Research Article

On the General Erdős-Turán Conjecture

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The general Erdős-Turán conjecture states that if A is an infinite, strictly increasing sequence of natural numbers whose general term satisfies $a_n \leq cn^2$, for some constant $c > 0$ and for all n , then the number of representations functions of A is unbounded. Here, we introduce the function $\psi(n)$, giving the minimum of the maximal number of representations of a finite sequence $A = \{a_k : 1 \leq k \leq n\}$ of n natural numbers satisfying $a_k \leq k^2$ for all k . We show that $\psi(n)$ is an increasing function of n and that the general Erdős-Turán conjecture is equivalent to $\lim_{n \rightarrow \infty} \psi(n) = \infty$. We also compute some values of $\psi(n)$. We further introduce and study the notion of capacity, which is related to the ψ function by the fact that $\lim_{n \rightarrow \infty} \psi(n)$ is the capacity of the set of squares of positive integers, but which is also of intrinsic interest.

1. Introduction

In 1941, Erdős and Turán conjectured [1] that if A is an additive basis of the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, that is, if every $n \in \mathbb{N}$ is the sum of two elements of A , then the number of representations functions of A , defined by

$$r(A, n) = |\{(a, b) \in A \times A : a + b = n\}|, \quad n \in \mathbb{N}, \quad (1)$$

is unbounded. As is well known [2], if $A = \{a_1 < a_2 < \dots < a_n < \dots\}$ is a basis of $\mathbb{N}$, then its general term satisfies $a_n \leq cn^2$, for a constant $c > 0$ and for all positive integers n . So, a few years later, Erdős stated a more general conjecture, known as the general Erdős-Turán conjecture, to the effect that if A is an infinite (strictly increasing) sequence in $\mathbb{N}$ whose general term satisfies $a_n \leq cn^2$, then the number of representations functions of A is unbounded (e.g., see [3]).

In a previous paper [4], we defined an order relation between infinite (strictly increasing) sequences $A = \{a_n : n \geq 1\}$ and $B = \{b_n : n \geq 1\}$ of natural numbers, by setting $A \ll B$ if $a_n \leq b_n$ for all n . We called B an *Erdős-Turán set* if the relation $A \ll B$ implies that the number of representations functions of A is unbounded. Then, denoting

by $\mathbb{S}$ the set of squares of positive integers, the general Erdős-Turán conjecture amounts to the following.

Conjecture 1 (GET). *For any positive integer c , the set $c \cdot \mathbb{S} = \{cs : s \in \mathbb{S}\}$ is an Erdős-Turán set.*

Moreover, we showed that (GET) is in fact equivalent to the statement.

Conjecture 2 (GET). *$\mathbb{S}$ is an Erdős-Turán set.*

The present paper is a sequel to [4], to which we refer for broader background and further motivation, but the necessary prerequisite material is here recalled as needed, to make the present text self-contained and clear.

In Section 3, we define the capacity of an infinite set (identified to a sequence) A of natural numbers as the infimum of the supremum of the number of representations functions of P , for all infinite sequences P such that $P \ll A$; that is,

$$\text{cap}(A) = \inf_{P \ll A} s(P), \quad \text{where } s(P) = \sup_{n \in \mathbb{N}} r(P, n). \quad (2)$$

It provides a measure of the deviation of A from being an Erdős-Turán set, and it allows to characterize the Erdős-Turán sets as those whose capacity is infinite. The notion of capacity is also of intrinsic interest, and we give several of its properties. We thus prove that if the capacity of A is finite, then there exists a positive integer n such that, for any infinite sequence $P = \{p_n : n \geq 1\}$ satisfying $P \ll A$, we have $s(\{p_1, \dots, p_n\}) \geq \text{cap}(A)$. And if the capacity of A is infinite, then, for any natural number e , there exists a positive integer n such that, for any infinite sequence $P = \{p_n : n \geq 1\}$ satisfying $P \ll A$, we have $s(\{p_1, \dots, p_n\}) > e$. Moreover, if two infinite sequences A and B are “close”, in the sense that there exists a positive integer d such that $|a_n - b_n| \leq d$ for all n , then we have

$$\frac{\text{cap}(A)}{4d+1} \leq \text{cap}(B) \leq (4d+1) \text{cap}(A). \quad (3)$$

We also establish that the range of the capacity function is the interval $[2, \infty]$ of $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. There are therefore sets of any finite capacity ≥ 2 , and those are the ones for which the established properties are most significant.

In another paper [5], we introduced a function $\sigma(n)$, giving the minimum of the maximal number of representations of a set $A = \{a_1 < a_2 < \dots < a_n\}$ of n natural numbers forming a basis of the interval $[0, a_n]$ of $\mathbb{N}$. We showed that $\sigma(n)$ is an increasing function of n and that the original Erdős-Turán conjecture is equivalent to $\lim_{n \rightarrow \infty} \sigma(n) = \infty$. Here, in Section 4, we introduce the analogue for the general Erdős-Turán conjecture of $\sigma(n)$, namely, the function $\psi(n)$, giving the minimum of the maximal number of representations of a set $A = \{a_1 < a_2 < \dots < a_n\}$ of n natural numbers satisfying $a_k \leq k^2$ for all k . We show that $\psi(n)$ is an increasing function of n and that

$$\lim_{n \rightarrow \infty} \psi(n) = \text{cap}(\mathbb{S}), \quad (4)$$

so that the conjecture (GET) is equivalent to the statement.

Conjecture 3 (GET). Consider $\lim_{n \rightarrow \infty} \psi(n) = \infty$.

We also give the results of some computations of values of $\psi(n)$.

We note that there is an extensive literature related to the Erdős-Turán conjectures, for example, [6–16].

2. Notation and Preliminaries

Notation 1. (1) The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$, while the set of positive integers is $\mathbb{N}^* = \{1, 2, \dots\}$.

(2) A subset $A = \{a_1 < a_2 < \dots < a_n < \dots\}$ of $\mathbb{N}$, finite or infinite, is identified with the strictly increasing sequence of its elements (denoted by the corresponding lowercase letter).

(3) For any $n \in \mathbb{N}$, let $r(A, n) = |\{(a, b) \in A \times A : a + b = n\}|$, where $|E|$ denotes the cardinality of a set E . Further, let $s(A) = \sup\{r(A, n) : n \in \mathbb{N}\}$, considered as an element of $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

(4) For any $t \in \mathbb{N}$, let $A[t] = A \cap [0, t]$ denote an initial segment of A .

(5) For any $t \in \mathbb{N}$, let $A[t[= A \cap [t, \infty[$ denote a terminal segment of A .

(6) For any $t \in \mathbb{N}$, let $t + A = \{t + a : a \in A\}$ denote a translate of A .

(7) For any $t \in \mathbb{N}^*$, let $t \cdot A = \{ta : a \in A\}$ denote a homothetic of A .

(8) The set of all infinite subsets of $\mathbb{N}$ is denoted by $\mathcal{P}$.

(9) A partial order relation $\ll$ is defined in $\mathcal{P}$ by $A \ll B$ if $a_n \leq b_n$ for all n .

(10) An element C of $\mathcal{P}$ is called an Erdős-Turán set if, for any $A \in \mathcal{P}$, the relation $A \ll C$ implies that $s(A) = \infty$.

(11) The class of all Erdős-Turán sets C is denoted by $\mathcal{C}(\text{ET})$.

