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Research Article

Several New Third-Order and Fourth-Order Iterative Methods for Solving Nonlinear Equations

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In order to find the zeros of nonlinear equations, in this paper, we propose a family of third-order and optimal fourth-order iterative methods. We have also obtained some particular cases of these methods. These methods are constructed through weight function concept. The multivariate case of these methods has also been discussed. The numerical results show that the proposed methods are more efficient than some existing third- and fourth-order methods.

1. Introduction

Newton's iterative method is one of the eminent methods for finding roots of a nonlinear equation:

$$f(x) = 0. \tag{1}$$

Recently, researchers have focused on improving the order of convergence by evaluating additional functions and first derivative of functions. In order to improve the order of convergence and efficiency index, many modified third-order methods have been obtained by using different approaches (see [1–3]). Kung and Traub [4] presented a hypothesis on the optimality of the iterative methods by giving 2^{n-1} as the optimal order. It means that the Newton iteration by two function evaluations per iteration is optimal with 1.414 as the efficiency index. By using the optimality concept, many researchers have tried to construct iterative methods of optimal higher order of convergence. The order of the methods discussed above is three with three function evaluations per full iteration. Clearly its efficiency index is $3^{1/3} \approx 1.442$, which is not optimal. Very recently, the concept of weight functions has been used to obtain different classes of third- and fourth-order methods; one can see [5–7] and the references therein.

This paper is organized as follows. In Section 2, we present a new class of third-order and fourth-order iterative methods by using the concept of weight functions, which includes some existing methods and also provides some new methods. We have extended some of these methods for multivariate case. Finally, we employ some numerical examples and compare the performance of our proposed methods with some existing third- and fourth-order methods.

2. Methods and Convergence Analysis

First we give some definitions which we will use later.

Definition 1. Let $f(x)$ be a real valued function with a simple root α and let x_n be a sequence of real numbers that converge towards α. The order of convergence m is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^m} = \zeta \neq 0, \tag{2}$$

where ζ is the asymptotic error constant and $m \in R^+$.

Definition 2. Let n be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [8, 9] and defined as

$$m^{1/n}, \tag{3}$$

where m is the order of convergence of the new method.

2.1. Third-Order Iterative Methods. To improve the order of convergence of Newton's method, some modified methods are given by Grau-Sánchez and Díaz-Barrero in [10], Weerakoon and Fernando in [1], Homeier in [2], Chun and Kim in [3], and so forth. Motivated by these papers, we consider the following two-step iterative method:

$$y_n = x_n - a\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - A(t)\frac{f(x_n)}{f'(x_n)}, \tag{4}$$

where $t = f'(y_n)/f'(x_n)$ and a is a real constant. Now we find under what conditions it is of order three.

Theorem 3. *Let α be a simple root of the function f and let f have sufficient number of continuous derivatives in a neighborhood of α. The method (4) has third-order convergence, when the weight function $A(t)$ satisfies the following conditions:*

$$A(1) = 1, \qquad A'(1) = -\frac{1}{2a}, \qquad \left|A''(1)\right| \le +\infty. \tag{5}$$

Proof. Suppose $e_n = x_n - \alpha$ is the error in the nth iteration and $c_h = f^{(h)}(\alpha)/h!f'(\alpha), h \ge 1$. Expanding $f(x_n)$ and $f'(x_n)$ around the simple root α with Taylor series, then we have

$$\begin{aligned} f(x_n) &= f'(\alpha) \\ &\quad\times\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O\left(e_n^6\right)\right], \\ f'(x_n) &= f'(\alpha) \\ &\quad\times\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \tag{6}$$

Now it can be easily found that

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + \left(2c_2^2 - 2c_3\right)e_n^3 + O\left(e_n^4\right). \tag{7}$$

By using (7) in the first step of (4), we obtain

$$y_n = \alpha + (1-a)e_n + ac_2e_n^2 + 2a\left(c_3 - c_2^2\right)e_n^3 + O\left(e_n^4\right). \tag{8}$$

At this stage, we expand $f'(y_n)$ around the root by taking (8) into consideration. We have

$$\begin{aligned} &f'(y_n) \\ &= f'(\alpha)\left[1 + 2(1-a)c_2e_n + \left(2ac_2^2 + 3(1-a)^2c_3\right)e_n^2\right. \\ &\quad + \left(6(1-a)ac_2c_3 + 4ac_2\left(-c_2^2 + c_3\right) + 4(1-a)^3c_4\right) \\ &\quad \left.\times e_n^3 + O\left(e_n^4\right)\right]. \end{aligned} \tag{9}$$

