Research Article

On Liouville Sequences in the Non-Archimedean Case

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We study Liouville numbers in the non-Archimedean case. We deal with the concept of a Liouville sequence in the non-Archimedean case and we give some results both in the $\mathbb{Q}_p$ and the functions field $K\langle x \rangle$.

1. Introduction

It is well known that if a complex number $\alpha$ is a root of a nonzero polynomial equation

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0, \quad (1)$$

where the $a_i$ are integers (or equivalently, rational numbers) and $\alpha$ satisfies no similar equation of degree $< n$, then $\alpha$ is said to be an algebraic number of degree $n$. A complex number that is not algebraic is said to be transcendental. Liouville's theorem states that, for any algebraic number $\alpha$ with degree $n > 1$, there exists $C(\alpha) > 0$ such that

$$|\alpha - \frac{a}{b}| > \frac{C(\alpha)}{b^n}, \quad (2)$$

for all rational numbers $a/b$ with $b > 1$. The construction of transcendental numbers has been usually shown using Liouville's theorem. For instance, the transcendence of the number $\xi = \sum_{n=1}^{\infty} 10^{-n!}$ can be easily proved from Liouville's theorem. Also, Liouville's theorem can be applied to prove the transcendence of a large class of real numbers which are called Liouville numbers.

A real number $\xi \in \mathbb{R}$ is called a Liouville number if, for every positive real number $\omega$, there exist integers $a$ and $b(> 1)$ such that

$$|\xi - \frac{a}{b}| < \frac{1}{b^\omega}, \quad (3)$$

It is easy to prove that any real number $\sum_{n=1}^{\infty} a_n 10^{-n!}$ with $a_n \in \{1, 2\}$ is a Liouville number (see [1, 2]). Real Liouville numbers have many interesting properties and have been investigated by many authors (see [3–8]). In 1975, Erdős [9] proved a very interesting criterion for Liouville series.

Theorem 1 (see Erdős [9]). Let $a_1 < a_2 < a_3 < \cdots$ be an infinite sequence of integers satisfying

$$\lim_{n \to \infty} \sup_{t > 0} a_n^{1/t} = \infty$$

for every $t > 0$ and

$$a_n > n^{1+\epsilon}$$

for fixed $\epsilon > 0$ and $n > n_0(\epsilon)$. Then,

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n c_n}$$

is a Liouville number.

Hančl [8] defined the concept of Liouville sequences and generalized the above theorem of Erdős. Now, we recall the definition of Liouville sequences.

Definition 2 (see [8]). Let $(a_n)$ be a sequence of positive real numbers. If, for every $(c_n)$ of positive integers, the sum

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$$

is a Liouville number.
is a Liouville number, then the sequence \((a_n)\) is called a Liouville sequence.

The properties of Liouville sequences were investigated in [8] and some criteria were given for them. In the present work, we define the concept of Liouville sequences in non-Archimedean case and obtain some properties for them.

2. \(p\)-adic Numbers and \(p\)-adic Liouville Numbers

Recall that a norm on a field \(K\) is a function \(|\cdot|: K \to [0, \infty)\) satisfying the following conditions:

(i) \(|x| = 0\) if and only if \(x = 0\),
(ii) \(|xy| = |x||y|\), for all \(x, y \in K\),
(iii) \(|x + y| \leq |x| + |y|\), for all \(x, y \in K\).

A norm on \(K\) is called non-Archimedean if it satisfies the extra condition

(iv) \(|x + y| \leq \max\{|x|, |y|\}\) for all \(x, y \in K\);

otherwise, we say that the norm is Archimedean.

It is well known that the usual absolute value on the rational numbers field \(\mathbb{Q}\) (or the real numbers field \(\mathbb{R}\)) is Archimedean. There are interesting non-Archimedean numbers field \(\mathbb{Q}\). It is Archimedean. Otherwise, we say that the norm is non-Archimedean.

As a special case, the \(p\)-adic Liouville numbers have been studied in [17–21] and others. As a special case, the \(p\)-adic Liouville numbers have been studied in [17–21] and others.