(12) The set of squares of positive integers is denoted by $\mathbb{S}$; that is, $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$.

The following results, some of which are in [4], are used several times in the sequel.

Lemma 4. Let $A, B \in \mathcal{P}$ and $t \in \mathbb{N}$.

- (1) If $B \subset A$, then $A \ll B$.
- (2) In particular, $\mathbb{N} \ll A$, and $A \ll A[t[$.
- (3) Also, $A \ll t + A$, and $A \ll t \cdot A$ when $t > 0$.

For $B \subset A$, $b_n = a_{k_n}$ with $1 \leq k_1 < k_2 < \dots$, so $n \leq k_n$ and $a_n \leq b_n$ for all n .

Lemma 5. For any subset A of $\mathbb{N}$ and any $t \in \mathbb{N}$, one has $s(t + A) = s(A)$.

We have $r(t + A, n) = r(A, n - 2t)$, for all n , and the result follows.

Lemma 6. Assume that A has at least two elements.

- (1) If $a_{n+1} > 2a_n$ for all n , then $s(A) = 2$.
- (2) If $a_1 > 0$ and $a_{n+1} \geq 2a_n$ for all n , then $s(A) = 2$.

Proof. If $r(A, x) > 2$ for some integer x , there would exist indices $i < m \leq n < j$ such that $a_i + a_j = a_m + a_n = x$. Therefore

$$a_{n+1} \leq a_j \leq a_m + a_n \leq 2a_n. \quad (5)$$

Moreover, if $a_i > 0$, then the inequality becomes

$$a_{n+1} \leq a_j < a_m + a_n \leq 2a_n. \quad (6)$$

In either case, this yields a contradiction.

Thus $s(A) \leq 2$, and since $|A| \geq 2$, $s(A) = 2$. $\square$

Example 7. Let $A = \{2^n : n \in \mathbb{N}\}$. Then, by Lemma 6, we have $s(A) = 2$.

Notation 2. For any subsets A and B of $\mathbb{N}$, and any $n \in \mathbb{N}$, let

- (1) $R(A, B; n) = |\{(a, b) \in A \times B : a + b = n\}|$, $r(A, B; n) = |R(A, B; n)|$,
- (2) $s(A, B) = \sup\{r(A, B; n) : n \in \mathbb{N}\}$, in $\overline{\mathbb{N}}$,

- (3) $R(A, n) = R(A, A; n)$; so $r(A, n) = r(A, A; n) = |R(A, n)|$, $s(A) = s(A, A)$,
 (4) An element (a, b) of $R(A, B; n)$ will be called an (A, B) -representation of n .

The following properties can be directly verified.

Lemma 8. For all subsets A and B of $\mathbb{N}$, and all $n \in \mathbb{N}$, one has

- (1) $r(B, A; n) = r(A, B; n)$,
- (2) $s(B, A) = s(A, B)$,
- (3) $R(A \cup B, n) = R(A, n) \cup R(B, n) \cup R(A, B; n) \cup R(B, A; n)$,
- (4) $\max(r(A, n), r(B, n), r(A, B; n)) \leq r(A \cup B, n) \leq r(A, n) + r(B, n) + 2r(A, B; n)$,
- (5) $\max(s(A), s(B), s(A, B)) \leq s(A \cup B) \leq s(A) + s(B) + 2s(A, B)$,
- (6) $\sum_{k=0}^n r(A, B; k) \cdot r(A, B; n-k) = \sum_{k=0}^n r(A, k) \cdot r(B, n-k) \leq (n+1)s(A)s(B)$ (both sums being equal to $|\{(a, b, a', b') \in A \times B \times A \times B : a + b + a' + b' = n\}|$).

Lemma 9. Let F and G be nonempty subsets of $\mathbb{N}$ such that F is finite with largest element m and $G = \{g_1 < g_2 < \dots\}$ is either finite or infinite, satisfying

- (1) $g_1 > 2m$,
- (2) $g_{n+1} > 2g_n$, for all n .

Then $s(F \cup G) = \max(s(F), 2)$.

Proof. Set $H = F \cup G$. Since m is the largest element in F , we have

$$F + F \subset [0, 2m]. \quad (7)$$

And, since $g_1 > 2m$, we also have

$$H + G \subset]2m, \infty[. \quad (8)$$

For $n \in [0, 2m]$, we therefore have

$$r(H, n) = r(F, n) \leq s(F). \quad (9)$$

Given $f \in F$, set $g_0 = f$. We have $g_1 > 2g_0$. Thus $s(\{f\} \cup G) = 2$, due to assumption (2) and Lemma 6.

If $f + g = f' + g'$, with f, f' in F , and $g \leq g'$ in G , then $f = f'$ and $g = g'$; otherwise, we would have $g < g'$ and

$$f - f' = g' - g > g > 2m, \quad (10)$$

which is impossible. This means that any integer n has at most one (F, G) -representation. Moreover, if it has an (F, G) -representation (f, g) , then it cannot have a (G, G) -representation; otherwise we would have $s(\{f\} \cup G) > 2$.

Hence, in view of Lemma 8, and since $H \times H$ is the union of four pairwise disjoint product sets, with H replaced by F or G , we have

$$\begin{aligned} r(H, n) &= r(F, n) + r(G, n) + 2r(F, G; n) \\ &= \max(r(F, n), r(G, n), 2r(F, G; n)) \\ &\leq \max(s(F), 2). \end{aligned} \quad (11)$$

The result then follows. $\square$

3. The Capacity Function

Definition 10. For any $A \in \mathcal{P}$, the capacity of A is the element of $\overline{\mathbb{N}}$ defined by

$$\text{cap}(A) = \inf \{s(P) : P \in \mathcal{P}, P \ll A\}. \quad (12)$$

Remark 11. For any $A \in \mathcal{P}$, the capacity of A is an attained minimum. More precisely, there exists $P \in \mathcal{P}$ such that

$$P \ll A, \quad \text{cap}(A) = s(P) = \text{cap}(P). \quad (13)$$

Indeed, if $\text{cap}(A) = \infty$, then, for every $P \ll A$ in $\mathcal{P}$, we have

$$\text{cap}(A) = s(P) = \text{cap}(P) = \infty. \quad (14)$$

If $\text{cap}(A)$ is finite, then at least one of the $s(P)$'s, for $P \ll A$ in $\mathcal{P}$, is finite, and the set

$$\{s(P) : P \in \mathcal{P}, P \ll A, s(P) < \infty\} \subset \mathbb{N} \quad (15)$$

has smallest element m . Taking one of those $P \ll A$ such that $s(P) = m$, we have

$$s(P) \geq \text{cap}(P) \geq \text{cap}(A) = s(P), \quad (16)$$

in view of Lemma 12, (3).

Lemma 12. For any $A, B \in \mathcal{P}$ and any $t \in \mathbb{N}$, one has

- (1) $2 \leq \text{cap}(A) \leq s(A)$,
- (2) $\text{cap}(A) = \infty$ if and only if $A \in \mathcal{C}(\text{ET})$,
- (3) if $A \ll B$ then $\text{cap}(B) \leq \text{cap}(A)$,
- (4) if $B \subset A$ then $\text{cap}(B) \leq \text{cap}(A)$,
- (5) $\text{cap}(A[t]) \leq \text{cap}(A)$,
- (6) $\text{cap}(t + A) \leq \text{cap}(A)$,
- (7) if $t > 0$, then $\text{cap}(t \cdot A) \leq \text{cap}(A)$.

