Research Article

Summation Formulas Obtained by Means of the Generalized Chain Rule for Fractional Derivatives

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In 1970, several interesting new summation formulas were obtained by using a generalized chain rule for fractional derivatives. The main object of this paper is to obtain a presumably new general formula. Many special cases involving special functions of mathematical physics such as the generalized hypergeometric functions, the Appell $F_1$ function, and the Lauricella functions of several variables $F^{(n)}_D$ are given.

1. Introduction

The fractional derivative of arbitrary order $\alpha$, $\alpha \in \mathbb{C}$, is an extension of the familiar $n$th derivative $D^n g(z) F(\zeta) = \frac{d^n F(\zeta)}{d g(z)^n}$ of the function $F(\zeta)$ with respect to $g(z)$ to nonintegral values of $n$ and is denoted by $D^\alpha g(z) F(\zeta)$. The aim of this concept is to generalize classical results of the $n$th order derivative to fractional order. Most of the properties of the classical calculus have been expanded to fractional calculus, for instance, the composition rule [1], the Leibniz rule [2, 3], the chain rule [4], and Taylor’s and Laurent’s series [5–7]. Fractional calculus also provides tools that make it easier to deal with special functions of mathematical physics [8].

The most familiar representation for fractional derivative of order $\alpha$ of $f(z)$ is the Riemann-Liouville integral [9]; that is,

$$D^\alpha_z f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi) (\xi-z)^{-\alpha-1} d\xi,$$

(1)

which is valid for $\Re(\alpha) < 0$ and where the integration is done along a straight line from $0$ to $z$ in the $\xi$-plane. By integrating by part $m$ times, we obtain

$$D^m_z f(z) = \frac{d^m}{dz^m} D^{\alpha-m}_z f(z).$$

(2)

This allows modifying the restriction $\Re(\alpha) < 0$ to $\Re(\alpha) < m$ [10].

In 1970, Osler [2] introduced a more general definition of the fractional derivative of a function $f(z)$ with respect to another function $g(z)$ based on Cauchy’s integral formula.

Definition 1. Let $f(z)$ be analytic in the simply connected region $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior or boundary point of $\mathcal{R}$. Assume also that $\oint g(f(z)) dz = 0$ for any simple closed contour in $\mathcal{R} \cup \{0\}$ through $g^{-1}(0)$. Then if $\alpha$ is not a negative integer and $z$ is in $\mathcal{R}$, the fractional derivative of order $\alpha$ of $f(z)$ with respect to $g(z)$ is defined by

$$D^\alpha_{g(z)} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{g^{-1}(0)} \frac{f(\xi) g'(\xi)}{(g(\xi) - g(z))^{\alpha+1}} d\xi.$$

(3)

For nonintegral $\alpha$, the integrand has a branch line which begins at $\xi = z$ and passes through $\xi = g^{-1}(0)$. The notation on this integral implies that the contour of integration starts at $g^{-1}(0)$, encloses $z$ once in the positive sense, and returns to $g^{-1}(0)$ without cutting the branch line.

With the use of that representation based on the Cauchy integral formula for the fractional derivatives, Osler gave a generalization of the following result [11, page 19] involving...
the derivative of order \( N \) of the composite function \( f(z) = F(h(z)) \):

\[
D^N_z f(z) = \sum_{n=0}^{N} \frac{U_n(z) D^n_{h(z)} f(z)}{n!},
\]

where

\[
U_n(z) = \sum_{\nu=0}^{n} \frac{(-h(z))^\nu}{\nu! n!}.
\]

In particular, he found the following formula [4]:

\[
D^\alpha_{g(z)} f(z) = \sum_{n=-\infty}^{\infty} \left( \begin{array}{c} \alpha + n \\ y + n \end{array} \right) \frac{D^m_{h(z)} f(z)}{F(w, z)} \cdot D^{\alpha - y - n}_{h(z)} \left\{ \frac{F(w, z) g' (z)}{h'(z)} \left( \frac{h(w) - h(z)}{g(w) - g(z)} \right)^{\alpha + 1} \right\}_{w=z},
\]

where the notation \( D^\alpha_{g(z)} f(z) \) means the fractional derivative of order \( \alpha \) of \( f(z) \) with respect to \( g(z) \). Osler proved the generalized chain rule by applying the generalized Leibniz rule [2] for fractional derivatives to an important fundamental relation involving fractional derivatives discovered also by Osler [4, page 290, Theorem 2]. The fundamental relation which is the central point of this paper is given by the next theorem.