Let \(p\) be a prime number. Every nonzero rational number \(x\) can be written uniquely under the form

\[ x = p^{v_p(x)} \frac{a}{b}, \]

where \(v_p(x)\), \(a, b \in \mathbb{Z}\), and \(a\) and \(b\) are not divided by \(p\). Here, \(v_p(x)\) is the \(p\)-adic valuation of \(x\). The \(p\)-adic norm is defined by

\[ |x|_p = \begin{cases} p^{-v_p(x)}, & x \neq 0 \\ 0, & x = 0. \end{cases} \]

It is clear that the \(p\)-adic norm is non-Archimedean. The \(p\)-adic numbers field \(\mathbb{Q}_p\) is the completion of the rational numbers field \(\mathbb{Q}\) with respect to the \(p\)-adic norm. Every nonzero \(p\)-adic number \(x \in \mathbb{Q}_p\) is uniquely represented in the canonical form

\[ x = p^m (a_0 + a_1 p + a_2 p^2 + \cdots), \]

where \(m = v_p(x)\), \(a_j \in \mathbb{Z}\), such that \(0 \leq a_j \leq p - 1\) and \(a_0 \neq 0\) \((j = 0, 1, 2, \ldots)\). The unit ball (or the ring of \(p\)-adic integers) is denoted by \(\mathbb{Z}_p\) and defined by

\[ \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}. \]

Similarly, every nonzero \(p\)-adic integer \(x \in \mathbb{Z}_p\) is uniquely represented in the canonical form

\[ x = a_0 + a_1 p + a_2 p^2 + \cdots, \]

where \(a_j \in \mathbb{Z}\) and \(0 \leq a_j \leq p - 1\) \((j = 0, 1, 2, \ldots)\). The natural numbers set \(\mathbb{N}\) is dense in \(\mathbb{Z}_p\).

Although the classical Liouville numbers are real numbers that can be rapidly approximated by rational numbers, the \(p\)-adic Liouville numbers are those numbers that can be rapidly approximated by positive integers in the \(p\)-adic norm. The \(p\)-adic Liouville numbers are defined as follows.

Definition 3 (see [10, 11]). Let \(\lambda\) be a \(p\)-adic integer. If

\[ \lim_{n \to \infty} \inf\sqrt[n]{|n - \lambda|_p} = 0, \]

then \(\lambda\) is called a \(p\)-adic Liouville number.

According to this definition, \(\lambda \in \mathbb{Z}_p\) is a \(p\)-adic Liouville number if and only if there exists a sequence of positive integers \(a_n\) such that

\[ \lim_{n \to \infty} \sqrt[n]{|a_n - \lambda|_p} = 0. \]

Example 4. Consider the series \(\alpha = \sum_{n=0}^{\infty} p^n\). It is easy to see that the sum is a \(p\)-adic Liouville number.

The definition above is first introduced by Clark [11] and it is better adapted to differential equations. In fact, consider the differential equation

\[ xf'(x) - \lambda f(x) = \frac{1}{1-x} \]

on a neighborhood \(D\) of \(0\) in \(\mathbb{Z}_p\), where \(\lambda \in \mathbb{Z}_p \setminus \{0, 1, 2, \ldots\}\). This equation has a unique formal solution; namely, \(f(x) = \sum_{n=1}^{\infty}(1/(n - \lambda))x^n\). It is clear that this solution diverges if and only if \(\lambda\) is a \(p\)-adic Liouville number (for details, see [12]). We note that the set of \(p\)-adic Liouville numbers forms a dense subset of \(\mathbb{Z}_p\) and every \(p\)-adic Liouville number is transcendental over \(\mathbb{Q}\) (for details, see [10]).

In general case, the \(p\)-adic transcendental numbers have been studied by Mahler [13], Adams [14], X. Long Xin [15], Nishioka [16], and others. As a special case, the \(p\)-adic Liouville numbers have been studied in [17–21] and others.

3. Liouville Sequences in the \(p\)-adic Numbers Fields

We define the Liouville sequence in \(\mathbb{Q}_p\) as follows.

Definition 5. Let \((a_n)\) be a sequence of \(p\)-adic integers. If, for every \((c_n)\) of positive integers, the sum

\[ \sum_{n=1}^{\infty} a_n c_n \]

is a \(p\)-adic Liouville number, then the sequence \((a_n)\) is called a \(p\)-adic Liouville sequence.

Example 6. Let \(p\) be a prime number. It is easy to see that

\[ (a_n) = (p^n) \subset \mathbb{Z}_p \]