Proof. These properties follow from the definitions and Lemma 4. $\square$

The following notion of *caliber* was introduced and studied in [4, item 5.5].

Definition 13. For any $A \in \mathcal{P}$, the caliber of A is the element of

$$\overline{\mathbb{R}}^+ = \{x \in \mathbb{R} : x \geq 0\} \cup \{\infty\} \quad (17)$$

defined by

$$\text{cal}(A) = \liminf_{n \rightarrow \infty} \frac{a_n}{n^2}. \quad (18)$$

We recall the following result from [4, item 5.9] that will be needed in the proof of the lemma following it.

Lemma 14. For any $A \in \mathcal{P}$, one has

$$s(A) \geq \frac{1}{2 \cdot \text{cal}(A)}. \quad (19)$$

Lemma 15. For any $A \in \mathcal{P}$, one has

$$\text{cap}(A) \geq \frac{1}{2 \cdot \text{cal}(A)}. \quad (20)$$

Proof. For any $P \in \mathcal{P}$ such that $P \ll A$, we have, from the definition,

$$0 \leq \text{cal}(P) \leq \text{cal}(A). \quad (21)$$

Moreover, by the previous lemma, we have

$$s(P) \geq \frac{1}{2 \cdot \text{cal}(P)} \geq \frac{1}{2 \cdot \text{cal}(A)}. \quad (22)$$

Therefore $\text{cap}(A)$, which is the infimum of all such $s(P)$, satisfies

$$\text{cap}(A) \geq \frac{1}{2 \cdot \text{cal}(A)}. \quad (23) \quad \square$$

Remark 16. It follows from Lemma 15 that if $A \in \mathcal{P}$ has zero caliber, that is, if $\text{cal}(A) = 0$, then $\text{cap}(A) = \infty$; that is, A is an Erdős-Turán set. Thus, for example, if

$$A = \{[n^{2-\epsilon}] : n \in \mathbb{N}^*\}, \quad \text{with } 0 < \epsilon < 2, \quad (24)$$

where $[x]$ is the integer part of a real number x , then $\text{cal}(A) = 0$ and therefore A is an Erdős-Turán set. In particular, any arithmetic progression is an Erdős-Turán set.

The infinite sets A of natural numbers for which $\text{cal}(A) = 0$ are the only known Erdős-Turán sets, because, in [4], we proved that the general Erdős-Turán conjecture (GET) is equivalent to the statement that there exists a subset A of $\mathbb{N}$ such that A is an Erdős-Turán set and $\text{cal}(A) \neq 0$ [4, item 5.16].

Remark 17. For any $e, s \in \overline{\mathbb{N}}$ such that $2 \leq e \leq s$, there exists $A \in \mathcal{P}$ such that $\text{cap}(A) \leq e$ and $s(A) = s$.

In particular, there exists $A \in \mathcal{P}$ such that $\text{cap}(A) = 2$ and $s(A) = \infty$, which is a nontrivially finite capacity set. We provide, below, an explicit example of such a set.

Indeed, since $e \geq 2$, there exists $P \in \mathcal{P}$ such that $s(P) \leq e$ (e.g., $P = \{2^n : n \in \mathbb{N}\}$ for which $s(P) = 2$, as in Example 7). Then, by [4, item 2.3], there exists $A \in \mathcal{P}$ such that $P \ll A$ and $s(A) = s$. Hence $\text{cap}(A) \leq s(P) \leq e$, while $s(A) = s$.

In particular, taking $e = 2$ and $s = \infty$ yields the latter special case. We next proceed to explicitly construct such a set $A \in \mathcal{P}$ having $s(A) = \infty$ and $\text{cap}(A) = 2$, following the procedure described in the proof of [4, item 2.3].

We start with the set

$$P = \{2^n : n \in \mathbb{N}\} = \{p_n : n \in \mathbb{N}^*\}, \quad (25)$$

where $p_n = 2^{n-1}$ for $n \in \mathbb{N}^*$.

For any $n \in \mathbb{N}^*$, let m, r be the unique natural numbers such that

$$n = 2^m + r, \quad 0 \leq r < 2^m, \quad (26)$$

i.e., $2^m \leq n < 2^{m+1}$, $r = n - 2^m$,

and let

$$a_n = p_{2^{m+1}} + r = 2^{2^{m+1}-1} + r. \quad (27)$$

Then

$$a_{n+1} = a_n + 1 > a_n, \quad \text{if } 2^m \leq n < 2^{m+1} - 1, \quad (28)$$

while if $n = 2^{m+1} - 1 = 2^m + (2^m - 1)$, then

$$a_{n+1} = a_{2^{m+1}} = 2^{2^{m+2}-1} > a_n = 2^{2^{m+1}-1} + 2^m - 1. \quad (29)$$

Thus

$$A = \{a_1 < a_2 < \dots < a_n < \dots\} \in \mathcal{P}. \quad (30)$$

Moreover, for $n = 2^m + r$, with $0 \leq r < 2^m$,

$$p_n = 2^{2^m+r-1} \leq 2^{2^{m+1}-2} < a_n = 2^{2^{m+1}-1} + r. \quad (31)$$

So $P \ll A$, and therefore

$$\text{cap}(A) \leq s(P) = 2, \quad \text{i.e., } \text{cap}(A) = 2. \quad (32)$$

Finally, if

$$k = 2p_{2^{m+1}} + r = 2^{2^{m+1}} + r, \quad \text{with } 0 \leq r < 2^m, \quad (33)$$

then

$$k = (p_{2^{m+1}} + q) + (p_{2^{m+1}} + r - q) = a_{2^m+q} + a_{2^m+r-q}, \quad (34)$$

for $0 \leq q \leq r$,

so that $r(A, k) \geq r + 1$. In particular, taking $r = 2^m - 1$, we get

$$r(A, 2^{2^{m+1}} + 2^m - 1) \geq 2^m, \quad \text{for any } m \in \mathbb{N}. \quad (35)$$

Hence $s(A) = \infty$.

Proposition 18. If $C \in \mathcal{P}$ and $e \in \mathbb{N}$ are such that $\text{cap}(C) > e$, then there exists an integer $m \in \mathbb{N}^*$ such that, for any $A \in \mathcal{P}$ satisfying $A \ll C$, one has $s(\{a_1, \dots, a_n\}) > e$ for all $n \geq m$.