Furthermore, we have

$$\begin{aligned} \frac{f'(y_n)}{f'(x_n)} &= 1 + \left\{-2c_2 + 2(1-a)c_2\right\}e_n \\ &\quad + \left\{4c_2^2 - 4(1-a)c_2^2 + 2ac_2^2 - 3c_3 + 3(1-a)^2c_3\right\} \\ &\quad \times e_n^2 + \cdots + O\left(e_n^4\right). \end{aligned} \tag{10}$$

By virtue of (10) and (4), we get

$$\begin{aligned} A(t)\times\frac{f(x_n)}{f'(x_n)} &= e_n - \left[\left(\frac{3a}{2} - 1\right)c_3 - 2c_2^2\left(-1 + a^2A''(1)\right)\right]e_n^3 \\ &\quad + O\left(e_n^4\right). \end{aligned} \tag{11}$$

Hence, from (11) and (4) we obtain the following general equation, which has third-order convergence:

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha \\ &= x_n - A(t)\times\frac{f(x_n)}{f'(x_n)} - \alpha \\ &= \left[\left(\frac{3a}{2} - 1\right)c_3 - 2c_2^2\left(-1 + a^2A''(1)\right)\right]e_n^3 + O\left(e_n^4\right). \end{aligned} \tag{12}$$

This proves the theorem. □

Particular Cases. To find different third-order methods we take $a = 2/3$ in (4).

Case 1. If we take $A(t) = (7-3t)/4$ in (4), then we get the formula:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left(\frac{7}{4} - \frac{3}{4}\frac{f'(y_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{13}$$

and its error equation is given by

$$2c_2^2e_n^3 + \left(-9c_2^3 + 7c_2c_3 + \frac{c_4}{9}\right)e_n^4 + O\left(e_n^5\right). \tag{14}$$

Case 2. If we take $A(t) = 4t/(7t-3)$ in (4), then we get the formula:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left(\frac{4f'(y_n)}{7f'(y_n) - 3f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{15}$$

and its error equation is given by

$$-\frac{1}{3}c_2^2e_n^3 + \frac{1}{9}\left(17c_2^3 - 21c_2c_3 + c_4\right)e_n^4 + O\left(e_n^5\right). \tag{16}$$

Case 3. If we take $A(t) = 4/(1+3t)$ in (4), then we get the formula:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 3f'(y_n)}, \tag{17}$$

and its error equation is given by

$$c_2^2 e_n^3 + \left(-3c_2^3 + 3c_2c_3 + \frac{c_4}{9}\right)e_n^4 + O\left(e_n^5\right). \tag{18}$$

Case 4. If we take $A(t) = (t+7)/(1+7t)$ in (4), then we get the formula:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left(\frac{f'(y_n) + 7f'(x_n)}{f'(x_n) + 7f'(y_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{19}$$

and its error equation is given by

$$\frac{5}{6}c_2e_n^3 + \frac{1}{36}\left(-79c_2^3 + 84c_2c_3 + 4c_4\right)e_n^4 + O\left(e_n^5\right). \tag{20}$$

Case 5. If we take $A(t) = (t+3)/4t$ in (4), then we get the formula:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{4}\left(\frac{1}{f'(x_n)} + \frac{3}{f'(y_n)}\right), \tag{21}$$

which is Huen's formula [11].

Remark 4. By taking different values of a and weight function $A(t)$ in (4), one can get a number of third-order iterative methods.

2.2. Optimal Fourth-Order Iterative Methods. The order of convergence of the methods obtained in the previous subsection is three with three function evaluations (one function and two derivatives) per step. Hence its efficiency index is $3^{1/3} \approx 1.442$, which is not optimal. To get optimal fourth-order methods we consider

$$y_n = x_n - a\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \{A(t) \times B(t)\}\frac{f(x_n)}{f'(x_n)}, \tag{22}$$

where $A(t)$ and $B(t)$ are two real-valued weight functions with $t = f'(y_n)/f'(x_n)$ and a is a real constant. The weight functions should be chosen in such a way that the order of convergence arrives at optimal level four without using additional function evaluations. The following theorem indicate the required conditions for the weight functions and constant a in (22) to get optimal fourth-order convergence.

