Research Article

The Concept of q -Cycle and Applications

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The concept of q -cycle is investigated for its properties and applications. Connections with irreducible polynomials over a finite field are established with emphases on the notions of order and degree. The results are applied to deduce new results about primitive and self-reciprocal polynomials.

1. Introduction

Let $\mathbb{F}_q$ denote the finite field of q elements and let $N \in \mathbb{N}$, $\gcd(q, N) = 1$. Let $a_0, a_1, \dots, a_{\ell-1}$ be ℓ distinct numbers chosen from $\mathbb{Z}_N := \{0, 1, \dots, N-1\}$. If

$$\begin{aligned} a_i q &\equiv a_{i+1} \pmod{N} \quad (i = 0, 1, \dots, \ell-2), \\ (a_0 q^\ell) &\equiv a_{\ell-1} q \equiv a_0 \pmod{N}, \end{aligned} \quad (1)$$

then we say that $(a_0, a_1, \dots, a_{\ell-1})$ forms a q -cycle mod N with leading element a_0 , abbreviated by $q(a_0)$ -cycle(N) or q -cycle(N) when the leading element in the cycle is immaterial, and call ℓ the length of this $q(a_0)$ -cycle(N). The notion of q -cycles was introduced by Wan in his book [1, page 203]. Since $q^{o_N(q)} \equiv 1 \pmod{N}$, where $o_N(q)$ is the order of q in $\mathbb{Z}_N^* := \mathbb{Z}_N \setminus \{0\}$ (the multiplicative group of nonzero integers modulo N), it clearly follows that each $q(a)$ -cycle(N) always has a unique length ℓ which is the least positive integer ℓ for which $a \equiv a q^\ell \pmod{N}$. Observe that q -cycles are nothing but $\mathbb{Z}$ -orbits $q^{\mathbb{Z}} a$, $a \in \mathbb{Z}_N$; that is, $1 \in \mathbb{Z}$ acts on $\mathbb{Z}_N$ by multiplication with q . The concept of q -cycles is important because of the following connections with irreducible polynomials.

(A) (see [1, Theorem 9.11]) Let α be a primitive N th root of unity (if the order of q in $\mathbb{Z}_N^*$ is m , then there exists a primitive

N th root of unity in $\mathbb{F}_{q^m}$). If $(a_0, a_1, \dots, a_{\ell-1})$ is a q -cycle(N), then

$$h(x) = (x - \alpha^{a_0})(x - \alpha^{a_1}) \cdots (x - \alpha^{a_{\ell-1}}) \quad (2)$$

is a monic irreducible factor of $x^N - 1$ in $\mathbb{F}_q[x]$. Conversely, if $h(x)$ is a monic irreducible factor of $x^N - 1$ in $\mathbb{F}_q[x]$, then all the roots of $h(x)$ are powers of α whose exponents form a q -cycle(N). We henceforth refer to these two facts as the α -correspondence.

(B) (see [1, Corollary 9.12]) The number of distinct irreducible polynomials dividing $x^N - 1$ in $\mathbb{F}_q[x]$ is equal to the number of q -cycles(N) formed with the leading N elements taken from $\{0, 1, \dots, N-1\}$.

Our objectives here are to illustrate the versatility of q -cycles by using them to prove new results about irreducible polynomials. General properties of q -cycles are given in the next section; specific details in two special cases corresponding to $N = q^n - 1$, $q^m + 1$ are worked out for applications to primitive and self-reciprocal polynomials in the last section. Section 3 deals with results about the order of a polynomial, while Section 4 does the same for the degree of a polynomial. Section 5 shows that knowing a q -cycle is equivalent to knowing all coefficients of the corresponding

polynomial (2). The last section provides applications of q -cycles to primitive and self-reciprocal polynomials.

Notation and Terminology. Throughout, we fix the following symbols and their meanings.

- (i) p is a fixed prime, q is a power of p , and $\mathbb{F}_q$ is the finite field with q elements.
- (ii) N is a fixed positive integer such that $\gcd(q, N) = 1$.
- (iii) $d := d(N)$ is the number of (positive integer) divisors of N .
- (iv) $N = N_1 > N_2 > \dots > N_d = 1$ (all divisors of N).
- (v) For $n \in \mathbb{N}$, the Möbius function μ is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is a squarefree product of } \\ & k \text{ distinct primes,} \\ 0, & \text{if } n \text{ is not squarefree.} \end{cases} \quad (3)$$

- (vi) For $n \in \mathbb{N}$, the Euler's function $\varphi(n)$ is the number of integers $k \in \{1, 2, \dots, n\}$ with $\gcd(k, n) = 1$.
- (vii) For $M \in \mathbb{N}$, denote by $o_M(q)$ the order of q in $\mathbb{Z}_M^* := \mathbb{Z}_M \setminus \{0\}$; that is,

$$q^{o_M(q)} \equiv 1 \pmod{M}, \quad (4)$$

$$\text{but } q^i \not\equiv 1 \pmod{M} \quad \forall i \in \mathbb{N}, i < o_M(q).$$

- (viii) For $r \in \{1, 2, \dots, d\}$, set $\mathcal{C}_{N_r} = \{q\text{-cycles}(N_r)\}$, the set of all q -cycles mod N_r .

2. Properties of q -Cycles

Let $(a_0, a_1, \dots, a_{\ell-1})$ be a $q(a_0)$ -cycle(N). It is easy to see that $\mathbb{Z}_N$ can be decomposed into a finite union of disjoint q -cycles(N), namely,

$$\mathbb{Z}_N = (0) \cup \left(1, q, q^2, \dots, q^{o_N(q)-1}\right) \bigcup_{a \in \mathbb{Z}_N \setminus \{0, 1\}} (a, aq, aq^2, \dots). \quad (5)$$

Since $q^{o_N(q)} \equiv 1 \pmod{N}$ and $aq^{o_N(q)} \equiv a \pmod{N}$ for all $a \in \mathbb{Z}_N$, each q -cycle(N) is of length $\leq o_N(q)$ with the one containing 1 having the largest length. We collect in our first theorem further properties of q -cycles, whose straightforward proofs are omitted.

Theorem 1. Let $a, b \in \mathbb{Z}_N$, $N_0 := \mathbb{N} \cup \{0\}$.

- (i) The element $b \in q(a)$ -cycle(N) if and only if $b \equiv aq^m \pmod{N}$ for some $m \in \mathbb{N}_0$.
- (ii) Two q -cycles, $q(a)$ -cycle(N) and $q(b)$ -cycle(N), are identical if and only if $b \in q(a)$ -cycle(N).
- (iii) For $b \in \mathbb{Z}_N^*$, each $q(b)$ -cycle(N) has length ℓ if and only if $N/\gcd(b, N)$ divides $q^\ell - 1$ but does not divide $q - 1, q^2 - 1, \dots, q^{\ell-1} - 1$ (if $\ell = 1$, only the first divisibility needs to be checked). The $q(0)$ -cycle(N) has length 1.