Proof. We argue by contradiction. Assume that, for every $n \in \mathbb{N}^*$, the set

$$F_n = \{A \in \mathcal{P} : A \ll C, s(\{a_1, \dots, a_n\}) \leq e\} \quad (36)$$

is not empty. Clearly, $F_{n+1} \subset F_n$. For $m, n \in \mathbb{N}^*$, let

$$F_{n,m} = \{(a_1, \dots, a_m) : A \in F_n\}. \quad (37)$$

We have $F_{n,m} \neq \emptyset$ and $F_{n+1,m} \subset F_{n,m}$ for all $m, n \in \mathbb{N}^*$, in view of the corresponding properties of the F_n 's. Moreover, each $F_{n,m}$ is finite since it is contained in $\prod_{k=1}^m \mathbb{N}[c_k]$. We thus have, for every $m \in \mathbb{N}^*$, a decreasing sequence of finite nonempty sets $(F_{n,m})_{n \in \mathbb{N}^*}$, which is therefore stationary; that is, there exists some $n_m \in \mathbb{N}^*$ such that, for all $n \geq n_m$, we have $F_{n,m} = F_{n_m,m}$, and therefore the set

$$\mathcal{F}_m = \bigcap_{n \in \mathbb{N}^*} F_{n,m} = F_{n_m,m} \neq \emptyset. \quad (38)$$

Now, consider an $m \in \mathbb{N}^*$ and fix an $n \geq \max(n_m, n_{m+1})$. Note that if

$$(a_1, \dots, a_{m+1}) \in \mathcal{F}_{m+1} = F_{n, m+1}, \quad (39)$$

then

$$(a_1, \dots, a_m) \in F_{n, m} = \mathcal{F}_m. \quad (40)$$

Conversely, given any $(a_1, \dots, a_m) \in \mathcal{F}_m$, there is some $A \in F_n$ whose first m terms are the given ones in $\mathcal{F}_m = F_{n, m}$, and then, since $n \geq n_{m+1}$, the first $m+1$ terms of A lie in $F_{n, m+1} = \mathcal{F}_{m+1}$.

Therefore $\mathcal{F}_m$ consists exactly of the first m terms of the elements of $\mathcal{F}_{m+1}$. Hence the existence, by induction on m , of a set $A \in \mathcal{P}$ such that $A \ll C$ and $(a_1, \dots, a_m) \in \mathcal{F}_m$ for all $m \in \mathbb{N}^*$, so that $s(\{a_1, \dots, a_m\}) \leq e$ for all $m \in \mathbb{N}^*$. It follows that

$$s(A) \leq e < \text{cap}(C), \quad (41)$$

while $A \ll C$, contradicting the definition of $\text{cap}(C)$.

So the original assumption does not hold; that is, there exists some $m \in \mathbb{N}^*$ such that F_m is empty, which means that for all $A \in \mathcal{P}$ satisfying $A \ll C$ we have $s(\{a_1, \dots, a_m\}) > e$, and thus $s(\{a_1, \dots, a_n\}) > e$ for all $n \geq m$. $\square$

Corollary 19. *Let $C \in \mathcal{P}$.*

- (1) *If $C \in \mathcal{C}(\text{ET})$, then for every $e \in \mathbb{N}$, there exists an integer $m \in \mathbb{N}^*$ such that, for any $A \in \mathcal{P}$ satisfying $A \ll C$, we have $s(\{a_1, \dots, a_n\}) > e$ for all $n \geq m$.*
- (2) *If $\text{cap}(C) < \infty$, then there exists an integer $m \in \mathbb{N}^*$ such that, for any $A \in \mathcal{P}$ satisfying $A \ll C$, we have $s(\{a_1, \dots, a_n\}) \geq \text{cap}(C)$ for all $n \geq m$.*

Proof. If $C \in \mathcal{C}(\text{ET})$, then $\text{cap}(C) = \infty$ and Proposition 18 applies with any $e \in \mathbb{N}$ to give (1).

If $\text{cap}(C) < \infty$, then Proposition 18 applies with $e = \text{cap}(C) - 1$ to give (2). $\square$

Lemma 20. *For every integer $e \geq 2$, as well as for $e = \infty$, there exists some set $A \in \mathcal{P}$ such that $\text{cap}(A) = s(A) = e$.*

Thus the range of the capacity function is the closed interval $[2, \infty]$ of $\mathbb{N}$.

Proof. For a given integer $e \geq 2$, let

$$A = \{0, 1, \dots, e-1\} \cup \{2^{2n+e} : n \in \mathbb{N}^*\}. \quad (42)$$

Thus A is the union of the finite interval

$$F = \mathbb{N}[e-1] = \{0, 1, \dots, e-1\} \subset \mathbb{N}, \quad (43)$$

for which $s(F) = e$, and of the infinite set G with elements

$$g_n = 2^{2n+e}, \quad n \in \mathbb{N}^*. \quad (44)$$

Since

$$g_1 = 2^{e+2} > 2(e-1), \quad (45)$$

where $e-1$ is the largest element of F , and since, for all $n \in \mathbb{N}^*$, we have

$$g_{n+1} = 2^{2n+e+2} > 2^{2n+e+1} = 2g_n, \quad (46)$$

Lemma 9 applies and gives

$$s(A) = \max(e, 2) = e. \quad (47)$$

Moreover, if $P \in \mathcal{P}$ is such that $P \ll A$, then, by definition and in view of Lemma 4, we have

$$n-1 \leq p_n \leq a_n, \quad \forall n \in \mathbb{N}^*. \quad (48)$$

In particular, for $1 \leq n \leq e$, we get

$$n-1 \leq p_n \leq a_n = n-1, \quad (49)$$

so that

$$F = \mathbb{N}[e-1] \subset P \quad (50)$$

and therefore

$$s(P) \geq s(F) = e = s(A). \quad (51)$$

It follows that

$$\text{cap}(A) = \inf \{s(P) : P \in \mathcal{P}, P \ll A\} = s(A) = e. \quad (52)$$

For $e = \infty$, we may take $A = \mathbb{N}$. Indeed, $\mathbb{N}$ being the smallest element of $\mathcal{P}$ for the order $\ll$, the definition immediately yields

$$\text{cap}(\mathbb{N}) = s(\mathbb{N}) = \infty. \quad (53) \quad \square$$

Lemma 21. *For any $A \in \mathcal{P}$, one has*

$$\text{cap}(A) - 2 \leq \text{cap}(A[a_2]) \leq \text{cap}(1+A) \leq \text{cap}(A). \quad (54)$$

Proof. As the sequence (a_n) is strictly increasing, the inequalities

$$a_n \leq 1 + a_n \leq a_{n+1} \quad (55)$$

hold for all $n \in \mathbb{N}^*$, and they amount to

$$A \ll 1+A \ll A[a_2]. \quad (56)$$

So, by Lemma 12,

$$\text{cap}(A[a_2]) \leq \text{cap}(1+A) \leq \text{cap}(A). \quad (57)$$

Now, let $Q \in \mathcal{P}$ be such that

$$Q \ll A[a_2], \quad \text{i.e., } q_n \leq a_{n+1} \quad \forall n \in \mathbb{N}^*. \quad (58)$$

There are two possible cases.

- (i) Either $q_n \leq a_n$ for all $n \in \mathbb{N}^*$, that is, $Q \ll A$, and thus $s(Q) \geq \text{cap}(A)$.