is a \(p\)-adic Liouville sequence.
Proof. Let \((c_n) \subset \mathbb{Z}^+\) be an arbitrary sequence. We want to show that the sum
\[
\gamma = \sum_{n=1}^{\infty} c_n p^n
\]  
(18)
is a \(p\)-adic Liouville number. Since \(|c_n p^n|_p \leq |p^n|_p = p^{-n} \to 0\), the series \(\sum_{n=1}^{\infty} c_n p^n\) is convergent. We can write
\[
0 < \left| y - S_{n-1} \right|_p^{1/n} = \left| \sum_{i=n}^{\infty} c_{n+i} p^{i+1} \right|_p^{1/n} \leq \left| \sum_{i=0}^{\infty} c_{n+i} p^{i+1} \right|_p^{1/n} \leq \left| c_n p^n \right|_p^{1/n},
\]  
(19)
where \(S_n = \sum_{k=1}^{n} c_k p^k\). Since \(c_n\) is an integer, then we get
\[
0 < \left| y - S_{n-1} \right|_p^{1/n} \leq \left| p^n \right|_p^{1/n} = p^{-(n-1)!}.
\]  
(20)
Thus,
\[
\lim_{n \to \infty} \left| y - S_{n-1} \right|_p^{1/n} = \lim_{n \to \infty} p^{-(n-1)!} = 0.
\]  
(21)
This shows that the sum
\[
\gamma = \sum_{n=1}^{\infty} c_n p^n
\]  
(22)
is a \(p\)-adic Liouville number. \(\square\)

**Theorem 7.** Let \((a_n)\) be a sequence of positive integers satisfying the following conditions:
\[
y_p \left( a_n \right) < y_p \left( a_{n+1} \right),
\]  
(23)
for every \(n\), and
\[
y_p \left( a_{n+1} \right) \geq n^{1+\varepsilon}
\]  
(24)
for fixed \(\varepsilon > 0\) and \(n > n_0(\varepsilon)\). Then, \((a_n)\) is a \(p\)-adic Liouville sequence.

Proof. Let \((c_n)\) be an arbitrary sequence of positive integers and let \(\varepsilon > 0\) be a given arbitrary positive real number. First, we have to prove that the series \(\sum_{n=1}^{\infty} a_n c_n\) is convergent. By condition (24), we know that
\[
y_p \left( a_{n+1} \right) \geq n^{1+\varepsilon},
\]  
(25)
for all \(n > n_0(\varepsilon)\). It follows from \(|c_n|_p \leq 1\) that the relation holds
\[
|a_{n+1} c_{n+1}|_p \leq |a_{n+1}|_p p^{-y_p(a_{n+1})} \leq p^{-n^{1+\varepsilon}},
\]  
(26)
for all \(n > n_0(\varepsilon)\). Thus, \(\lim_{n \to 0} a_n c_n = 0\), so the series \(\sum_{n=1}^{\infty} c_n a_n\) is convergent. By the property \(\left| \sum_{n=1}^{\infty} c_n a_n \right|_p \leq \max_{n \in \mathbb{N}} \left| c_n a_n \right|_p\), we obtain that \(\alpha = \sum_{n=1}^{\infty} c_n a_n \in \mathbb{Z}_p\). Also, by condition (23), \(\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}\). Now, we want to show that the sum
\[
\alpha = \sum_{n=1}^{\infty} a_n c_n
\]  
(27)
is a \(p\)-adic Liouville number. Using (23) and (24), we have
\[
0 < |\alpha - S_n|_p^{1/n} = \left| \sum_{i=1}^{n} a_{i+1} c_{i+1} \right|_p^{1/n} \leq \left| \sum_{i=0}^{n} a_{i+1} c_{i+1} \right|_p^{1/n} = |a_{n+1} c_{n+1}|_p^{1/n},
\]  
(28)
where \(S_n = \sum_{k=1}^{n} a_k c_k\). Since \(|c_n|_p \leq 1\), for all \(i = 1, 2, \ldots\), then
\[
0 < |\alpha - S_n|_p^{1/n} = \sum_{i=1}^{n} a_{i+1} c_{i+1} \leq p^{-y_p(a_{n+1})} \leq p^{-n^{1+\varepsilon}},
\]  
(29)
Since \(S_n \in \mathbb{N}\), for all \(n\), this shows that \(\alpha\) is a \(p\)-adic Liouville number and the theorem is proved. \(\square\)

**Remark 8.** Since \(y_p(a_n) \in \mathbb{N}\), for all \(a_n \in \mathbb{Z}_p\), in Theorem 7, condition (24) can be replaced by the condition
\[
y_p \left( a_{n+1} \right) \geq n^2.
\]  
(30)
In similar way, we can give the following result.

**Corollary 9.** Let \((a_n)\) be a sequence of \(p\)-adic integers satisfying the following conditions:
\[
y_p \left( a_n \right) < y_p \left( a_{n+1} \right),
\]  
(31)
for every \(n\), and
\[
y_p \left( a_{n+1} \right) \geq n^2
\]  
(32)
for \(n > n_0\). Then, \((a_n)\) is a \(p\)-adic Liouville sequence.