- (iv) The length of each q -cycle(N) divides $o_N(q)$.

- (v) If $b \in \mathbb{Z}_N$ is such that $\gcd(b, N) = N_r$, then the $q(b)$ -cycle(N) has length $o_{N/N_r}(q)$, and there are altogether $\varphi(N/N_r)/o_{N/N_r}(q)$ distinct such q -cycles. In particular, if $b \in \mathbb{Z}_N$ is such that $\gcd(b, N) = 1$, then the $q(b)$ -cycle(N) has length $o_N(q)$, and there are altogether $\varphi(N)/o_N(q)$ distinct q -cycles with length $o_N(q)$.

- (vi) The length of any q -cycle(N) is of the form $o_{N_r}(q)$ for some $r \in \{1, \dots, d\}$.

- (vii) The number of q -cycles(N) of length ℓ is $(1/\ell) \sum_{1 \leq r \leq d, o_{N_r}(q) = \ell} \varphi(N_r)$.

- (viii) The total number of q -cycles(N) is $|\mathcal{C}_N| = \sum_{1 \leq r \leq d} (\varphi(N_r)/o_{N_r}(q))$ ($|\cdot|$ signifies the number of elements in the set).

2.1. Two Special Cases. There are two particular cases $N = q^n - 1$ and $N = q^m + 1$, which are closely related to primitive and self-reciprocal and will be needed in the last section.

2.1.1. The Case $N = q^n - 1$. When $N = q^n - 1$ ($n \in \mathbb{N}$), more precise information is now derived.

Theorem 2. Let $n \in \mathbb{N}$. Then

- (i) $o_{q^n-1}(q) = n$;
- (ii) n is the largest length among q -cycles($q^n - 1$) and any other length of q -cycles($q^n - 1$) divides n ;
- (iii) the number of q -cycles($q^n - 1$) having length ℓ is equal to

$$\frac{1}{\ell} \sum_{\substack{1 \leq r \leq d \\ o_{N_r}(q) = \ell}} \varphi(N_r) = \begin{cases} q - 1 & \text{if } \ell = 1 \\ \frac{1}{\ell} \sum_{k|\ell} \mu(k) q^{\ell/k} & \text{if } \ell > 1. \end{cases} \quad (6)$$

Proof. Assertions (i) and (ii) are immediate from the definition and Theorems 1(iii)-(iv). To verify (iii), observe that if $\ell = 1$, then it is easily checked that all the q -cycles($q^n - 1$) of length 1 are those starting with 0, $A := q^{n-1} + q^{n-2} + \dots + q + 1, 2A, \dots, (q-2)A$ whose total number is $q - 1$. Assume now that $\ell > 1$ and that the assertion holds up to $\ell - 1$. From Theorem 1(vii), the number of q -cycles($q^n - 1$) having length ℓ is $(1/\ell) \sum_{1 \leq r \leq d, o_{N_r}(q) = \ell} \varphi(N_r)$. Thus, to prove the desired result, we only need to verify that

$$\sum_{\substack{1 \leq r \leq d \\ o_{N_r}(q) = \ell}} \varphi(N_r) = \sum_{k|\ell} \mu(k) q^{\ell/k}. \quad (7)$$

Consider

$$\begin{aligned}
 \sum_{\substack{1 \leq r \leq d \\ o_{N_r}(q) = \ell}} \varphi(N_r) &= \sum_{\substack{N_r | (q^\ell - 1) \\ N_r \nmid (q^s - 1) \forall s | \ell, 1 \leq s < \ell}} \varphi(N_r) \\
 &= \sum_{N_r | (q^\ell - 1)} \varphi(N_r) - \sum_{\substack{N_r | (q^\ell - 1) \\ \forall s | \ell, 1 \leq s < \ell}} \varphi(N_r) \quad (8) \\
 &= q^\ell - 1 - \sum_{\substack{N_r | (q^\ell - 1) \\ \forall s | \ell, 1 \leq s < \ell}} \varphi(N_r).
 \end{aligned}$$

Using the induction hypothesis for the second term on the right-hand side of (8) with an extra -1 to take care of the case $\ell = 1$, we get

$$\begin{aligned}
 \sum_{\substack{N_r | (q^\ell - 1) \\ \forall s | \ell, 1 \leq s < \ell}} \varphi(N_r) &= \sum_{\substack{s | \ell \\ 1 \leq s < \ell}} \left(\sum_{t | s} \mu(t) q^{s/t} \right) - 1 \\
 &= \sum_{\substack{k | \ell \\ 1 \leq k < \ell}} q^k \sum_{\substack{i | k \\ 1 \leq i < k}} \mu(i) - 1 \\
 &= \sum_{\substack{k | \ell \\ 1 \leq k < \ell}} q^k \left(\sum_{i | (\ell/k)} \mu(i) - \mu\left(\frac{\ell}{k}\right) \right) - 1 \quad (9) \\
 &= - \sum_{\substack{k | \ell \\ 1 \leq k < \ell}} q^k \mu\left(\frac{\ell}{k}\right) - 1 \\
 &= - \sum_{\substack{k | \ell \\ 1 \leq k < \ell}} \mu(k) q^{\ell/k} - 1.
 \end{aligned}$$

The result now follows by substituting this into (8). $\square$

Implicit in the proof of Theorem 2 is the following nice identity.

Corollary 3. *With the same notation as in Theorem 2, one has*

$$\sum_{\substack{1 \leq r \leq d \\ o_{N_r}(q) = \ell}} \varphi(N_r) = \sum_{d | \ell} \mu(d) q^{\ell/d} - \left\lfloor \frac{1}{\ell} \right\rfloor. \quad (10)$$

2.1.2. The Case $N = q^m + 1$. When $N = q^m + 1$ ($m \in \mathbb{N}$), more precise information about q -cycles is now derived.

Theorem 4. *Let $m \in \mathbb{N}$. Then*

- (i) $o_{q^m+1}(q) = 2m$;
- (ii) $2m$ is the largest length among q -cycles($q^m + 1$) and any other length divides $2m$;
- (iii) the number of q -cycles($q^m + 1$) having length ℓ is $(1/\ell) \sum_{1 \leq r \leq d, o_{N_r}(q) = \ell} \varphi(N_r)$.

Proof. Since $q^{2m} \equiv 1 \pmod{q^m + 1}$, we have $o_{q^m+1}(q) \mid 2m$. Assertion (i) thus follows from the observation that $o_{q^m+1}(q) > m$. Assertions (ii) and (iii) are direct consequences of Theorems 1(iii)–(vii). $\square$

In [2], Yucas and Mullen introduced the following set:

$$D_m := \left\{ n \in \mathbb{N}; n \mid (q^m + 1), \right. \\ \left. n \nmid (q^k + 1), k \in \{0, 1, \dots, m-1\} \right\}. \quad (11)$$

We show next that elements in this set can also be described through order.