(ii) Or there exists $n \in \mathbb{N}^*$ such that $q_n > a_n$, and we let m be the smallest such n , so that

$$\begin{aligned} q_n &\leq a_n \quad \text{for } 1 \leq n \leq m-1, \\ a_m &< q_m \leq a_{m+1}, \\ q_n &\leq a_{n+1} \quad \text{for } n \geq m. \end{aligned} \quad (59)$$

In this case, we define a sequence $(p_n)_{n \in \mathbb{N}^*}$ by

$$p_n = \begin{cases} q_n & \text{if } 1 \leq n \leq m-1, \\ a_m & \text{if } n = m, \\ q_{n-1} & \text{if } n \geq m+1. \end{cases} \quad (60)$$

Since the sequence (q_n) is strictly increasing and since

$$p_{m-1} = q_{m-1} \leq a_{m-1} < a_m = p_m < q_m = p_{m+1} \quad (61)$$

(assuming $m \geq 2$ for the first two inequalities, otherwise they are unnecessary), the sequence (p_n) is also strictly increasing; that is, $P = \{p_n : n \in \mathbb{N}^*\}$ lies in $\mathcal{P}$. Also, from the definition of (p_n) , we have $p_n \leq a_n$ for all $n \in \mathbb{N}^*$; that is, $P \ll A$, so that $s(P) \geq \text{cap}(A)$. Moreover, $P = Q \cup \{a_m\}$ and therefore, by [4, item 2.9], $s(P) \leq s(Q) + 2$. Hence

$$s(Q) \geq s(P) - 2 \geq \text{cap}(A) - 2. \quad (62)$$

Thus, in any case, and for all $Q \in \mathcal{P}$ such that $Q \ll A[a_2]$, we have

$$s(Q) \geq \text{cap}(A) - 2. \quad (63)$$

It follows that

$$\text{cap}(A[a_2]) \geq \text{cap}(A) - 2. \quad (64)$$

□

Corollary 22. For any $A \in \mathcal{P}$ and for any $n \in \mathbb{N}^*$, one has

$$\text{cap}(A) - 2n \leq \text{cap}(A[a_{n+1}]) \leq \text{cap}(n + A) \leq \text{cap}(A). \quad (65)$$

Proof. For $n = 1$, the stated inequalities are given by Lemma 21. For general n , the result follows by induction on n . □

Remark 23. For any $A \in \mathcal{P}$, there exists $m \in \mathbb{N}^*$, such that, for all $n \geq m$, we have

$$\begin{aligned} \text{cap}(n + A) &= \text{cap}(m + A), \\ \text{cap}(A[a_n]) &= \text{cap}(A[a_m]). \end{aligned} \quad (66)$$

Indeed, first, if $\text{cap}(A) = \infty$, then, in view of Lemma 12, $A \in \mathcal{C}(\text{ET})$, and therefore, by [4, item 2.14] and [4, item 2.16], both $n + A \in \mathcal{C}(\text{ET})$ and $A[a_n] \in \mathcal{C}(\text{ET})$, so that

$$\text{cap}(A[a_n]) = \text{cap}(n + A) = \text{cap}(A) = \infty, \quad \forall n \in \mathbb{N}^*. \quad (67)$$

Second, if $\text{cap}(A) < \infty$, then $(\text{cap}(n + A))_{n \in \mathbb{N}^*}$ and $(\text{cap}(A[a_n]))_{n \in \mathbb{N}^*}$ are decreasing sequences in $\mathbb{N}^*$, since if $m \leq n$, then

$$\begin{aligned} A &\ll m + A \ll n + A, \\ A &\ll A[a_m] \ll A[a_n], \end{aligned} \quad (68)$$

and therefore, in view of Lemma 12,

$$\begin{aligned} \text{cap}(n + A) &\leq \text{cap}(m + A) \leq \text{cap}(A), \\ \text{cap}(A[a_n]) &\leq \text{cap}(A[a_m]) \leq \text{cap}(A). \end{aligned} \quad (69)$$

Thus, the two sequences are stationary.

The following definition is taken from [4, 3.1].

Definition 24. For $A, B \in \mathcal{P}$, one sets

$$\delta(A, B) = \sup \{|a_n - b_n| : n \in \mathbb{N}^*\}, \quad \text{in } \overline{\mathbb{N}}. \quad (70)$$

If $\delta(A, B) \leq d$ for some $d \in \mathbb{N}^*$, we say that A and B are d -close.

More generally, if $\delta(A, B) < \infty$, we say that A and B are close.

Lemma 25. Let $A, B \in \mathcal{P}$ and $d \in \mathbb{N}^*$. If A and B are d -close, then

$$\frac{\text{cap}(A)}{4d + 1} \leq \text{cap}(B) \leq (4d + 1) \text{cap}(A). \quad (71)$$

Proof. Let $P \in \mathcal{P}$ be such that $P \ll A$. Define a sequence $(q_n)_{n \in \mathbb{N}^*}$ in $\mathbb{N}$ by

$$q_n = \max(n - 1, p_n - d), \quad \forall n \in \mathbb{N}^*. \quad (72)$$

Since the sequences $(p_n - d)_{n \in \mathbb{N}^*}$ and $(n - 1)_{n \in \mathbb{N}^*}$ are strictly increasing, so is $(q_n)_{n \in \mathbb{N}^*}$; that is,

$$Q = \{q_n : n \in \mathbb{N}^*\} \in \mathcal{P}. \quad (73)$$

Since A and B are d -close, and $P \ll A$, we have

$$p_n - d \leq a_n - d \leq b_n, \quad (74)$$

and since $n - 1 \leq b_n$ (Lemma 4), $q_n \leq b_n$, for all $n \in \mathbb{N}^*$. So $Q \ll B$, and $s(Q) \geq \text{cap}(B)$. Moreover, from the definition of q_n (and using Lemma 4 again),

$$p_n - d \leq q_n \leq p_n, \quad \forall n \in \mathbb{N}^*, \quad (75)$$

so that P and Q are d -close. It then follows from [4, item 3.3] that

$$s(P) \geq \frac{s(Q)}{(4d + 1)} \geq \frac{\text{cap}(B)}{(4d + 1)}. \quad (76)$$

Thus

$$s(P) \geq \frac{\text{cap}(B)}{(4d + 1)}, \quad \forall P \in \mathcal{P} \text{ such that } P \ll A, \quad (77)$$

and therefore

$$\text{cap}(A) \geq \frac{\text{cap}(B)}{(4d+1)}. \quad (78)$$

Furthermore, by exchange of A and B , we similarly get

$$\text{cap}(B) \geq \frac{\text{cap}(A)}{(4d+1)}. \quad (79)$$

□

Corollary 26. For any $A, B \in \mathcal{P}$, one has

$$(1) \text{cap}(B) \leq (4\delta(A, B) + 1)\text{cap}(A) \text{ and } \text{cap}(A) \leq (4\delta(A, B) + 1)\text{cap}(B),$$

(2) if $\text{cap}(A)$ and $\text{cap}(B)$ are finite, then

$$|\text{cap}(A) - \text{cap}(B)| \leq 4 \min(\text{cap}(A), \text{cap}(B)) \cdot \delta(A, B). \quad (80)$$

Proof. If $\delta(A, B) = \infty$, then all the inequalities hold trivially. We may thus assume that $\delta(A, B) = d < \infty$, with $d \in \mathbb{N}^*$, so that A and B are d -close.

The inequalities in (1) immediately follow from Lemma 25.

Assuming $\text{cap}(A)$ and $\text{cap}(B)$ are finite, we deduce from (1) the two inequalities

$$\begin{aligned} \text{cap}(B) - \text{cap}(A) &\leq 4d \cdot \text{cap}(A), \\ \text{cap}(A) - \text{cap}(B) &\leq 4d \cdot \text{cap}(B), \end{aligned} \quad (81)$$

of which one is precisely the inequality in (2). □

Lemma 27. For any $A \in \mathcal{P}$ and any $x \in \mathbb{N}$, one has

$$\text{cap}(A) \leq \text{cap}(A \cup \{x\}) \leq \text{cap}(A) + 2. \quad (82)$$

Proof. Let $B = A \cup \{x\}$. The case where $x \in A$ being trivial, we assume that $x \notin A$. Let $m \in \mathbb{N}^*$ be the smallest positive integer such that $a_m > x$, so that $a_{m-1} < x < a_m$, where if $m = 1$, we set $a_0 = -1$. Thus

$$b_n = \begin{cases} a_n & \text{if } 1 \leq n \leq m-1, \\ x & \text{if } n = m, \\ a_{n-1} & \text{if } n \geq m+1. \end{cases} \quad (83)$$

Since $A \subset B$, by Lemma 12, $\text{cap}(A) \leq \text{cap}(B)$, which yields the first inequality.