Theorem 5. *Let α be a simple root of the function f and let f have sufficient number of continuous derivatives in a neighborhood of α. The method (22) has fourth-order convergence, when $a = 2/3$ and the weight functions $A(t)$ and $B(t)$ satisfy the following conditions:*

$$A(1) = 1, \qquad A'(1) = -\frac{3}{4}, \qquad \left|A^{(3)}(1)\right| \le +\infty,$$
$$B(1) = 1, \qquad B'(1) = 0, \tag{23}$$
$$B''(1) = \frac{9}{4} - A''(1), \qquad \left|B^{(3)}(1)\right| \le +\infty.$$

Proof. Using (6) and putting $a = 2/3$ in the first step of (22), we have

$$y_n = \alpha + \frac{e_n}{3} + \frac{2c_2e_n^2}{3} + \frac{4\left(c_3 - c_2^2\right)e_n^3}{3} + \cdots + O\left(e_n^5\right). \tag{24}$$

Now we expand $f'(y_n)$ around the root by taking (24) into consideration. Thus, we have

$$f'(y_n) = f'(\alpha)\left[1 + \frac{2c_2e_n}{3} + \frac{\left(4c_2^2 + c_3\right)e_n^2}{3} + \cdots + O\left(e_n^5\right)\right]. \tag{25}$$

Furthermore, we have

$$\frac{f'(y_n)}{f'(x_n)} = 1 - \frac{4c_2}{3}e_n + \left(4c_2^2 - \frac{8c_3}{3}\right)e_n^2 + \cdots + O\left(e_n^5\right). \tag{26}$$

By virtue of (26) and (22), we obtain

$$\begin{aligned}
&\{A(t) \times B(t)\}\frac{f(x_n)}{f'(x_n)} \\
&= e_n - \frac{1}{81}\left[-81c_2c_3 + 9c_4 \right. \\
&\qquad \left. + \left(243 + 72A''(1) + 32A'''(1) + 32B'''(1)\right)c_2^3\right] \\
&\quad \times e_n^4 + O\left(e_n^5\right).
\end{aligned} \tag{27}$$

Finally, from (27) and (22) we can have the following general equation, which reveals the fourth-order convergence:

$$\begin{aligned}
e_{n+1} &= x_{n+1} - \alpha \\
&= x_n - \{A(t) \times B(t)\}\frac{f(x_n)}{f'(x_n)} - \alpha \\
&= \frac{1}{81}\left[-81c_2c_3 + 9c_4 + \left(243 + 72A''(1)\right.\right. \\
&\qquad \left.\left. + 32A'''(1) + 32B'''(1)\right)c_2^3\right]e_n^4 + O\left(e_n^5\right).
\end{aligned} \tag{28}$$

It proves the theorem. □

Particular Cases

Method 1. If we take $A(t) = (t+3)/4t$ and $B(t) = ((11/8) - (3/4)t + (3/8)t^2)$, where $t = f'(y)/f'(x)$, then the iterative method is given by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left[\frac{11}{8} - \frac{3}{4}\frac{f'(y_n)}{f'(x_n)} + \frac{3}{8}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right] \times \left(\frac{1}{f'(x_n)} + \frac{3}{f'(y_n)}\right)\frac{f(x_n)}{4}, \tag{29}$$

and its error equation is given by

$$e_{n+1} = \frac{1}{9}\left[23c_2^3 - 9c_2c_3 + c_4\right]e_n^4 + O\left(e_n^5\right). \tag{30}$$

Method 2. If we take $A(t) = (7-3t)/4$ and $B(t) = ((17/8) - (9/4)t + (9/8)t^2)$, where $t = f'(y)/f'(x)$, then the iterative method is given by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left[\frac{17}{8} - \frac{9}{4}\frac{f'(y_n)}{f'(x_n)} + \frac{9}{8}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right] \times \left(\frac{7}{4} - \frac{3}{4}\frac{f'(y_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{31}$$

and its error equation is given by

$$e_{n+1} = \left[3c_2^3 - c_2c_3 + \frac{c_4}{9}\right]e_n^4 + O\left(e_n^5\right). \tag{32}$$