Lemma 5. *Let $m, n (> 2) \in \mathbb{N}$. Then*

$$n \in D_m \Leftrightarrow \text{the multiplicative order of } q \text{ mod } n \text{ is } 2m, \text{ that is, } o_n(q) = 2m, \text{ and there is a positive integer } t \text{ dividing } m \text{ such that } n \mid (q^t + 1).$$

Proof. If $n \in D_m$, then $n \mid (q^{2m} - 1)$. If $n \mid (q^s - 1)$ ($s \in \mathbb{N}$), then [2, Proposition 1] tells us that s must be an even multiple of m . Thus, $o_n(q) = 2m$. Conversely, assume that $o_n(q) = 2m$ and t is the least positive integer for which $n \mid (q^t + 1)$. Thus, [2, Proposition 1] tells us that $2m$ is an even multiple of t , and so $t \mid m$. If $t < m$, the divisibility $n \mid (q^{2t} - 1)$ contradicts $o_n(q) = 2m$. Hence, $t = m$. $\square$

The next proposition, which is of independent interest, connects the sum in Theorem 4(iii) with one involving the Möbius function.

Proposition 6. *One has*

$$\begin{aligned}
 \sum_{n \in D_m} \varphi(n) &= \begin{cases} q^m - 1 & \text{if } m \text{ is a power of } 2, q \text{ is odd,} \\ q^m & \text{if } m \text{ is a power of } 2, q \text{ is even,} \\ \sum_{\substack{i | m \\ i \text{ odd}}} \mu(i) q^{m/i} & \text{if } m \text{ is not a power of } 2. \end{cases} \quad (12)
 \end{aligned}$$

To prove Proposition 6, we need some arithmetical facts about greatest common divisors.

Lemma 7. *Let $e, f, s, m \in \mathbb{N}$. Then*

$$\begin{aligned}
 \gcd(q^e + 1, q^f - 1) &= \begin{cases} 1, & \text{if } \gcd(2e, f) = \gcd(e, f), 2 \mid q \\ 2, & \text{if } \gcd(2e, f) = \gcd(e, f), 2 \nmid q \\ q^{\gcd(e, f)} + 1, & \text{if } \gcd(2e, f) = 2 \gcd(e, f), \end{cases} \quad (13)
 \end{aligned}$$

$$\begin{aligned}
 \gcd(q^s + 1, q^m + 1) &= \begin{cases} 1, & \text{if } v(m) \geq v(m-s), 2 \mid q \\ 2, & \text{if } v(m) \geq v(m-s), 2 \nmid q \\ q^{\gcd(m, s)} + 1, & \text{if } v(m) < v(m-s), \end{cases} \quad (14)
 \end{aligned}$$

where $v(j) = v_2(j)$ denotes the highest power of 2 that divides $j \in \mathbb{N}$.

Proof. To prove (13), we consider two separate cases.

Case 1 ($\gcd(2e, f) = \gcd(e, f)$). Observe that

$$\begin{aligned} \gcd(q^e + 1, q^f - 1) & \mid \gcd(q^{2e} - 1, q^f - 1) \\ & = q^{\gcd(2e, f)} - 1 = (q^{\gcd(e, f)} - 1) \mid (q^e - 1). \end{aligned} \quad (15)$$

If $2 \mid q$, then $\gcd(q^e + 1, q^e - 1) = 1$, and we infer that $\gcd(q^e + 1, q^f - 1) = 1$.

If $2 \nmid q$, then $\gcd(q^e + 1, q^e - 1) = 2$, and we infer that $\gcd(q^e + 1, q^f - 1) = 2$.

Case 2 ($\gcd(2e, f) = 2\gcd(e, f)$). Observe that

$$\begin{aligned} \gcd(q^e + 1, q^f - 1) & \mid \gcd(q^{2e} - 1, q^f - 1) \\ & = q^{2\gcd(e, f)} - 1 = q^{2\gcd(e, f)} - 1 \\ & = (q^{\gcd(e, f)} - 1)(q^{\gcd(e, f)} + 1). \end{aligned} \quad (16)$$

If $2 \mid q$, then $\gcd(q^e + 1, q^{\gcd(e, f)} - 1) = 1$, and from (16), we have

$$\gcd(q^e + 1, q^f - 1) \mid (q^{\gcd(e, f)} + 1). \quad (17)$$

On the other hand,

$$(q^{\gcd(e, f)} + 1) \mid (q^{2\gcd(e, f)} - 1) = (q^{\gcd(2e, f)} - 1) \mid (q^f - 1). \quad (18)$$

Since $e/\gcd(e, f)$ is odd, we have $(q^{\gcd(e, f)} + 1) \mid (q^e + 1)$. This last relation together with (17) and (18) yields $\gcd(q^e + 1, q^f - 1) = q^{\gcd(e, f)} + 1$, as required.

If $2 \nmid q$, then $\gcd((q^e + 1)/2, (q^{\gcd(e, f)} - 1)/2) = 1$, and so we infer from (16) that

$$\gcd\left(\frac{q^e + 1}{2}, \frac{q^f - 1}{2}\right) \mid (q^{\gcd(e, f)} + 1). \quad (19)$$

Since (18) holds, we have

$$\frac{q^{\gcd(e, f)} + 1}{2} \mid \frac{q^f - 1}{2}. \quad (20)$$

Since $e/\gcd(e, f)$ is odd, we also have

$$\frac{q^{\gcd(e, f)} + 1}{2} \mid \frac{q^e + 1}{2}. \quad (21)$$

The relations (20) and (21) show that

$$\gcd\left(\frac{q^e + 1}{2}, \frac{q^f - 1}{2}\right) = \frac{q^{\gcd(e, f)} + 1}{2} \gcd(k_1, k_2) \quad (22)$$

for some $k_1, k_2 \in \mathbb{N}$. Taking (19) into account, we deduce that $\gcd(k_1, k_2) = 1$ or 2 . To complete the proof, it suffices to verify that $\gcd(k_1, k_2) = 1$. To this end, suppose that $\gcd(k_1, k_2) = 2$. Thus, (22) yields

$$2(q^{\gcd(e, f)} + 1) \mid (q^e + 1). \quad (23)$$

Since $e/\gcd(e, f)$ is odd, we see that $q^e + 1 = (q^{\gcd(e, f)} + 1)G$, where $G \in \mathbb{N}$ is odd. Combining this last relation with (23), we arrive that $2 \mid G$, which is untenable.