On the other hand, by Remark 11, there exists $P \in \mathcal{P}$ such that $P \ll A$ and $s(P) = \text{cap}(A)$. Thus, to prove the second inequality, it is enough to prove the existence of a set $Q \in \mathcal{P}$ such that $Q \ll B$ and $s(Q) \leq s(P) + 2$. To this end, we distinguish three cases.

(i) If $p_m \leq x$ and $p_{n+1} \leq a_n$ for $n \geq m$, then $P \ll B$, and we may take $Q = P$.

(ii) If $x < p_m$, then

$$p_{m-1} \leq a_{m-1} < x < p_m \leq a_m, \quad (84)$$

and we may take $Q = P \cup \{x\}$.

(iii) If $p_m \leq x$ and there exists $n \geq m$ such that $a_n < p_{n+1}$, let $k \in \mathbb{N}^*$ be the smallest such n , so that

$$p_{n+1} \leq a_n \quad \text{for } m \leq n \leq k-1, \quad a_k < p_{k+1}. \quad (85)$$

In particular,

$$p_k \leq a_{k-1} < a_k < p_{k+1}, \quad (86)$$

so that

$$p_k < p_k + 1 \leq a_k < p_{k+1}, \quad (87)$$

and we may take $Q = P \cup \{p_k + 1\}$.

In all three cases, we have

$$P \subset Q \subset P \cup \{y\}, \quad \text{with } y \in \mathbb{N}, \quad (88)$$

and therefore, by [4, item 2.9], $s(Q) \leq s(P) + 2$. Since, in addition, $Q \ll B$, we conclude that

$$\text{cap}(B) \leq s(Q) \leq s(P) + 2 = \text{cap}(A) + 2. \quad (89)$$

□

Corollary 28. For any $A \in \mathcal{P}$ and any finite subset F of $\mathbb{N}$, one has

$$\text{cap}(A) \leq \text{cap}(A \cup F) \leq \text{cap}(A) + 2|F|. \quad (90)$$

Proof. This follows from Lemma 27 by induction on the cardinality of F . □

4. The Function ψ and an Equivalent Formulation of (GET)

Definition 29. For $n \in \mathbb{N}^*$, let $\mathcal{A}_n$ denote the set of all subsets A of $\mathbb{N}$ satisfying

$$\begin{aligned} |A| &= n, \quad A = \{a_1 < a_2 < \dots < a_n\}, \\ a_k &\leq k^2 \quad \text{for } 1 \leq k \leq n. \end{aligned} \quad (91)$$

Also, define a function $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ by

$$\psi(n) = \min \{s(A) : A \in \mathcal{A}_n\}, \quad \text{for } n \in \mathbb{N}^*. \quad (92)$$

Lemma 30. The following properties hold.

- (1) The function $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ is increasing
- (2) It has a limit κ in $\overline{\mathbb{N}}$ satisfying

$$\kappa = \lim_{n \rightarrow \infty} \psi(n) = \sup \{\psi(n) : n \in \mathbb{N}^*\} \leq \text{cap}(\mathbb{S}). \quad (93)$$

Proof. For any $n \in \mathbb{N}^*$ and any $B \in \mathcal{A}_{n+1}$, the set $A = B \setminus \{\max B\}$ is a subset of B lying in $\mathcal{A}_n$, so that $s(B) \geq s(A) \geq \psi(n)$. Hence

$$\psi(n+1) = \min \{s(B) : B \in \mathcal{A}_{n+1}\} \geq \psi(n). \quad (94)$$

Thus ψ is an increasing function.

The sequence $(\psi(n))_{n \in \mathbb{N}^*}$ is increasing in $\mathbb{N}$ and is thus either stationary or unbounded, so that

$$\kappa = \lim_{n \rightarrow \infty} \psi(n) = \sup \{\psi(n) : n \in \mathbb{N}^*\} \quad (95)$$

exists in $\overline{\mathbb{N}}$. Moreover, for any $A \in \mathcal{P}$ such that $A \ll \mathbb{S}$, the truncated set

$$A \mid n = \{a_1 < a_2 < \dots < a_n\}, \quad (96)$$

consisting of the first n elements of A , lies in $\mathcal{A}_n$. Therefore

$$s(A) \geq s(A \mid n) \geq \psi(n), \quad \forall n \in \mathbb{N}^*, \quad (97)$$

and thus $s(A) \geq \kappa$. Hence

$$\text{cap}(\mathbb{S}) = \inf \{s(A) : A \in \mathcal{P}, A \ll \mathbb{S}\} \geq \kappa. \quad (98)$$

□

We recall the “diagonal lemma” and part of its corollary from [5, items 2.2 and 2.3] that will be used in the proof of the following proposition.

Lemma 31 (diagonal lemma and corollary). *Let $(P_i)_{i \in I}$ be a family of subsets P_i of $\mathbb{N}$ indexed by an infinite set I . There exists a subset A of $\mathbb{N}$, called a diagonal of the family $(P_i)_{i \in I}$, satisfying the following condition.*

(DL) *For any $n \in \mathbb{N}$, there are infinitely many indices $i \in I$ such that $A[n] = P_i[n]$.*

Moreover,

(CDL) *If $s(P_i) \leq s$, for some $s \in \overline{\mathbb{N}}$ and all $i \in I$, then $s(A) \leq s$.*

Proposition 32. *One has $\kappa = \text{cap}(\mathbb{S})$; that is,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(n) &= \sup \{\psi(n) : n \in \mathbb{N}^*\} \\ &= \inf \{s(A) : A \in \mathcal{P}, A \ll \mathbb{S}\}. \end{aligned} \quad (99)$$

Proof. For every $n \in \mathbb{N}^*$, let $A_n \in \mathcal{A}_n$ be such that $s(A_n) = \psi(n)$.

By the diagonal lemma, since $(A_n)_{n \in \mathbb{N}^*}$ is a family of subsets of $\mathbb{N}$, indexed by the infinite set $\mathbb{N}^*$, there exists a subset $A = \{a_1 < a_2 < \dots < a_k < \dots\}$ of $\mathbb{N}$ (called a diagonal of this family) satisfying the condition.

(DL) For any $x \in \mathbb{N}$, there exists an infinite subset I_x of $\mathbb{N}^*$ such that $A_n[x] = A[x]$ for every $n \in I_x$.

Moreover, since $s(A_n) = \psi(n) \leq \kappa$ for all $n \in \mathbb{N}^*$, we have $s(A) \leq \kappa$, by (CDL) in the previous lemma.

Furthermore, for every $k \in \mathbb{N}^*$, taking $x = a_k$ and $n \in I_x$, we get

$$A_n[a_k] = A[a_k] = \{a_1 < a_2 < \dots < a_k\}, \quad (100)$$

and since $A_n \in \mathcal{A}_n$, we conclude that $a_k \leq k^2$. Therefore $A \ll \mathbb{S}$.