**Theorem 2.** Let \( f(g^{-1}(z)) \) and \( f(h^{-1}(z)) \) be defined and analytic on the simply connected region \( \mathcal{R} \) and let the origin be an interior or boundary point of \( \mathcal{R} \). Suppose also that \( g^{-1}(z) \) and \( h^{-1}(z) \) are regular univalent functions on \( \mathcal{R} \) and that \( h^{-1}(0) = g^{-1}(0) \). Let \( f(g^{-1}(z))dz \) vanish over simple closed contour in \( \mathcal{R} \cup \{0\} \) through the origin. Then the following relation holds true:

\[
D^\alpha_{g(z)} f(z) = D^\alpha_{h(z)} \left\{ \frac{f(z) g'(z)}{h'(z)} \left( \frac{h(w) - h(z)}{g(w) - g(z)} \right)^{\alpha + 1} \right\}_{w=z}.
\]

This fundamental relation is very useful to obtain very easily known and new summation formulas involving special functions of mathematical physics. For example, set \( f(z) = z^{p-2}, g(z) = z^2, \) and \( h(z) = z \) in (7). One sees easily that \( g^{-1}(0) = h^{-1}(0) \). Thus, one has

\[
D^\alpha_z z^{p-2} = D^\alpha_z 2z^{p-1} (z + w)^{-\alpha - 1} \bigg|_{w=z^{-1}}.
\]

The left-hand side is evaluated by using the well-known formula [12, page 83, Equation (2.4)]

\[
D^\alpha_z z^q = \Gamma(1 + q) \Gamma(1 + q - \alpha)^{-1} z^{q - \alpha} \quad (\Re q > -1)
\]

after replacing \( z \) by \( z^2 \). Expanding \( (w+z)^{\alpha - 1} \) in power series, using (9), and replacing \( w \) by \( z \) after operation, one obtains Kummer’s summation formula

\[
_{2}F_{1} \left[ \begin{array}{c} \alpha + 1, \ p; \\ p - \alpha \end{array} \right| -1 \right] = \frac{\Gamma(p/2) \Gamma(p - \alpha)}{2 \Gamma(p) \Gamma(p/2 - \alpha)},
\]

where

\[
_{2}F_{1} \left[ \begin{array}{c} a, \ b; \\ c; \ z \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}
\]

denotes the Gauss hypergeometric function [13] and \((\lambda)_n\) holds for the Pochhammer symbol defined, in terms of the Gamma function, by

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} \lambda (\lambda + 1) \cdots (\lambda + n - 1) \quad (n \in \mathbb{N}; \lambda \in \mathbb{C}) \\ 1 \quad (n = 0; \lambda \in \mathbb{C} \setminus \{0\}) \end{cases}
\]

In this paper, we present several new summation formulas involving special functions of mathematical physics obtained by using the fundamental relation (7). In Section 2, we introduce the Pochhammer based representation for fractional derivatives and we recall a well-poised fractional calculus operator given by Tremblay [14]. This well-poised operator will be used, throughout this paper, in order to ease the computations of fractional derivatives. Finally, Section 3 is devoted to the presentation of the main results. Many presumably new summation formulas are also given as special cases.

### 2. Pochhammer Contour Integral Representation for Fractional Derivative and the Well-Poised Fractional Calculus Operator \( g(z) O^\alpha_{\beta} \)

The less restrictive representation of fractional derivative according to parameters is Pochhammer’s contour definition introduced in [14, 15].