To prove (14), we first observe that

$$\begin{aligned} \gcd(2e, f) = \gcd(e, f) & \iff \nu(e) \geq \nu(f), \\ \gcd(2e, f) = 2\gcd(e, f) & \iff \nu(e) < \nu(f). \end{aligned} \quad (24)$$

Putting $m = e, m - s = f$ in (13), the desired result (14) follows at once if we can show that

$$\gcd(q^m + 1, q^{m-s} - 1) = \gcd(q^m + 1, q^s + 1). \quad (25)$$

To this end, let $d_1 = \gcd(q^m + 1, q^{m-s} - 1)$, $d_2 = \gcd(q^m + 1, q^s + 1)$. It is easily checked that $d_1 = d_2$ because $d_2 \mid d_1 \iff d_2 \mid (q^{m-s} - 1)$ and $d_1 \mid d_2 \iff d_1 \mid (q^s + 1)$. $\square$

We are now ready to prove Proposition 6.

Case 1 (m is a power of 2). Clearly, in this case $\nu(m) \geq \nu(m-s)$ for $0 \leq s < m$. For brevity, let

$$\begin{aligned} S_m &:= \{n \in \mathbb{N}; n \mid (q^s + 1) \\ &\quad \text{for some } s \in \{0, 1, \dots, m-1\}, n \mid (q^m + 1)\}. \end{aligned} \quad (26)$$

Thus, $\sum_{n \in D_m} \varphi(n) = \sum_{n \mid (q^m + 1)} \varphi(n) - \sum_{n \in S_m} \varphi(n)$. If $2 \mid q$, then using (14), we see that those $s \in \{0, 1, \dots, m-1\}$ for which $n \mid (q^s + 1)$ must satisfy $\gcd(q^s + 1, q^m + 1) = 1$, showing that $n = 1$. On the other hand, if $2 \nmid q$, then using (14), we see those $s \in \{0, 1, \dots, m-1\}$ for which $n \mid (q^s + 1)$ must satisfy $\gcd(q^s + 1, q^m + 1) = 2$, showing that $n = 1, 2$. Combining the two possibilities, we get

$$\begin{aligned} \sum_{n \in D_m} \varphi(n) &= \sum_{n \mid (q^m + 1)} \varphi(n) - \begin{cases} \varphi(1), & \text{if } 2 \mid q, \\ \varphi(1) + \varphi(2), & \text{if } 2 \nmid q, \end{cases} \\ &= \begin{cases} q^m, & \text{if } 2 \mid q, \\ q^m - 1, & \text{if } 2 \nmid q. \end{cases} \end{aligned} \quad (27)$$

Case 2 (m is not a power of 2). Then $m = 2^\nu u$, where $\nu := \nu(m)$ and $u \geq 3$ is odd. From (14), we need to find those $s \in \{0, 1, \dots, m-1\}$ for which $\nu(m-s) > \nu(m)$. These are s such that $m-s = 2^{\nu+1}w$ ($w \in \mathbb{N}$); that is, $s = 2^\nu(u-2w)$. Thus, all such s are of the form

$$s = 2^\nu \cdot 1, 2^\nu \cdot 3, \dots, 2^\nu \cdot (u-2) \quad (28)$$

and correspondingly, $\gcd(s, m) = 2^v c$, for some $c \mid u$, $1 \leq c < u$. We proceed now by induction on u , noting that the case $u = 1$ (and $u = 2$) is contained in Case 1. We have

$$\begin{aligned}
& \sum_{n \in D_m} \varphi(n) \\
&= \sum_{n \mid (q^m + 1)} \varphi(n) - \sum_{n \in S_m} \varphi(n) \\
&= q^m + 1 - \sum_{\substack{c \mid u \\ 1 \leq c < u}} \sum_{n \in D_{2^v c}} \varphi(n) + \begin{cases} \varphi(1) & \text{if } 2 \mid q \\ \varphi(1) + \varphi(2) & \text{if } 2 \nmid q \end{cases} \\
&= q^m - \sum_{\substack{c \mid u \\ 1 \leq c < u}} \sum_{i \mid c} \mu(i) q^{2^v c/i} - \begin{cases} 0 + q^{2^v} & \text{if } 2 \mid q \\ 1 + q^{2^v} - 1 & \text{if } 2 \nmid q \end{cases} \\
&\quad \quad \quad \text{(using induction)} \\
&= q^m - \sum_{\substack{c \mid u \\ 1 \leq c < u}} \sum_{i \mid c} \mu(i) q^{2^v c/i} \\
&= q^m - \sum_{\substack{k \mid u \\ 1 \leq k < u}} q^{2^v k} \sum_{\substack{i \mid (u/k) \\ 1 \leq i < u/k}} \mu(i) \\
&= q^m + \sum_{\substack{k \mid u \\ 1 \leq k < u}} q^{2^v k} \mu\left(\frac{u}{k}\right) = \sum_{\substack{j \mid m \\ j \text{ odd}}} \mu(j) q^{m/j}.
\end{aligned} \tag{29}$$

3. Order of a Polynomial

Following [3, page 84], for $f(x) \in \mathbb{F}_q[x] \setminus \{0\}$ with $f(0) \neq 0$, the least $e \in \mathbb{N}$ for which $f(x)$ divides $x^e - 1$ is called the *order* of $f(x)$, denoted by $\text{ord}(f)$. If $f(0) = 0$, then $f(x) = x^h g(x)$, where $h \in \mathbb{N}$ and $g \in \mathbb{F}_q[x]$ with $g(0) \neq 0$ are uniquely determined, and $\text{ord}(f)$ is defined to be $\text{ord}(g)$. The order of a polynomial is also called the period or the exponent of that polynomial. The following facts are well-known.

- (i) (see [3, Theorem 3.3]) Let $f(x) \in \mathbb{F}_q[x]$ be irreducible with $\deg(f) = m \in \mathbb{N}$, $f(0) \neq 0$. Then $\text{ord}(f)$ is equal to the order of any root of f in the multiplicative group $\mathbb{F}_{q^m}^*$.
- (ii) (see [3, Lemma 3.6]) Let $c \in \mathbb{N}$. Then $f \in \mathbb{F}_q[x]$ with $f(0) \neq 0$ divides $x^c - 1$ if and only if $\text{ord}(f) \mid c$.
- (iii) (see [3, Theorem 3.8]) Let $g \in \mathbb{F}_q[x]$ be irreducible over $\mathbb{F}_q$ with $g(0) \neq 0$ and $\text{ord}(g) = e$, and let $f = g^b$ with $b \in \mathbb{N}$. Let t be the smallest integer with $p^t \geq b$, where p is the characteristic of $\mathbb{F}_q$. Then $\text{ord}(f) = ep^t$.
- (iv) (see [3, Theorem 3.9]) Let $g_1, \dots, g_k$ be pairwise relatively prime nonzero polynomials over $\mathbb{F}_q$, and let $f = g_1 \cdots g_k$. Then $\text{ord}(f) = \text{lcm}\{\text{ord}(g_1), \dots, \text{ord}(g_k)\}$.

