, and \( S_1 = a_1 + a_2 + \cdots + a_i \); then

\[
\sum_{i=1}^{n} a_i b_i = S_n b_n - \sum_{i=1}^{n-1} (b_{i+1} - b_i) S_i.
\]

(11)

3. Spectrum of \( r \)-Circulant Matrix with Generalized \( k \)-Horadam Numbers

We start this section by giving the following lemma.

Lemma 5. Suppose that \( \{H_{kj}\}_{k \in \mathbb{N}} \) is a generalized \( k \)-Horadam sequence defined in (1). The following conclusions hold.

(1) If \( f(k) + g(k) \neq 1 \), then

\[
\sum_{i=0}^{n} H_{kj} = \frac{H_{k,n+1} + g(k) H_{k,n} + f(k)a - a - b}{f(k) + g(k) - 1}.
\]

(12)

(2) If \( f(k) + g(k) = 1 \), then

\[
\sum_{i=0}^{n} H_{kj} = \frac{g(k) H_{k,n} + n [g(k) a + b] + a}{g(k) + 1}.
\]

(13)

Proof. (1) According to (1), we have

\[
\sum_{i=0}^{n} H_{kj} = f(k) \sum_{i=0}^{n} H_{k,i-1} + g(k) \sum_{i=0}^{n} H_{k,i-2}.
\]

Changing the summation index in (14), we have

\[
\sum_{i=0}^{n} H_{kj} = f(k) \left( \sum_{j=0}^{n} H_{k,j-1} - H_{k,n} + H_{k,-1} - H_{k,-2} \right) + g(k) \left( \sum_{j=0}^{n} H_{k,j-2} - H_{k,n} + H_{k,-1} - H_{k,-2} \right).
\]

(15)

By direct calculation, together with recursive relation (1), one can obtain that

\[
[f(k) + g(k) - 1] \sum_{j=0}^{n} H_{kj} = H_{k,n+1} + g(k) H_{k,n} + f(k) a - a - b.
\]

(16)

Therefore we immediately obtain (12) from \( f(k) + g(k) \neq 1 \).

(2) Suppose that \( f(k) + g(k) = 1 \); we first illustrate that \( H_{k,j+1} + g(k) H_{k,j} = g(k)a + b \). Let \( V_i = H_{k,i+1} + g(k) H_{k,i} \); then \( V_0 = g(k) a + b \). Combining (1) and \( f(k) + g(k) = 1 \), one can obtain that

\[
V_{i+1} = H_{k,j+2} + g(k) H_{k,j+1} = (f(k) H_{k,j+1} + g(k) H_{k,j}) + g(k) H_{k,j+1}.
\]

(17)
which shows that \(\{V_i\}\) is a constant sequence, and therefore

\[
H_{k,i+1} + g(k)H_{k,i} = V_i = V_0 = g(k) a + b. \tag{18}
\]

Evaluating summation from 0 to \(n\), we have

\[
\sum_{j=0}^{n} H_{k,j+1} + g(k) \sum_{i=0}^{n} H_{k,i} = (n+1) [g(k) a + b]. \tag{19}
\]

Changing the summation index in (19) gives

\[
\left( \sum_{j=0}^{n} H_{k, j} + H_{k,n+1} - a \right) + g(k) \sum_{i=0}^{n} H_{k,i} = (n+1) [g(k) a + b]. \tag{20}
\]

Therefore

\[
[g(k) + 1] \sum_{j=0}^{n} H_{k,j} = g(k) H_{k,n} + n [g(k) a + b] + a. \tag{21}
\]

In view of assumptions \(f^2(k) + g(k) > 1\) and \(f(k) + g(k) = 1\), we know that \(g(k) + 1 \neq 0\). Thus we obtain (13) from (21).

From Lemma 5 we have the following theorem.

**Theorem 6.** Let \(A = C_r(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1})\) be an \(r\)-circulant matrix with eigenvalues \(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\); then for \(i = 0, 1, 2, \ldots, n-1\) the following hold.

(1) If \(f(k) + g(k) \neq 1\), then

\[
\lambda_i = \left( r H_{k,n} + g(k) r^{1/n} \omega^i H_{k,n-1} \right. \right.

\[
+ r^{1/n} \left[ f(k) a - b \right] \omega^i - a \right.

\[
\times \left( r^{1/n} \omega^i f(k) + r^{2/n} \omega^{2i} g(k) - 1 \right)^{-1}. \tag{22}
\]

(2) If \(f(k) + g(k) = 1\), then

\[
\lambda_i = \left( (g(k) r H_{k,n-1} + a) \left( 1 - r^{1/n} \omega^i \right) \right. \right.

\[
+ [g(k) a + b] \left( r^{1/n} \omega^i - r \right) \right.

\[
\times \left( \left( 1 - r^{1/n} \omega^i \right) [g(k) r^{1/n} \omega^i + 1] \right)^{-1}. \tag{23}
\]

**Proof.** According to Lemma 2, we have

\[
\lambda_i = \sum_{j=0}^{n-1} H_{k,j} \mu^j_i, \quad \mu_i = r^{1/n} \omega^i. \tag{24}
\]

Using Abel transformation (Lemma 4), we have

\[
\lambda_i = \mu_i^{n-1} \sum_{j=0}^{n-1} H_{k,j} \mu_i^j - \sum_{j=0}^{n-2} \left( \mu_i^{j+1} - \mu_i^j \right) \sum_{s=0}^{j} H_{k,s} \mu_i^s \tag{25}
\]