**Definition 3.** Let \( f(z) \) be analytic in a simply connected region \( \mathcal{R} \). Let \( g(z) \) be regular and univalent on \( \mathcal{R} \) and let \( g^{-1}(0) \) be an interior point of \( \mathcal{R} \) then if \( \alpha \) is a nonnegative integer, \( p \) is not an integer, and \( z \) is in \( \mathcal{R} - \{g^{-1}(0)\} \), one defines the fractional derivative of order \( \alpha \) of \( g(z)^p f(z) \) with respect to \( g(z) \) by

\[
D^\alpha_{g(z)} g(z)^p f(z) = e^{zp \pi} \Gamma(1 + p) \quad 4\pi \sin(p) \times \int_{C_{z}, \cdot \cdot} \left( \frac{f(\xi) g(\xi)^p g'(\xi)}{g(\xi) - g(z)} \right)^{\alpha + 1} d\xi.
\]
The operator \( g(\xi)O^\alpha_\beta \) is defined in terms of the fractional calculus operator \( D^\alpha_{g(\xi)} \) as

\[
g(\xi)O^\alpha_\beta \equiv \frac{\Gamma(\beta)}{\Gamma(\alpha)} g(z)^{1-\beta} D^\alpha_{g(\xi)} g(z)^{\alpha-1}
\]  
with \( \beta \neq 0, -1, -2, \ldots \)

This operator has been used very recently in [3] to prove a new generalized Leibniz rule for fractional derivatives as well as in [17] to obtain some new series involving special functions.

This operator has many very useful properties. We chose to give only one of them which will be used in the proofs of the main results; that is,

\[
g(\xi)O^\alpha_\beta g(z)^{\lambda} f(z) = \frac{\Gamma(\beta) \Gamma(\alpha + \lambda)}{\Gamma(\beta + \lambda) \Gamma(\alpha)} g(z)^{\lambda} g(z)^{\alpha+\lambda} f(z).
\]  
In terms of the fractional calculus operator \( g(\xi)O^\alpha_\beta \), the modified fundamental relation (7) holds the following form:

\[
g(\xi)O^\alpha_\beta f(z) = \left( \frac{g(z)}{h(z)} \right)^{1-\beta} h(z)O^\alpha_\beta f(z)
\]

\[
\times \left\{ f(z) \left( \frac{g(z)}{h(z)} \right)^{\alpha-1} \frac{g'(z)}{h'(z)} \left( \frac{g(z) - g(z)}{h(z) - h(z)} \right)^{\beta-\alpha-1} \right\} \bigg|_{z=a}^{z=b}
\]

with \( \beta \neq 0, -1, -2, \ldots \)

It is worthy to mention that the operator \( g(\xi)O^\alpha_\beta \) has a lot more interesting properties and applications.

### 3. Main Results and Special Cases

In this section, we present a new general formula related to the generalized chain rule. We give many special cases involving special functions such as the first Appell function \( F_1 \), the Lauricella function of several variables \( L^{(n)}_D \), and the generalized hypergeometric functions. These functions are evaluated most of the time at arguments related to the roots of unity.

**Main Formula.** Consider

\[
z^\alpha O^\alpha_\beta f(z) = \frac{p \Gamma(\beta) \Gamma(pa/q)}{q \Gamma(\alpha) \Gamma(p - q - \alpha/q)}
\]

\[
\times z^\alpha O^\alpha_\beta f(z) \left\{ f(z) \sum_{s=1}^{p-1} \left( 1 - \frac{z}{w} e^{-2\pi is/p} \right)^{\beta-\alpha-1} \right\} \bigg|_{z=a}^{z=b}
\]

with \( \beta \neq 0, -1, -2, \ldots \) and \( \beta - (p - q)\alpha/q \neq 0, -1, -2, \ldots \)
Proof. Let \( g(z) = z^p \) and let \( h(z) = z^q \) with \( p \) and \( q \) two positive integers in (16). We have

\[
z^pO^\alpha_\beta f(z) = (z^{p-q})^\beta \Gamma(\beta) \Gamma(\alpha/2) \Gamma(\beta - \alpha/2) / \Gamma(\alpha) \Gamma(\beta) = \sum_{n=0}^{\infty} \frac{(\beta - \alpha - 1)^n}{n!z^{2n}} z^{\beta - \alpha/2} (z^4)^{n/2}
\]