3.1. Counting Formulae. Working with q -cycles, apart from the known result, [3, Theorem 3.5], that the number of monic irreducible $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\text{ord}(f) = N_r$ ($r \in$

$\{1, \dots, d\}$) and $\deg(f) = o_{N_r}(q)$ is $\varphi(N_r)/o_{N_r}(q)$, some new information about the number of monic irreducible polynomials is now derived.

Theorem 8. (i) The polynomial $f \in \mathbb{F}_q[x]$ constructed via (2) from a $q(a)$ -cycle(N) with $\gcd(a, N) = N_r$ ($r \in \{1, \dots, d\}$) is a monic irreducible polynomial with $f(0) \neq 0$, $\text{ord}(f) = N/N_r$ and $\deg(f) = o_{N/N_r}(q)$. Conversely, each irreducible polynomial $f \in \mathbb{F}_q[x]$ with $f(0) \neq 0$, $\text{ord}(f) = N_r$ and $\deg(f) = o_{N_r}(q)$ is an irreducible factor of $x^N - 1$ arisen, through (2), from an $q(a)$ -cycle(N) with $\gcd(a, N) = N/N_r$.

(ii) The set of monic, irreducible $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\text{ord}(f) \mid N$ is identical with the set of irreducible factors of $x^N - 1$.

(iii) The number of q -cycles(N), the number of monic irreducible factors of $x^N - 1$, and the number of monic irreducible $f \in \mathbb{F}_q[x]$ with $f(0) \neq 0$ and $\text{ord}(f) \mid N$ are all equal to $\sum_{r=1}^d \varphi(N_r)/o_{N_r}(q)$.

Proof. From Theorem 1(v), each $q(a)$ -cycle(N) with $\gcd(a, N) = N_r$ has length $o_{N/N_r}(q)$ and it thus gives rise, through the α -correspondence, to a monic irreducible $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$, of degree $o_{N/N_r}(q)$. Each root of f is of the form α^{aq^i} ($i \in \{0, 1, \dots, o_{N/N_r}(q) - 1\}$) where α is a primitive N th root of unity, and so its order is N/N_r . Part (i) is thus a consequence of [3, Theorem 3.3] and the α -correspondence.

To prove (ii), note that if the monic irreducible $f \in \mathbb{F}_q[x]$ satisfies $f(0) \neq 0$ and $\text{ord}(f) \mid N$, then $\text{ord}(f) = N_r$ for some $r \in \{1, \dots, d\}$. Thus, $f \mid (x^{N_r} - 1)$, and so $f \mid (x^N - 1)$. On the other hand, if f is an irreducible factor of $x^N - 1$, the α -correspondence assures us that f is constructed from a q -cycle(N) and the desired conclusion follows from (i).

Part (iii) follows (ii), the α -correspondence, and Theorem 1(viii). $\square$

There is another straightforward technique to derive a formula for the number of monic irreducible polynomials of fixed order using q -cycles based on the inclusion-exclusion principle. Although the formula so obtained is not easy to use, we give a proof to illustrate this different approach. For $r \in \{1, 2, \dots, d\}$, set (keeping the earlier notation in the first section)

$$\mathcal{J}_{N_r} = \{\text{monic irreducible polynomials arising from the } q\text{-cycles}(N_r) \text{ as constructed in (2)}\},$$

$$\mathcal{O}_{N_d} = \mathcal{O}_1 = |\mathcal{J}_1|,$$

$$\mathcal{O}_{N_r} = |\mathcal{J}_{N_r}| - \sum_{M \mid N_r, M < N_r} \mathcal{O}_M. \tag{30}$$

For a given $s \in \mathbb{N}$ and distinct primes $p_1, p_2, \dots, p_k$, define

$$\begin{aligned}
\langle s \rangle &:= \{p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}; n_1, n_2, \dots, n_k \in \mathbb{N}, \\
&\quad n_1 + n_2 + \cdots + n_k = s\}.
\end{aligned} \tag{31}$$

Proposition 9. Let $N = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, the unique prime factorization of N . The number of monic, irreducible polynomials $f \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$, $f(0) \neq 0$ having order N is equal to

$$\mathcal{O}_N = |\mathcal{I}_{N_1}| - \left| \bigcup_{r=2}^d \mathcal{I}_{N_r} \right| \quad (32)$$

$$\begin{aligned} &= |\mathcal{E}_{N_1}| \\ &\quad - \left\{ \sum_{r=2}^d |\mathcal{E}_{N_r}| \right. \\ &\quad - \sum_{x_1, x_2 \in \{N_2, N_3, \dots, N_d\}, x_1 < x_2} |\mathcal{E}_{\gcd(x_1, x_2)}| \\ &\quad + \cdots + (-1)^d \\ &\quad \times \left. \sum_{\substack{x_1, x_2, \dots, x_{d-1} \in \{N_2, N_3, \dots, N_d\}, \\ x_1 < x_2 < \cdots < x_{d-1}}} |\mathcal{E}_{\gcd(x_1, x_2, \dots, x_{d-1})}| \right\} \end{aligned} \quad (33)$$

$$\begin{aligned} &= |\mathcal{E}_{N_1}| - \sum_{m \in \langle s-1 \rangle} |\mathcal{E}_m| \\ &\quad + \sum_{m \in \langle s-2 \rangle} |\mathcal{E}_m| + \cdots + (-1)^k \sum_{m \in \langle s-k \rangle} |\mathcal{E}_m|. \end{aligned} \quad (34)$$

Proof. Since the number of distinct irreducible factors of $x^N - 1$ in $\mathbb{F}_q[x]$ is equal to the number of q -cycles(N) [1, Corollary 9.12], from its definition, the α -correspondence and [3, Lemma 3.6], $\mathcal{O}_{N_r}$ is simply the number irreducible factors (in $\mathbb{F}_q[x]$ constructed through (2)) of $x^{N_r} - 1$ which are not irreducible factors of $x^M - 1$ for any $M \mid N_r$, $M < N_r$, which is in turn equivalent to saying that $\mathcal{O}_{N_r}$ is the number of monic, irreducible polynomials $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$, having order N_r , and this proves (32).

Again using [1, Corollary 9.12] and [3, Lemma 3.6] to translate the number of q -cycles to those of corresponding irreducible polynomials, the inclusion-exclusion formulae on the right-hand sides of (33) and (34) clearly yield the number of irreducible factors of $x^N - 1$ which are not irreducible factors of $x^{N_r} - 1$ for all $r \in \{2, 3, \dots, d\}$, and this verifies (33) and (34). $\square$

Remark 10. Let us remark that there are no irreducible polynomials in $\mathbb{F}_q[x]$ of order n with $\gcd(q, n) > 1$. This is seen as follows: for $f(x) \in \mathbb{F}_q[x]$ with $f(0) \neq 0$, since $\text{ord}(f)$ is the least positive integer for which $f(x)$ divides $x^{\text{ord}(f)} - 1$, monic irreducible polynomials of order n are contained in the set of irreducible factors of $x^n - 1$ in $\mathbb{F}_q[x]$. Putting $n = Np^e$ ($e \in \mathbb{N}$, $\gcd(q, N) = 1$), since $x^n - 1 = (x^N - 1)^{p^e}$, the set of all irreducible factors of $x^n - 1$ is identical with the set of all irreducible factors of $x^N - 1$, and so each irreducible factor of $x^n - 1$ is of order $\leq N < n$.