Thus

$$\text{cap}(\mathbb{S}) \leq s(A) \leq \kappa, \quad (101)$$

and since, by Lemma 30, $\kappa \leq \text{cap}(\mathbb{S})$, it follows that $\text{cap}(\mathbb{S}) = \kappa$. □

Corollary 33. *The conjecture (GET) is equivalent to the following condition:*

$$(\text{GET } \psi) : \lim_{n \rightarrow \infty} \psi(n) = \infty. \quad (102)$$

Proof. By [4, item 4.14], (GET) is equivalent to the condition $\mathbb{S} \in \mathcal{C}(\text{ET})$, which, by Lemma 12, is equivalent to $\text{cap}(\mathbb{S}) = \infty$, which, by Proposition 32, amounts to $\kappa = \infty$. □

Remark 34. The function ψ is the analogue, for the conjecture (GET), of the function σ , for the conjecture (ET), introduced in [5, item 3.12].

Similarly, the statement (GET ψ), equivalent to (GET), is the analogue of the statement (ET σ), equivalent to (ET).

Indeed, we just need to recall some notations and results.

(i) For any $x \in \mathbb{N}$, $\mathcal{B}(x)$ denotes the set of all subsets P of $\mathbb{N}$, called bases of $\mathbb{N}[x]$, satisfying $P \subset \mathbb{N}[x] \subset P + P$.

(ii) For any $n \in \mathbb{N}^*$, $\mathcal{B}(\#n)$ denotes the set of all the subsets P of $\mathbb{N}$, called finite bases of cardinality n , satisfying $|P| = n$ and $P \in \mathcal{B}(\max P)$.

(iii) Furthermore, $\mathcal{B}(\mathbb{N})$ denotes the set of all subsets P of $\mathbb{N}$, called bases of $\mathbb{N}$, satisfying $P + P = \mathbb{N}$.

(iv) The function $\sigma : \mathbb{N}^* \rightarrow \mathbb{N}^*$ is defined, in [5], by

$$\sigma(n) = \min \{s(P) : P \in \mathcal{B}(\#n)\}, \quad \text{for } n \in \mathbb{N}^*. \quad (103)$$

It is an increasing function, like ψ .

(v) In [5], we proved that

$$\lim_{n \rightarrow \infty} \sigma(n) = \Lambda = \inf \{s(P) : P \in \mathcal{B}(\mathbb{N})\}, \quad \text{in } \overline{\mathbb{N}}, \quad (104)$$

and that the conjecture (ET) is equivalent to

$$(\text{ET } \sigma) : \lim_{n \rightarrow \infty} \sigma(n) = \infty. \quad (105)$$

(vi) Here, we proved that

$$\lim_{n \rightarrow \infty} \psi(n) = \kappa = \inf \{s(P) : P \in \mathcal{P}, P \ll \mathbb{S}\}, \quad \text{in } \overline{\mathbb{N}}, \quad (106)$$

and that the conjecture (GET) is equivalent to

$$(\text{GET } \psi) : \lim_{n \rightarrow \infty} \psi(n) = \infty. \quad (107)$$

Hence the analogy.

Lemma 35. *The following properties hold.*

(1) *If $A = \{a_1 < a_2 < \dots < a_n\}$ lies in $\mathcal{B}(\#n)$, then*

$$a_k \leq \frac{9}{16}k^2, \quad \text{for } 1 \leq k \leq n. \quad (108)$$

(2) *For any $n \in \mathbb{N}^*$, we have $\mathcal{B}(\#n) \subset \mathcal{A}_n$.*

(3) *For any $n \in \mathbb{N}^*$, we have $\psi(n) \leq \sigma(n)$.*

Proof. Let $A \in \mathcal{B}(\#n)$, so that $A = \{a_1 < a_2 < \dots < a_n\}$ and $A \in \mathcal{B}(a_n)$. Then, for every $1 \leq k \leq n$, the truncated set

$$A_k = A \mid k = \{a_1 < a_2 < \dots < a_k\} \in \mathcal{B}(a_k). \quad (109)$$

Therefore, by [5, item 3.15], we have

$$k = |A_k| \geq \frac{\sqrt{8a_k + 9} - 1}{2}, \quad (110)$$

so that

$$a_k \leq \frac{k(k+1)}{2} - 1 \leq \frac{9}{16}k^2 < k^2. \quad (111)$$

In particular, $A \in \mathcal{A}_n$. Hence $\mathcal{B}(\#n) \subset \mathcal{A}_n$, and therefore

$$\begin{aligned} \psi(n) &= \min \{s(A) : A \in \mathcal{A}_n\} \\ &\leq \min \{s(A) : A \in \mathcal{B}(\#n)\} = \sigma(n). \end{aligned} \quad (112) \quad \square$$

Corollary 36. *One has $\kappa \leq \Lambda$; that is,*

$$\lim_{n \rightarrow \infty} \psi(n) \leq \lim_{n \rightarrow \infty} \sigma(n). \quad (113)$$

Corollary 37. *If A is an additive basis of $\mathbb{N}$, then*

$$a_n \leq \frac{9}{16}n^2 < n^2, \quad \forall n \in \mathbb{N}^*, \quad (114)$$

so that $A \ll \mathbb{S}$.

Remark 38. Either Corollary 36 or Corollary 37 confirms that (GET) implies (ET), as stated in Section 1.

Remark 39. It would be interesting to find a large lower bound for κ . However, the function ψ seems to increase very slowly. We found by computer calculations that

$$\psi(n) = 2, \quad \text{for } 2 \leq n \leq 22, \quad (115)$$

and that

$$2 \leq \psi(n) \leq 4, \quad \text{for } 23 \leq n \leq 435. \quad (116)$$

Indeed, we have

$$\psi(1) = 1, \quad \psi(2) = 2, \quad \psi(3) = 2. \quad (117)$$

Moreover, the set

$$A = \{0, 1, 3, 8, 14, 18, 40, 49, 68, 91, 112, 124, 167, 183, 219, 253, 280, 300, 338, 400, 425, 478\} \quad (118)$$

lies in $\mathcal{A}_{22}$ and satisfies $s(A) = 2$, so that

$$\psi(n) = 2, \quad \text{for } 2 \leq n \leq 22. \quad (119)$$

Also, the following set B lies in $\mathcal{A}_{435}$ and satisfies $s(B) = 4$, so that

$$2 \leq \psi(n) \leq 4, \quad \text{for } 23 \leq n \leq 435. \quad (120)$$

This set B was constructed recursively by setting $b_1 = 0$ and then systematically taking b_{n+1} to be the least integer $> b_n$ satisfying the two conditions

$$b_{n+1} \leq (n+1)^2, \quad s(\{b_1 < b_2 < \dots < b_{n+1}\}) \leq 4; \quad (121)$$