(21)

The right-hand side of (21) can be split in two parts

\[
\sum_{n=0}^{\infty} \frac{(\beta - \alpha - 1)^n}{n!} \Gamma(\beta - \alpha/2) \Gamma((\alpha + n)/2) / \Gamma(\alpha/2) \Gamma(\beta - (\alpha - n)/2).
\]

(22)

Expanding \((1 + z^2/w^2)^{1+\beta} - \beta - \alpha - \beta \) in power series, (20) becomes

\[
2\Gamma(\alpha) \Gamma(\beta - \alpha/2) / \Gamma(\beta) \Gamma(\alpha/2) = \sum_{n=0}^{\infty} \frac{(\beta - \alpha - 1)^n}{n!z^{2n}} z^{\beta - \alpha/2} (z^4)^{n/2}
\]

(23)

Converting the terms involving Gamma function in the last expression into Pochhammer’s symbol and making some simplifications, we find

\[
\sum_{n=0}^{\infty} \frac{(\beta - \alpha - 1)^n}{n!} \Gamma(\beta - \alpha/2) \Gamma((\alpha + n)/2) / \Gamma(\alpha/2) \Gamma(\beta - (\alpha - n)/2) = \sum_{n=0}^{\infty} \frac{(\beta - \alpha - 1)^n}{(2n)!} \Gamma(\beta - \alpha/2) \Gamma((\alpha + 1)/2 + n) / \Gamma(\alpha/2) \Gamma(\beta - (\alpha - 1)/2 + n).
\]

(24)

Finally, rewriting the right-hand side of (23) in terms of generalized hypergeometric function and combining with (21), we obtain the following summation formula:

\[
2\Gamma(\alpha) \Gamma(\beta - \alpha/2) / \Gamma(\beta) \Gamma(\alpha/2) = \sum_{n=0}^{\infty} \frac{(\beta - \alpha - 1)^n}{n!} \Gamma(\beta - \alpha/2) \Gamma((\alpha + n)/2) / \Gamma(\alpha/2) \Gamma(\beta - (\alpha - n)/2).
\]

(25)
Consider $z_0 O^a_p f(z)$.

**Example 8.** Setting $f(z) = 1$ and $p = 3$ in (25) gives

$$
1 = 3z^{-3} \pi O^a_p \times \left( \frac{e^{2\pi i/3}}{w} \right)^{\beta - \alpha - 1}(1 - \frac{z}{w} e^{-2\pi i/3})^{\beta - \alpha - 1} \left\{ \left. f(z) \right| \frac{z}{w} e^{-2\pi i/3} \right\}_{w = z}.
$$

Observe that

$$
z_0 O^a_p (1 - yz)^{-\nu}(1 - \lambda z)^{-\nu} = \sum_{i=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j}}{i! j!} (yz)^j (\lambda z)^i
$$

denotes the first Appell function $F_1(a, \mu; \nu; b, y, z, \lambda z)$.

Furthermore, with the help of the following reduction formulas for the Appell $F_1$ functions given by Nagel [21],

$$
F_1(a, b, b; y; x, y) = \left( \frac{xy}{(x-1)(y-1)} \right)^{\beta - \alpha - 1} \times \sum_{n=0}^{\infty} \frac{(y-a)_n (b)_n}{(y)_n n!} \left( \frac{1}{(x-1)(y-1)} \right)^n
$$

in conjunction with (28) give, respectively, after simplifications the two summation formulas

$$
\sum_{n=0}^{\infty} \frac{(\beta - \alpha)_n (1 + \alpha - \beta)_n}{(\beta + 2\alpha)_n n!} \left( \frac{2}{3} \right)^n
$$