In contrast to the preceding remark, there always exists a (reducible) polynomial in $\mathbb{F}_q[x]$ of order n with $\gcd(q, n) > 1$.

This is seen as follows: using the terminology of the last remark, since $\gcd(q, N) = 1$, Theorem 8 assures us that there always exists a monic, irreducible polynomial $g(x) \in \mathbb{F}_q[x]$, $g(0) \neq 0$, having $\text{ord}(g) = N$. Consequently, Theorem 3.8 in [3] tells us that $\text{ord}(g(x)^{p^e}) = Np^e = n$.

Our next task is to compute the number of monic polynomials in $\mathbb{F}_q[x]$ (both reducible and irreducible) having order N . Let $f(x) \in \mathbb{F}_q[x]$ be a factor of the polynomial $x^N - 1$ with $\text{ord}(f) = N$. Since $\gcd(q, N) = 1$, the polynomial $x^N - 1$ has no multiple factors. Assume that $f(x)$ is decomposed into k distinct monic irreducible factors (in $\mathbb{F}_q[x]$), say, $f(x) = f_1(x) \cdots f_k(x)$. Since $\text{ord}(f) = N = \text{lcm}\{\text{ord}(f_1), \dots, \text{ord}(f_k)\}$, each $\text{ord}(f_i) \in \{N = N_1 > N_2 > \cdots > N_d = 1\}$ (the set of all divisors of N). Thus, the number of all such f 's is equal to $\sum_{e_1 + e_2 + \cdots + e_d = k, e_r \geq 0, \text{lcm}(N_r, e_r \neq 0) = N} \binom{\mathcal{O}_{N_1}}{e_1} \cdots \binom{\mathcal{O}_{N_d}}{e_d}$. We have thus proved.

Theorem 11. The total number of monic polynomials (both reducible and irreducible) $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$, having order N is

$$\sum_{k=1}^m \sum_{\substack{e_1 + e_2 + \cdots + e_d = k, e_r \geq 0 \\ \text{lcm}(N_r, e_r \neq 0) = N}} \binom{\mathcal{O}_{N_1}}{e_1} \cdots \binom{\mathcal{O}_{N_d}}{e_d}, \quad m := \sum_{r=1}^d \mathcal{O}_{N_r}. \quad (35)$$

Having determined the number of polynomials with fixed order, it is natural to find out how many of them have orders dividing a fixed $n \in \mathbb{N}$.

Theorem 12. Let $n = p^e N \in \mathbb{N}$ with $e \in \mathbb{N}_0$ and $\gcd(q, N) = 1$. Then the product of all monic irreducible polynomials $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\text{ord}(f) \mid n$ is equal to $x^n - 1$ and their number is equal to $|\mathcal{E}_N| = \sum_{r=1}^d \varphi(N_r) / o_{N_r}(q)$.

Proof. Since $x^n - 1 = (x^N - 1)^{p^e}$, the set of all monic irreducible factors of $x^n - 1$ is identical with the set of all monic irreducible factors of $x^N - 1$. From [3, Theorem 3.6], for $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$, we know that $\text{ord}(f) \mid n$ if and only if $f(x) \mid (x^n - 1)$. Thus, the set of all monic irreducible polynomials $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\text{ord}(f) \mid n$ is identical with the set of all irreducible factors of $x^n - 1$ which is also equal to the set of all irreducible factors of $x^N - 1$. Since $x^N - 1$ has no multiple root, the product of all monic irreducible polynomials $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\text{ord}(f) \mid n$ is $x^n - 1$ and the first assertion is established. The second assertion follows immediately from the first assertion and Theorem 8(iv). $\square$

3.2. Some Explicit Shapes. Having counted the number of monic irreducible polynomials of fixed order, we proceed to determine their explicit shapes. From [1, Corollary 9.12], we know that the number of irreducible factors of $x^N - 1$ in $\mathbb{F}_q[x]$ is equal to the number of q -cycles(N) formed by the N numbers $0, 1, \dots, N-1$. To determine explicit shapes of all

monic, irreducible polynomials $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$, having order N , we consider the polynomial

$$f(x) = (x - \alpha^{a_0})(x - \alpha^{a_1}) \cdots (x - \alpha^{a_{\ell-1}}) \in \mathbb{F}_q[x] \quad (36)$$

associated with the q -cycle(N) $(a_0, a_1, \dots, a_{\ell-1})$, where α is a primitive N th root of unity. By definition, [3, Lemma 3.6] and [1, Theorem 9.11], among all such polynomials, those which do not divide $x^{N_r} - 1$, $r \in \{2, 3, \dots, d\}$, are all the sought after polynomials having order N .

Although, the procedure just described is satisfactory in principle, in certain cases, more precise shapes can be given. For a positive integer n not divisible by p , recall that the n th cyclotomic polynomial [3, page 64] is defined as

$$Q_n(x) = \prod_{\substack{i=1 \\ \gcd(i,n)=1}}^n (x - \zeta^i) = \prod_{i|n} (x^i - 1)^{\mu(n/i)} \in \mathbb{F}_q[x], \quad (37)$$

where ζ is a primitive n th root of unity and μ is the Möbius function. It is known that

$$x^N - 1 = \prod_{r=1}^d Q_{N_r}(x). \quad (38)$$

Proposition 13. *Let $d = d(N)$ be the number of divisors of N .*

- (i) *If the number of q -cycles(N) is equal to d , then in (38) each $Q_{N_r}(x)$ is irreducible in $\mathbb{F}_q[x]$ and $\text{ord}(Q_{N_r}) = N_r$. In particular, $Q_N(x) \in \mathbb{F}_q[x]$ is irreducible and of order N . The same conclusion holds if q is a primitive root mod N_r for all $r \in \{1, 2, \dots, d\}$.*
- (ii) *If $Q_{N_r}(x)$ is irreducible in $\mathbb{F}_q[x]$ for all $r \in \{1, 2, \dots, d\}$, then the number of q -cycles(N) is equal to d .*

Proof. We prove only part (i), for the other part is trivial from (38). If the number of q -cycles(N) is equal to d , the α -correspondence shows that the number of monic irreducible factors of $x^N - 1$ is also d . The first assertion that each $Q_{N_r}(x) \in \mathbb{F}_q[x]$ is irreducible follows at once from (38). This is also the case if q is a primitive root mod N_r for all $r \in \{1, 2, \dots, d\}$ which is a direct consequence of [3, Theorem 2.47(ii)].