Method 3. If we take $A(t) = 4t/(7t-3)$ and $B(t) = ((13/16) + (3/8)t - (3/16)t^2)$, where $t = f'(y)/f'(x)$, then the iterative method is given by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left[\frac{13}{16} + \frac{3}{8}\frac{f'(y_n)}{f'(x_n)} - \frac{3}{16}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right] \times \left(\frac{4f'(y_n)}{7f'(y_n) - 3f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{33}$$

and its error equation is given by

$$e_{n+1} = \frac{1}{9}\left[-c_2^3 - 9c_2c_3 + c_4\right]e_n^4 + O\left(e_n^5\right). \tag{34}$$

Method 4. If we take $A(t) = 4/(1+3t)$ and $B(t) = ((25/8) - (9/8)t + (9/16)t^2)$, where $t = f'(y)/f'(x)$, then the iterative method is given by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left[\frac{25}{16} - \frac{9}{8}\frac{f'(y_n)}{f'(x_n)} + \frac{9}{16}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right] \times \left[\frac{4f(x_n)}{f'(x_n) + 3f'(y_n)}\right], \tag{35}$$

and its error equation is

$$e_{n+1} = \left[3c_2^3 - c_2c_3 + \frac{c_4}{9}\right]e_n^4 + O\left(e_n^5\right). \tag{36}$$

Method 5. If we take $A(t) = 4/(1+3t)$ and $B(t) = 1 + (9/16)(t-1)^2$, where $t = f'(y)/f'(x)$, then the iterative method is given by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left[1 + \frac{9}{16}\left(\frac{f'(y_n)}{f'(x_n)} - 1\right)^2\right] \times \left[\frac{4f(x_n)}{f'(x_n) + 3f'(y_n)}\right], \tag{37}$$

which is same as the formula (11) of [12].

Method 6. If we take $A(t) = (t+7)/(1+7t)$ and $B(t) = ((47/32) - (15/16)t - (15/32)t^2)$, where $t = f'(y)/f'(x)$, then the iterative method is given by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left[\frac{47}{32} - \frac{15}{16}\frac{f'(y_n)}{f'(x_n)} + \frac{15}{32}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right] \times \left(\frac{7f'(x_n) + f'(y_n)}{f'(x_n) + 7f'(y_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{38}$$

and its error equation is

$$e_{n+1} = \left[\frac{101c_2^3}{36} - c_2c_3 + \frac{c_4}{9}\right]e_n^4 + O\left(e_n^5\right). \tag{39}$$

Remark 6. By taking different values of $A(t)$ and $B(t)$ in (22), one can obtain a number of fourth-order iterative methods.

3. Further Extension to Multivariate Case

In this section, we extend some third- and fourth-order methods from our proposed methods to solve the nonlinear

systems. Similarly we can extend other methods also. The multivariate case of our third-order method (15) is given by

$$\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}\left[F'\left(X^{(k)}\right)\right]^{-1} F\left(X^{(k)}\right), \\ X^{(k+1)} &= X^{(k)} - 4\left[7F'\left(Y^{(k)}\right) - 3F'\left(X^{(k)}\right)\right]^{-1} \\ &\quad \times F'\left(Y^{(k)}\right)\left\{\left[F'\left(X^{(k)}\right)\right]^{-1} F\left(X^{(k)}\right)\right\}, \end{aligned} \tag{40}$$

where $X^{(k)} = [x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}]^T$, $(k = 0, 1, 2, \ldots)$; similarly $Y^{(k)}$; I is $n \times n$ identity matrix; $F(X^{(k)}) = [f_1(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}), f_2(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}), \ldots, f_n(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})]$; and $F'(X^{(k)})$ is the Jacobian matrix of F at $X^{(k)}$. Let $\xi + H \in \mathfrak{R}^n$ be any point of the neighborhood of exact solution $\xi \in \mathfrak{R}^n$ of the nonlinear system $F(X) = 0$. If Jacobian matrix $F'(\xi)$ is nonsingular, then Taylor's series expansion for multivariate case is given by

$$\begin{aligned} F(\xi + H) &= F'(\xi)\left[H + C_2H^2 + C_3H^3 + \cdots + C_{p-1}H^{p-1}\right] \\ &\quad + O\left(H^p\right), \end{aligned} \tag{41}$$