(1) In the light of (12) and (25), one can obtain that

\[
\lambda_i = \mu_i^{n-1} \sum_{j=0}^{n-1} H_{k,j} - \frac{\mu_i - 1}{f(k) + g(k) - 1} \times \sum_{j=0}^{n-2} \left[ H_{k,j+1} + g(k) H_{k,j} + f(k) a - a - b \right] \tag{26}
\]

\[
= \mu_i^{n-1} H_{k,n} + g(k) H_{k,n-1} + f(k) a - a - b \frac{f(k) + g(k) - 1}{f(k) + g(k) - 1}
\]

\[
- \frac{\mu_i - 1}{f(k) + g(k) - 1} \times \left( \sum_{j=0}^{n-2} H_{k,j+1} \mu_i^j + g(k) \sum_{j=0}^{n-2} H_{k,j} \mu_i^j \right)
\]

\[
+ [f(k) a - a - b] \sum_{j=0}^{n-2} \mu_i^j. \tag{27}
\]

It is clear that

\[
\sum_{j=0}^{n-2} H_{k,j+1} \mu_i^j = \lambda_i - a \frac{1}{\mu_i}, \tag{27}
\]

\[
\sum_{j=0}^{n-2} H_{k,j} \mu_i^j = \lambda_i - \mu_i^{n-1} H_{k,n-1}. \tag{27}
\]
Substituting (27) into (26), we obtain that
\[
\lambda_i = \mu_i^{-1} H_{k,n} + g(k) H_{k,n-1} + f(k) a - a - b \\
- \frac{\mu_i - 1}{f(k) + g(k) - 1} \\
\times \left( \lambda_i - a \mu_i + g(k) H_{k,n-1} \right) \\
+ \left[ f(k) a - a - b \right] \sum_{j=0}^{n-2} \mu_i^j
\]
\begin{align*}
&= \frac{(1 - \mu_i) \left[ 1 + g(k) \mu_i \right]}{\mu_i \left[ f(k) + g(k) - 1 \right]} \lambda_i \\
&\quad + \mu_i^{-1} H_{k,n} + \mu_i^2 g(k) H_{k,n-1} \\
&\quad + \frac{\mu_i^2 \left( f(k) a - a - b \right) - a (1 - \mu_i)}{\mu_i \left[ f(k) + g(k) - 1 \right]} \\
&\quad + \frac{(1 - \mu_i^2) (f(k) a - a - b)}{f(k) + g(k) - 1}.
\end{align*}

Therefore we have
\[
\left[ g(k) \mu_i^2 + f(k) \mu_i - 1 \right] \lambda_i
= \mu_i H_{k,n} + g(k) \mu_i^{n+1} H_{k,n-1} + \mu_i^n \left( f(k) a - a - b \right) \\
- a (1 - \mu_i) + (\mu_i - r) \left( f(k) a - a - b \right) \\
= r H_{k,n} + g(k) r^{j+1} m^j H_{k,n-1} \\
+ r^{1/n} \left( f(k) a - a - b \right) \omega^j - a.
\]

(28)

We immediately obtain formula (22) from (29).

4. Spectral Norms of Normal \( r \)-Circulant Matrices

In this section, we consider the spectral norms of normal \( r \)-circulant matrix whose entries are generalized \( k \)-Horadam numbers. Our results generalize and improve the results in [1, 2, 4, 5]. The following lemma can be found in [9], and we give a concise proof.

**Lemma 7.** Let \( A = C_r(a_0, a_1, \ldots, a_{n-1}) \) be an \( r \)-circulant matrix. If \(|r| = 1\), then \( A \) is normal matrix.

**Proof.** It is well known that
\[
A = \sum_{j=0}^{n-1} a_j P^j, \quad P = \begin{pmatrix} 0 \ 1_{n-1} \end{pmatrix}.
\]
(32)
If $|r| = 1$, then
\[ p^H = \begin{pmatrix} 0 & I_{n-1} \\ r & 0 \end{pmatrix} \begin{pmatrix} 0 & r \\ I_{n-1} & 0 \end{pmatrix} = I_n. \] (33)

That is, $P^H \equiv P^{-1}$. According to (32), we obtain that
\[ AA^H = \left( \sum_{j=0}^{n-1} a_j^2 \right) \left( \sum_{j=0}^{n-1} a_j^2 \right). \]

Therefore $AA^H = A^H A$, which shows that $A$ is normal. ∎

According to Theorem 6 and Lemma 7, we have the following theorem.

**Theorem 8.** Suppose that $A = C_{r}(H_{k_0}, H_{k_1}, \ldots, H_{k_{n-1}})$ is an $r$-circulant matrix. If $|r| = 1$ and $H_{k_j} \geq 0$, $i = 0, 1, 2, \ldots, n - 1$, then the spectral norm of $A$ is
\[ \|A\|_2 = \max_{\omega \in \mathbb{C} \cap \mathbb{N} - 1} \left| g(k) r^{1/n} \omega^i [f(k) a - b] \right|, \]
\[ f(k) + g(k) \neq 1, \]
\[ f(k) + g(k) = 1. \] (35)

Taking into account formulae (4)–(6), we have the following corollary.

**Corollary 10.** Let $A_1 = C_{r}(F_0, F_1, \ldots, F_{n-1})$ be a circulant matrix; then
\[ \|A_1\|_2 = F_{n+1} - 1. \] (40)

**Corollary 11.** Let $A_2 = C(L_0, L_1, \ldots, L_{n-1})$ be a circulant matrix; then
\[ \|A_2\|_2 = F_{n+2} + F_n - 1. \] (41)

**Corollary 12.** Let $A_3 = C(J_0, J_1, \ldots, J_{n-1})$ be a circulant matrix; then
\[ \|A_3\|_2 = \frac{F_{n+1} - 1}{2}. \] (42)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