From (38), we see that Q_{N_r} divides $x^{N_r} - 1$. If $\text{ord}(Q_{N_r}) = j < N_r$, then Q_{N_r} divides $x^j - 1$. Further, Lemma 3.6 of [3] shows that $j \mid N$, and so $x^j - 1$ divides $x^N - 1$. The first assertion then implies that each cyclotomic factor of $x^{N_r} - 1$ and of $x^j - 1$ (as in (38)) is irreducible. Since $\gcd(N_r, q) = 1$, the polynomial $x^{N_r} - 1$ contains no multiple root and so all its irreducible factors are distinct. But the cyclotomic factorization of $x^j - 1$ (as in (38)) does not contain the irreducible factor Q_{N_r} (because $N_r \nmid j$), which is a contradiction. $\square$

Our next proposition, which shows how the number of q -cycles can be used to generate irreducible polynomials, is based on the following known results.

- (i) (Berlekamp's factorization theorem, [3, Theorem 4.1]). If $f \in \mathbb{F}_q[x]$ is monic and $h \in \mathbb{F}_q[x]$ is such that $h^q \equiv h \pmod{f}$, then $f(x) = \prod_{c \in \mathbb{F}_q} \gcd(f(x), h(x) - c)$.

- (ii) (see [1, Theorem 9.13]) For each q -cycle(N) $(a_0, a_1, \dots, a_{\ell-1})$, if $g(x) = x^{a_0} + x^{a_1} + \dots + x^{a_{\ell-1}}$, then $g(x)^q \equiv g(x) \pmod{(x^N - 1)}$.

Combining these two results, we get the following.

Proposition 14. *Let $(a_0, a_1, \dots, a_{\ell-1})$ be a q -cycle(N), and $g(x) = x^{a_0} + x^{a_1} + \dots + x^{a_{\ell-1}}$. If the number of q -cycles(N) is equal to q , and if $\gcd(x^N - 1, g(x) - c) \neq 1$ for all $c \in \mathbb{F}_q$, then each polynomial $\gcd(x^N - 1, g(x) - c)$ is irreducible over $\mathbb{F}_q$.*

4. Degree of a Polynomial

We begin by listing (without proofs) known facts involving degrees of irreducible polynomials, which can be proved using q -cycles.

- (i) (see [3, Theorem 3.20]) The product of all monic irreducible $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\deg(f) \mid n$ is equal to $x^{q^n - 1} - 1$.
- (ii) (see [3, Theorem 3.25]) The number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree ℓ is equal to $(1/\ell) \sum_{k|\ell} \mu(k) q^{\ell/k}$.
- (iii) (see [3, Corollary 3.4]) The set of monic irreducible $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\deg(f) \mid n$ is identical with the set of monic irreducible $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\text{ord}(f) \mid (q^n - 1)$. Moreover, for $f \in \mathbb{F}_q[x]$, $f(0) \neq 0$, we have $\deg(f) \mid n$ if and only if $\text{ord}(f) \mid (q^n - 1)$.

Using q -cycles, that is, Theorem 1(viii) and the assertion [3, Theorem 3.20] the following new result is immediate.

Theorem 15. *The number of monic irreducible polynomials $f(x) \in \mathbb{F}_q[x]$, $f(0) \neq 0$ for which $\deg(f) \mid n$ is equal to $|\mathcal{C}_{q^n-1}| = \sum_{M|(q^n-1)} \varphi(M)/o_M(q)$, where the definition of $\mathcal{C}_{q^n-1}$ can be found in the heading "Notation and Terminology".*

Combining Theorems 15 and 2(iii), we obtain a nice arithmetical identity.

Corollary 16. *One has $\sum_{M|(q^n-1)} (\varphi(M)/o_M(q)) = \sum_{M|n} ((1/M) \sum_{k|M} \mu(k) q^{M/k}) - 1$.*

5. Coefficients and q -Cycle

Since a given q -cycle gives rise, through the α -correspondence, to a unique polynomial, we illustrate now how to uniquely determine all coefficients of its corresponding polynomial (2) and conversely.

Theorem 17. *Let α be a primitive N th root of unity, let m be the order of $q \pmod{N}$, and let ξ be a generator of $\mathbb{F}_{q^m}$.*

- (i) *If the $q(a_0)$ -cycle(N) $(a_0, a_1, \dots, a_{\ell-1})$ is given, then the coefficients of its corresponding polynomial in (2)*

$h(x) = (x - \alpha^{a_0})(x - \alpha^{a_1}) \cdots (x - \alpha^{a_{\ell-1}}) := \sum_{k=0}^{\ell} (-1)^k c_k x^{\ell-k} \in \mathbb{F}_q[x]$ are given by

$$c_0 = 1,$$

$$c_k = \sum_{0 \leq i_0 < i_1 < \cdots < i_{k-1} \leq \ell-1} \alpha^{a_{i_0}} \alpha^{a_{i_1}} \cdots \alpha^{a_{i_{k-1}}} \quad (k = 1, \dots, \ell). \quad (39)$$

(ii) If the coefficients of $h(x)$ are given, then the corresponding q -cycle is uniquely determined through its leading element which is found by solving for s , A , and a_0 , respectively, from the system

$$\xi^s = (-1)^\ell c_\ell, \quad (40)$$

$$\left(\frac{q^m - 1}{N} \right) A \equiv s \pmod{q^m - 1}, \quad (41)$$

$$a_0 (1 + q + \cdots + q^{\ell-1}) \equiv A \pmod{N}. \quad (42)$$

Proof. (i) The relations in (39) are simply the symmetric polynomial relations between the roots and coefficients of a polynomial.

(ii) It is trivial that each q -cycle is uniquely determined from its leading element. Next, note that (40) is simply a representation of element in $\mathbb{F}_{q^m}$. Through the α -correspondence, the q -cycle $(a_0, a_1, \dots, a_{\ell-1})$ associated with $h(x)$ must satisfy

$$\xi^s = (-1)^\ell c_\ell = \alpha^t,$$

$$t := a_0 + a_1 + \cdots + a_{\ell-1} \quad (43)$$

$$\equiv a_0 (1 + q + \cdots + q^{\ell-1}) \pmod{N}.$$

The conclusion follows from the facts that $\alpha = \xi^{(q^m-1)/N}$ and that solving (41) and (42) yields a_0 uniquely mod N . $\square$

6. Primitive and Self-Reciprocal Polynomials

We now apply preceding results to two special cases, corresponding to $N = q^n - 1$ and $N = q^m + 1$, to deduce a number of results about primitive and self-reciprocal polynomials.

6.1. Primitive Polynomials. Recall from [3, Definition 3.15] that a primitive polynomial of degree $n \in \mathbb{N}$ over $\mathbb{F}_q$ is the minimal polynomial over $\mathbb{F}_q$ of a primitive element of $\mathbb{F}_{q^n}$. We mention, without proofs, some known results provable using q -cycles.