where $C_i = [F'(\xi)]^{-1}(F^{(i)}(\xi)/i!)$, $i \geq 2$ and

$$\begin{aligned} &F'(\xi + H) \\ &\quad = F'(\xi)\left[I + 2C_2H + 3C_3H^2 + \cdots + (p-1)C_{p-1}H^{p-2}\right] \\ &\qquad + O\left(H^{p-1}\right), \end{aligned} \tag{42}$$

where I is an identity matrix. From the previous equation we can find

$$\begin{aligned} &\left[F'(\xi + H)\right]^{-1} \\ &\quad = \left[F'(\xi)\right]^{-1}\left[I + L_1H + L_2H^2 + L_3H^3 + \cdots + L_{p-2}H^{p-2}\right] \\ &\qquad + O\left(H^{p-1}\right), \end{aligned} \tag{43}$$

where $L_1 = -2C_2$, $L_2 = 4C_2^2 - 3C_3$, and $L_3 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4$. Here we denote the error in kth iteration by $E^{(k)}$, that is, $E^{(k)} = X^{(k)} - \xi$. The order of convergence of method (40) can be proved by the following theorem.

Theorem 7. *Let $F : D \subseteq \mathfrak{R}^n \to \mathfrak{R}^n$ be sufficiently Frechet differentiable in a convex set D, containing a root ξ of $F(X) = 0$. Let one suppose that $F'(X)$ is continuous and nonsingular in D and $X^{(0)}$ is close to ξ. Then the sequence $\{X^{(k)}\}_{k\geq 0}$ obtained by the iterative expression* (40) *converges to ξ with order three.*

Proof. For the convenience of calculation, we replace 2/3 by β in the first step of (40). From (41), (42), and (43), we have

$$\begin{aligned} F\left(X^{(k)}\right) &= F'(\xi) \\ &\quad \times \left[E^{(k)} + C_2E^{(k)2} + C_3E^{(k)3} + C_4E^{(k)4} + C_5E^{(k)5}\right] \\ &\quad + O\left(E^{(k)6}\right), \end{aligned} \tag{44}$$

$$\begin{aligned} &F'\left(X^{(k)}\right) \\ &\quad = F'(\xi) \\ &\qquad \times \left[I + 2C_2E^{(k)} + 3C_3E^{(k)2} + 4C_4E^{(k)3} + 5C_5E^{(k)4}\right] \\ &\qquad + O\left(E^{(k)5}\right), \end{aligned} \tag{45}$$

$$\begin{aligned} &\left[F'\left(X^{(k)}\right)\right]^{-1} \\ &\quad = \left[F'(\xi)\right]^{-1}\left\{I - 2C_2E^{(k)}\right. \\ &\qquad + \left(4C_2^2 - 3C_3\right)E^{(k)2} \\ &\qquad + \left(-8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4\right) \\ &\qquad \left. \times E^{(k)3}\right\} + O\left(E^{(k)4}\right), \end{aligned} \tag{46}$$

where $C_i = [F'(\xi)]^{-1}(F^{(i)}(\xi)/i!)$, $i \geq 2$. Now from (46) and (44), we can obtain

$$\begin{aligned} S &= \left[F'\left(X^{(k)}\right)\right]^{-1} F\left(X^{(k)}\right) \\ &= G_1E^{(k)} + G_2E^{(k)2} + G_3E^{(k)3} + G_4E^{(k)4} + O\left(E^{(k)5}\right), \end{aligned} \tag{47}$$

where

$$\begin{aligned} G_1 &= I, \\ G_2 &= -C_2, \\ G_3 &= -2C_3 + 2C_2^2, \\ G_4 &= -3C_4 - 4C_2C_3 + 3C_3C_2 - 4C_2^3. \end{aligned} \tag{48}$$

By virtue of (47) the first step of the method (40) becomes

$$\begin{aligned} Y^{(k)} &= (1-\beta)E^{(k)} + \beta C_2E^{(k)2} + \beta\left(-2C_2^2 + 2C_3\right)E^{(k)3} \\ &\quad + \beta\left(4C_2^3 - 4C_2C_3 - 3C_3C_2 + 3C_4\right)E^{(k)4} \\ &\quad + O\left(E^{(k)5}\right). \end{aligned} \tag{49}$$