**Theorem 10.** Let \((a_n)\) be a sequence of positive integers and assume that the relation
\[
0 < |a_n|_p^{1/n} < \varepsilon
\]  
(33)
holds for every positive real number \(\varepsilon > 0\) and \(n > n_0(\varepsilon)\). Then,
\begin{enumerate}
\item \(\alpha = \sum_{n=1}^{\infty} a_n\) is a \(p\)-adic Liouville number,
\item \((a_n)\) is a \(p\)-adic Liouville sequence.
\end{enumerate}
Proof. Let \( \varepsilon > 0 \) be a given arbitrary positive real number.

(a) By condition (33), there exists \( n_0(\varepsilon) \in \mathbb{N} \) such that the relation

\[
0 < |a_n|_{p}^{1/n} < \varepsilon
\]  

(34)

holds, for all \( n > n_0(\varepsilon) \). Then, we get \( |a_n|_{p} < \varepsilon^n \) and \( |a_n|_{p} \to 0 \) as \( n \to \infty \). Hence, the series \( \sum_{n=1}^{\infty} a_n \) is convergent and by the inequality

\[
\left| \sum_{n=1}^{\infty} a_n p \right|_{p} \leq \max_{n \in \mathbb{N}}|a_n|_{p} \leq 1, 
\]

(35)

we have \( \alpha = \sum_{n=1}^{\infty} a_n \in \mathbb{Z}_p \). Now, we show that \( \alpha = \sum_{n=1}^{\infty} a_n \) is a \( p \)-adic Liouville number. Let \( \alpha_n = \sum_{k=1}^{n} a_k \). Then, we can write

\[
0 < |\alpha - \alpha_n|_{p}^{1/n} = \left| \sum_{k=n+1}^{\infty} a_k \right|_{p}^{1/n} \leq \max_{k \geq n+1}|a_k|_{p}^{1/n} < \varepsilon, 
\]

(36)

for all \( n > n_0(\varepsilon) \). It follows that

\[
|\alpha - \alpha_n|_{p}^{1/n} \to 0. 
\]

(37)

Hence, we obtain that \( \alpha = \sum_{n=1}^{\infty} a_n \) is a \( p \)-adic Liouville number.

(b) Let \( (c_n) \) be an arbitrary sequence of positive integers. We consider the sum:

\[
\beta = \sum_{n=1}^{\infty} c_n a_n. 
\]

(38)

We know that the relation

\[
0 < |a_n|_{p}^{1/n} < \varepsilon
\]  

(39)

holds, for all \( n > n_0(\varepsilon) \). Since \( |c_n|_{p} \leq 1 \), we get

\[
|c_n a_n|_{p} \leq |a_n|_{p} < \varepsilon^n 
\]

(40)

and \( |a_n c_n|_{p} \to 0 \) as \( n \to \infty \). Hence, the series \( \sum_{n=1}^{\infty} a_n c_n \) is convergent and by the inequality

\[
\left| \sum_{n=1}^{\infty} a_n c_n \right|_{p} \leq \max_{n \in \mathbb{N}}|a_n c_n|_{p} \leq 1, 
\]

(41)

we have \( \beta = \sum_{n=1}^{\infty} a_n c_n \in \mathbb{Z}_p \). Let \( \beta_n = \sum_{k=1}^{n} c_k a_k \). Then, we can write

\[
0 < |\beta - \beta_n|_{p}^{1/n} = \left| \sum_{k=n+1}^{\infty} c_k a_k \right|_{p}^{1/n} \leq \max_{k \geq n+1}|c_k a_k|_{p}^{1/n} \leq \max_{k \geq n+1}|a_k|_{p}^{1/n} < \varepsilon 
\]

(42)

for all \( n > n_0(\varepsilon) \). It follows that

\[
|\beta - \beta_n|_{p}^{1/n} \to 0. 
\]

(43)

Since \( \beta_n \in \mathbb{N} \), for all \( n \), we obtain that \( \beta = \sum_{n=1}^{\infty} c_n a_n \) is a \( p \)-adic Liouville number. So, the theorem is proved. \( \square \)

4. The Liouville Sequences in the Functions Field

Let \( K \) be an arbitrary field, \( x \) an indeterminate, \( K[x] \) the ring of all polynomials in \( x \) with coefficients in \( K \), \( K(x) \) the field of all rational functions in \( x \) with coefficients in \( K \), and \( K(x) \) the field of all formal series

\[
z = a_k x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \cdots
\]  

(44)

in \( x \), where the coefficients \( a_k, a_{k-1}, a_{k-2}, \ldots \) are in \( K \). Thus, \( K(x) \) is the quotient field of \( K[x] \) and a subfield of \( K(x) \).