- (1) (see [3, Theorem 3.16]) An irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ of degree n is primitive if and only if f is monic, $f(0) \neq 0$ and $\text{ord}(f) = q^n - 1$. In other words, each primitive polynomial in $\mathbb{F}_q[x]$ of degree n can be constructed from a $q(a)$ -cycle($q^n - 1$) with $\gcd(a, q^n - 1) = 1$ through (2). Conversely, for each $q(a)$ -cycle($q^n - 1$) with $\gcd(a, q^n - 1) = 1$, the polynomial (2) is a primitive polynomial in $\mathbb{F}_q[x]$ of degree n .

- (2) (see [1, Theorem 7.7]) The number of primitive polynomials in $\mathbb{F}_q[x]$ of degree n is $\varphi(q^n - 1)/n$, which equals to the number of $q(a)$ -cycle($q^n - 1$) with $\gcd(a, q^n - 1) = 1$.

6.2. Self-Reciprocal Polynomials. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x]$ with $a_n \neq 0$. The reciprocal polynomial f^* of f is defined by $f^*(x) = x^n f(1/x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$. A nonzero polynomial $f \in \mathbb{F}_q[x]$ is called *self-reciprocal* if $f(x) = f^*(x) := x^n f(1/x)$. We confine ourselves here to the study of *self-reciprocal irreducible monic* (srim) polynomials in $\mathbb{F}_q[x]$. Since there is only one first degree srim-polynomial, namely, $x + 1$, throughout the rest of this section we treat only srim-polynomials of degree ≥ 2 . We next mention a known characterization and results about srim-polynomials provable via q -cycles.

- (i) (see [2, page 275]) Let $f \in \mathbb{F}_q[x]$ be irreducible and monic of degree ≥ 2 . Then f is self-reciprocal if and only if its set of roots (each of which is evidently non-zero) is closed under inversion (and so its degree must be even).
- (ii) (see [2, Proposition 3]) If $f \in \mathbb{F}_q[x]$ is a srim-polynomial with $\deg(f) = 2m$, then $\text{ord}(f) \mid (q^m + 1)$ but $\text{ord}(f) \nmid (q^k + 1)$ for all $k \in \{0, 1, \dots, m-1\}$.
- (iii) (see [4, Lemma 2.3], [5, Theorem 1(i)], [2, Corollary 5]) For $a \in \mathbb{Z}_{q^m+1}$ with $\gcd(a, q^m + 1) = N_r$ ($r \in \{1, \dots, d\}$), each $q(a)$ -cycle($q^m + 1$), which is of length $o_{N/N_r}(q) = 2m$, gives rise through (2) to a srim-polynomial in $\mathbb{F}_q[x]$ of degree $2m$ and order dividing $q^m + 1$. Conversely, each srim-polynomial $f \in \mathbb{F}_q[x]$ of degree $2m$ arises via (2) from a $q(a)$ -cycle($q^m + 1$) with length $o_{(q^m+1)/N_r}(q) = 2m$ and $\text{ord}(f) \mid (q^m + 1)$, where $N_r = \gcd(a, q^m + 1)$.
- (iv) (see [5, Theorem 1(ii)]) Each irreducible factor $f(x)$ of $x^{q^m+1} - 1$ with $\deg(f) \geq 2$ is a srim-polynomial with $\deg(f) = 2k$, where $k \mid m$ and m/k is odd.

We next prove two new results using q -cycles.

Theorem 18. (i) For each divisor N_r of $q^m + 1$, the number of all srim-polynomial's $f \in \mathbb{F}_q[x]$ for which $\text{ord}(f) = N_r$ and $\deg(f) = o_{N_r}(q)$ is $\varphi(N_r)/o_{N_r}(q)$.

(ii) The number of srim-polynomials of degree $2m$ in $\mathbb{F}_q[x]$ is equal to $(1/2m) \sum_{1 \leq r \leq d, o_{N_r}(q)=2m} \varphi(N_r)$.

Proof. (i) and (ii) follow from [4, Lemma 2.3], [5, Theorem 1(ii)], Theorem 4, [3, Theorem 3.5], and Theorem 8(ii). $\square$

Combining [2, Proposition 3], [5, Theorem 1(i)-(ii)], and Theorem 18 with results in Section 2, we obtain the following known results about the set D_m as defined in (11).

- (i) (see [2, Proposition 4]) Assume that f is a srim-polynomial over $\mathbb{F}_q$ of degree $2m$ and let $\alpha \in \mathbb{F}_{q^{2m}}$ be a root of f . Then α is a primitive D th root of unity for some $D \in D_m$.

- (ii) (see [3, Exercise 3.15, page 141]) If f is a srin-polynomial in $\mathbb{F}_q[x]$ of degree >1 and order e , then every monic irreducible polynomial in $\mathbb{F}_q[x]$ of degree >1 whose order divides e is self-reciprocal.
- (iii) (see [2, Corollary 5], [6, Theorem 2]) If f is a srin-polynomial of degree $2m$ over $\mathbb{F}_q$, then $\text{ord}(f) \in D_m$.
- (iv) (see [2, Proposition 7]) Let $D \in D_m$ and let β be a primitive D th root of unity. If $f_\beta(x) = \prod_{i=0}^{2m-1} (x - \beta^{q^i})$, then the polynomial f_β is an srin-polynomial of degree $2m$ and order D .
- (v) (see [2, Theorem 8]) For a monic irreducible polynomial f of degree $2m$ over $\mathbb{F}_q$, the following statements are equivalent.
 - (1) f is self-reciprocal.
 - (2) $\text{ord}(f) \in D_m$.
 - (3) $f = f_\beta$ for some primitive D th root of unity β , with $D \in D_m$.
- (vi) (see [2, Theorem 9])
 - (1) There are $\varphi(D)/2m$ srin-polynomials in $\mathbb{F}_q[x]$ of degree $2m$ and order D for each $D \in D_m$.
 - (2) The number of srin-polynomials in $\mathbb{F}_q[x]$ of degree $2m$ is $(1/2m) \sum_{D \in D_m} \varphi(D)$.
- (vii) (see [2, Theorem 11]) For $D \in D_m$, the D th cyclotomic polynomial Q_D factors into the product of all srin-polynomials over $\mathbb{F}_q$ of degree $2m$ and order D .
- (viii) (see [5, Theorem 3]) Let $S_q(m)$ denote the number of srin-polynomials of degree $2m$ over $\mathbb{F}_q$. Then

$$S_q(m) = \begin{cases} \frac{q^m - 1}{2m}, & \text{if } q \text{ is odd, } m = 2^s, \\ \frac{1}{2m} \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d) q^{m/d}, & \text{otherwise.} \end{cases} \quad (44)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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