$$\begin{aligned} B = \{ & 0, 1, 2, 3, 5, 7, 11, 15, 23, 31, 40, 47, 48, 65, 72, 98, 99, \\ & 114, 133, 147, 156, 190, 199, 222, 233, 272, 304, \\ & 328, 356, 385, 411, 433, 459, 506, 512, 548, 565, \\ & 624, 635, 675, 725, 753, 790, 869, 884, 930, 956, \\ & 986, 1040, 1095, 1120, 1215, 1246, 1351, 1469, \\ & 1500, 1509, 1544, 1610, 1681, 1687, 1768, 1856, \\ & 1933, 1983, 2118, 2180, 2260, 2323, 2403, 2461, \\ & 2546, 2784, 2811, 2829, 2917, 2993, 3066, 3146, \\ & 3267, 3307, 3402, 3683, 3708, 3886, 4107, 4126, \\ & 4476, 4588, 4709, 4797, 4982, 5048, 5131, 5274, \\ & 5277, 5284, 5621, 5649, 5742, 5846, 6043, 6382, \\ & 6448, 6512, 6656, 6794, 6909, 6975, 7140, 7289, \\ & 7556, 7733, 7805, 7867, 8289, 8411, 8436, 8506, \\ & 8963, 8996, 9123, 9271, 9519, 9581, 9806, 9997, \\ & 10130, 10160, 10345, 10714, 10875, 10962, \\ & 11359, 11771, 11862, 12286, 12456, 12705, \\ & 12772, 13000, 13060, 13113, 13653, 13962, \\ & 14101, 14261, 14430, 14434, 14796, 14996, \\ & 15526, 15686, 15693, 15794, 16034, 16458, \\ & 16702, 17076, 17188, 17645, 17946, 18074, \\ & 18096, 18256, 18822, 19136, 19759, 19826, \\ & 19883, 20325, 20536, 20592, 20973, 21222, \\ & 21384, 21923, 22314, 22658, 22698, 23595, \\ & 23821, 24157, 24176, 24387, 24696, 25329, \\ & 25761, 25911, 26087, 26792, 26805, 26864, \\ & 27698, 27954, 28365, 28927, 29196, 29209, \\ & 29271, 30062, 30076, 30523, 30811, 31286, \\ & 31331, 31566, 31626, 32570, 32669, 33026, \\ & 33530, 33852, 34119, 34267, 35282, 36067, \\ & 36292, 36551, 36698, 37252, 37593, 37734, \\ & 38022, 38147, 38933, 39325, 39828, 40388, \\ & 40597, 41219, 41446, 41897, 42215, 42445, \\ & 43254, 43804, 44379, 44772, 45353, 45741, \\ & 46107, 46795, 46990, 47350, 47748, 48018, \\ & 48649, 49221, 50286, 50445, 50745, 51407, \end{aligned}$$

51998, 52473, 53193, 53310, 54280, 54748,
 54830, 55600, 55801, 56742, 57178, 57777,
 58373, 58567, 59371, 59654, 60407, 60959,
 61315, 61742, 62228, 63216, 63689, 63914,
 65241, 65358, 65576, 66053, 66237, 67695,
 67905, 68672, 70034, 70476, 70928, 71882,
 72284, 72724, 73752, 74126, 74348, 74448,
 75116, 76825, 78307, 78802, 79048, 80039,
 80530, 81065, 81092, 82135, 82370, 82624,
 84154, 84852, 85583, 87485, 87668, 87681,
 88079, 88744, 89081, 89386, 89516, 90363,
 91090, 91158, 91744, 93948, 94054, 95516,
 95534, 95894, 97026, 97062, 98265, 98556,
 99061, 100280, 101882, 102085, 103455,
 103494, 103605, 104616, 105939, 106603,
 106658, 107830, 108787, 109544, 110211,
 110227, 110601, 112111, 113022, 113822,
 114018, 114724, 115905, 116414, 117300,
 117552, 117719, 118928, 119910, 122518,
 123182, 124383, 124536, 125662, 125931,
 127370, 128302, 128850, 130498, 130536,
 131365, 132111, 132196, 133477, 133841,
 134494, 134877, 135277, 135862, 136204,
 137804, 138438, 138950, 140020, 140912,
 142845, 143577, 144255, 145242, 146902,
 148888, 148998, 149041, 149902, 151754,
 152956, 153202, 154042, 155618, 157013,
 158103, 158332, 158835, 159987, 160829,
 161657, 162331, 162929, 163691, 163874,
 164193, 165059, 165812, 168512, 170032,
 170232, 170771, 171764, 173767, 173838,
 175321, 177150, 177385, 177960, 178886,
 180725, 182329, 182774, 184137, 184302,
 184679, 187155, 188250, 188667}.

(122)

Remark 40. In view of the slow increase of the function ψ , it would be interesting to have a probabilistic model of the behavior of ψ , defined on the space $\mathcal{A}_n$. A first step is to determine the cardinality $u(n) = |\mathcal{A}_n|$ of $\mathcal{A}_n$, which is, in itself, a nontrivial problem of intrinsic interest.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] P. Erdős and P. Turán, "On a problem of Sidon in additive number theory," *Journal of the London Mathematical Society. Second Series*, vol. 16, pp. 212–215, 1941.
- [2] H. Halberstam and K. F. Roth, *Sequences*, Clarendon Press, Oxford, UK, 1966.
- [3] P. Erdős and W. H. Fuchs, "On a problem of additive number theory," *Journal of the London Mathematical Society*, vol. 31, pp. 67–73, 1956.
- [4] G. Grekos, L. Haddad, C. Helou, and J. Pihko, "The class of Erdős-Turán sets," *Acta Arithmetica*, vol. 117, no. 1, pp. 81–105, 2005.
- [5] G. Grekos, L. Haddad, C. Helou, and J. Pihko, "On the Erdős-Turán conjecture," *Journal of Number Theory*, vol. 102, no. 2, pp. 339–352, 2003.
- [6] G. A. Dirac, "Note on a problem in additive number theory," *Journal of the London Mathematical Society*, vol. 26, pp. 312–313, 1951.
- [7] M. Dowd, "Questions related to the Erdős-Turán conjecture," *SIAM Journal on Discrete Mathematics*, vol. 1, no. 1, pp. 142–150, 1988.
- [8] M. Helm, "Some remarks on the Erdős-Turán conjecture," *Acta Arithmetica*, vol. 63, no. 4, pp. 373–378, 1993.
- [9] M. B. Nathanson, "Generalized additive bases, König's lemma, and the Erdős-Turán conjecture," *Journal of Number Theory*, vol. 106, no. 1, pp. 70–78, 2004.
- [10] L. Haddad and C. Helou, "Bases in some additive groups and the Erdős-Turán conjecture," *Journal of Combinatorial Theory Series A*, vol. 108, no. 1, pp. 147–153, 2004.
- [11] J. Nešetřil and O. Serra, "The Erdős-Turán property for a class of bases," *Acta Arithmetica*, vol. 115, no. 3, pp. 245–254, 2004.
- [12] P. Borwein, S. Choi, and F. Chu, "An old conjecture of Erdős-Turán on additive bases," *Mathematics of Computation*, vol. 75, no. 253, pp. 475–484, 2006.
- [13] J. Nešetřil and O. Serra, "Semigroups with the Erdős-Turán property," in *Combinatorial Number Theory*, pp. 381–388, de Gruyter, Berlin, Germany, 2007.
- [14] Y.-G. Chen, "The analogue of Erdős-Turán conjecture in $\mathbb{Z}_m$," *Journal of Number Theory*, vol. 128, no. 9, pp. 2573–2581, 2008.
- [15] Y.-G. Chen, "On the Erdős-Turán conjecture," *Comptes Rendus de l'Académie des Sciences—Paris*, vol. 350, pp. 933–935, 2012.

- [16] Q.-H. Yang, “A generalization of Chen’s theorem on the Erdős-Turán conjecture,” *International Journal of Number Theory*, vol. 9, no. 7, pp. 1683–1686, 2013.