A valuation \(|z|\) in \( K(x) \) is now defined by putting \(|0| = 0\) and \(|z| = \varepsilon^k\) if

\[
z = a_k x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \cdots
\]

(45)

and \( a_k \neq 0 \).

If \( z \) lies in \( K[x] \), then \( \log |z| = \deg z \).

It is clear that this norm is a non-Archimedean and so \( K(x) \) is a non-Archimedean field with this norm.

The analogue of Liouville's theorem states that if \( \alpha \in K(x) \) is an algebraic number of degree \( n \geq 2 \) over \( K(x) \), then there exists a positive constant \( C(\alpha) \) depending only on \( \alpha \) such that

\[
|\alpha - a/b| \geq C(\xi) \frac{1}{|b|^n}
\]

(46)

for all \( a, b \in K[x] \) \( (b \neq 0) \) (see [22]). Some results on the Liouville numbers in the functions field were obtained in [20]. Now, we recall the definition of a Liouville number in this field.

Definition 11. An element \( \xi \in K(x) \) is called a Liouville number if, for every \( \omega \in \mathbb{R}^+ \), there existed integer \( a, b \in K[x] \setminus \{0\} \) with \(|b| > 1\) such that

\[
0 < |\xi - a/b| < \frac{1}{|b|^\omega}. 
\]

(47)

We define the concept of a Liouville sequence in the function fields as follows.

Definition 12. Let \( (z_n) \subset K(x) \). If, for every \( (a_n) \in K[x] \), the sum

\[
\sum_{n=1}^{\infty} \frac{1}{a_n z_n}
\]

(48)

is a Liouville number, then the sequence \( (z_n) \) is called a Liouville sequence.

Theorem 13. Let \( (z_n) \subset K(x) \) satisfying the following conditions:

\[
0 < \deg(z_n) < \deg(z_{n+1})
\]

(49)

for every \( n \), and

\[
\deg(z_{n+1}) > n^{1+\varepsilon},
\]

(50)

for fixed \( \varepsilon > 0 \) and \( n > n_0(\varepsilon) \). Then, \( (z_n) \) is a Liouville sequence.
Proof. Let \((a_n) \subset K[x]\) be an arbitrary sequence and \(\epsilon > 0\) be an arbitrary positive real number. First, we show that \(\sum_{n=1}^{\infty} (1/a_n z_n)\) is convergent in \(K(x)\). From condition (50), we have

\[
\left| \frac{1}{a_n z_n} \right| = e^{-\deg(a_n z_n)} \leq e^{-\deg(z_n)} < e^{-(n-1)^{1/\epsilon}} \tag{51}
\]

for all \(n > n_\epsilon(\epsilon)\). Then, we get \(1/a_n z_n \to 0\). Thus, the series \(\sum_{n=1}^{\infty} (1/a_n z_n)\) is convergent. Now, we want to show that the sum

\[
\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n z_n} \tag{52}
\]

is a Liouville number. Let \(S_n = \sum_{k=1}^{n} (1/a_k z_k)\). Then, we write

\[
0 < |\alpha - S_n|^{1/n} = \left| \sum_{k=1}^{\infty} \frac{1}{a_k z_k} \right|^{1/n} \leq \left[ \max_{k>n+1} \frac{1}{a_k z_k} \right]^{1/n}. \tag{53}
\]

Since \(\deg(a_k) \geq 0\), we obtain that

\[
\left[ \max_{k>n+1} \frac{1}{a_k z_k} \right]^{1/n} = \left[ \max_{k>n+1} e^{-\deg(a_k) - \deg(z_k)} \right]^{1/n} \leq \left[ \max_{k>n+1} e^{-\deg(z_k)} \right]^{1/n}. \tag{54}
\]

By (49), we can write

\[
\max_{k>n+1} e^{-\deg(z_k)} = e^{-\deg(z_{n+1})}, \tag{55}
\]

and by using (50) we get

\[
\left[ \max_{k>n+1} \frac{1}{a_k z_k} \right]^{1/n} \leq \left[ \max_{k>n+1} e^{-\deg(z_k)} \right]^{1/n} = e^{-\deg(z_{n+1})/n} \leq e^{-n^{1/\epsilon}/n} \tag{56}
\]

\[
e^{-n} \to 0 \quad (n \to \infty). \]

This shows that \(|\alpha - S_n|^{1/n} \to 0, n \to \infty\). Also, by condition (49), \(z_n \in K[x]\) and so \(S_n \in K(x)\). Thus, we prove that \(\alpha\) is a Liouville number.

\[\square\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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