Example 9. Let $f(z) = 1$ and let $p$ be an integer in (25). We have

$$
1 = p(z^{p-1})^{-\alpha} z_0 O^a_p \times \left( \frac{e^{2\pi i/p}}{w} \right)^{\beta - \alpha - 1}(1 - \frac{z}{w} e^{-2\pi i/p})^{\beta - \alpha - 1} \left\{ \left. f(z) \right| \frac{z}{w} e^{-2\pi i/p} \right\}_{w = z}.
$$
Using (27) and making some elementary simplifications, the last formula reduces to

\[
\frac{3\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta+\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha)_{3m}}{(2\alpha+\beta)_{3m}n!} \times F_1\left(3\alpha + 3n, 1 + \alpha - \beta, 1 + \alpha - \beta; \beta + 2\alpha + 3n; e^{-2ni/3}, e^{-4ni/3}\right).
\]

Combining (38) and (40) and putting \(z = 1\) provide the following presumably new summation formula:

\[
\sum_{n=0}^{\infty} \frac{(3\alpha)_n}{(2\alpha+\beta)_{3n}n!} \times F_1\left(3\alpha + 3n, 1 + \alpha - \beta, 1 + \alpha - \beta; \beta + 2\alpha + 3n; e^{-2ni/3}, e^{-4ni/3}\right).
\]

Cases with \(p = 1\). Consider

\[
q\Gamma(\alpha)\Gamma(\beta - \alpha + \alpha/q)\Gamma(\alpha/q)z^\alpha O_\beta^\alpha f(z)
\]

\[
= z^\alpha O_\beta^\alpha f(z)
\]

\[
= \Gamma(\alpha)\Gamma(\beta + (p-1)\alpha) \frac{p\Gamma(\beta)}{\Gamma(\alpha)}\Gamma(p\alpha) \times \prod_{s=1}^{p-1} \frac{(1 - z e^{-2\pi is/p})^{\beta-\alpha-1}}{w^\alpha}.
\]

Example II. Let \(f(z) = 1\) and \(q = 3\) in (42). We obtain

\[
\frac{3\Gamma(\alpha)\Gamma(\beta - 2\alpha/3)}{\Gamma(\alpha)\Gamma(\beta/3)} \times \prod_{s=1}^{3\alpha/3} \frac{(1 - z e^{-2\pi is/3})^{1+\alpha-\beta}}{w^{1+\alpha-\beta}}
\]

Expanding in power series, we find the right-hand side of (43)
\[= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-2\pi ik/3} \Gamma(\beta) e^{-4\pi i j/3}}{k!} \frac{\Gamma(\beta - 2\alpha/3) \Gamma((\alpha + k + j)/3)}{\Gamma(\beta - (2\alpha - k - j)/3) \Gamma(\alpha/3)} \]

We, thus, get the following (presumably) new summation formula:

\[= \sum_{k=0}^{\infty} \frac{e^{-2\pi ik/3}}{k!} \frac{\Gamma(\beta - (2\alpha - k)/3) \Gamma((\alpha + k)/3)}{\Gamma(\beta - (2\alpha - k)/3) \Gamma(\alpha/3)} \times \left[\begin{array}{c} -k, \\ \beta - \alpha - 1; \\ 2 + \alpha - \beta - k; \\ e^{-2\pi i j/3} \end{array}\right]. \]

(44)

We, thus, get the following (presumably) new summation formula:

\[= \sum_{k=0}^{\infty} \frac{e^{-2\pi ik/3} \Gamma(\alpha + k/3)}{k!} \frac{\Gamma(\beta - (2\alpha - k)/3)}{\Gamma(\beta - (2\alpha - k)/3) \Gamma(\alpha/3)} \times \left[\begin{array}{c} -k, \\ \beta - \alpha - 1; \\ 2 + \alpha - \beta - k; \\ e^{-2\pi i j/3} \end{array}\right]. \]

(45)

Remark 12. The cases where \(\beta = \alpha - n\) with \(n = 0, 1, 2, \ldots\) must be treated very carefully as \(\lim_{\beta \to \alpha - n^{-}}\).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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