Taylor's series expansion for Jacobian matrix $F'(Y^{(k)})$ can be given as

$$\begin{aligned} F'\left(Y^{(k)}\right) &= F'(\xi)\Big[I + 2C_2(1-\beta)E^{(k)} \\ &\quad + \left(2\beta C_2^2 + 3C_3(1-\beta)^2\right)E^{(k)^2} \\ &\quad + \Big(-4\beta C_2^3 + 4\beta C_2C_3 + 6\beta(1-\beta)C_3C_2 \\ &\quad + 4C_4(1-\beta)^3\Big)E^{(k)^3} \\ &\quad + \Big(8\beta C_2^4 - 8\beta C_2^2C_3 - 6\beta C_2C_3C_4 \\ &\quad + 6\beta C_2C_4 - 3\beta^2C_3C_2 \\ &\quad - 12\beta(1-\beta)C_3C_2^2 \\ &\quad + 12\beta(1-\beta)C_3^2 \\ &\quad + 12\beta(1-\beta)^2C_4C_2\Big) \\ &\quad + 5C_5(1-\beta)^4\Big)E^{(k)^4}\Big] + O\left(E^{(k)^5}\right). \end{aligned} \tag{50}$$

Now

$$\begin{aligned} &\left[7F'\left(Y^{(k)}\right) - 3F'\left(X^{(k)}\right)\right] \\ &\quad = 4\left[F'(\xi)\right] \\ &\quad \times \left[I + \frac{1}{4}\left[A_1E^{(k)} + A_2E^{(k)^2} + A_3E^{(k)^3}\right]\right] \\ &\quad + O\left(E^{(k)^4}\right), \end{aligned} \tag{51}$$

where

$$\begin{aligned} A_1 &= C_2(8-14\beta), \\ A_2 &= 14\beta C_2^2 + 21C_3(1-\beta)^2 - 9C_3, \\ A_3 &= -28\beta C_2^3 + 28\beta C_2C_3 + 42\beta(1-\beta)C_3C_2 \\ &\quad + 28C_4(1-\beta)^3 - 12C_4. \end{aligned} \tag{52}$$

Taking inverse of both sides of (51), we get

$$\begin{aligned} &4\left[7F'(Y^{(k)}) - 3F'\left(X^{(k)}\right)\right]^{-1} \\ &\quad = \left[F'(\xi)\right]^{-1}\left[I + B_1E^{(k)} + B_2E^{(k)^2} + B_3E^{(k)^3}\right] \\ &\quad + O\left(E^{(k)^4}\right), \end{aligned} \tag{53}$$

where

$$\begin{aligned} B_1 &= -\frac{A_1}{4}, \\ B_2 &= \left(-\frac{A_2}{4} + \frac{A_1^2}{16}\right), \\ B_3 &= \left(-\frac{A_3}{4} - \frac{A_1^3}{64} + \frac{A_1A_2}{16} + \frac{A_2A_1}{16}\right). \end{aligned} \tag{54}$$

By multiplying (53) and (50), we get

$$\begin{aligned} &4\left[7F'\left(Y^{(k)}\right) - 3F'\left(X^{(k)}\right)\right]^{-1}\left[F'\left(Y^{(k)}\right)\right] \\ &\quad = \left(I + E_1E^{(k)} + E_2E^{(k)^2} + E_3E^{(k)^3}\right) + O\left(E^{(k)^4}\right), \end{aligned} \tag{55}$$

where

$$\begin{aligned} E_1 &= B_1 + D_1, \\ E_2 &= B_1D_1 + B_2 + D_2, \\ E_3 &= B_2D_1 + B_1D_2 + B_3 + D_3, \end{aligned} \tag{56}$$

and the values of D_1, D_2, and D_3 are mentioned below:

$$\begin{aligned} D_1 &= 2C_2(1-\beta), \\ D_2 &= 2\beta C_2^2 + 3C_3(1-\beta)^2, \\ D_3 &= -4\beta C_2^3 + 4\beta C_2C_3 + 6\beta(1-\beta)C_3C_2 + 4C_4(1-\beta)^3. \end{aligned} \tag{57}$$

From multiplication of (47) and (55), we achieve

$$\begin{aligned} &4\left[7F'\left(Y^{(k)}\right) - 3F'\left(X^{(k)}\right)\right]^{-1}\left[F'\left(Y^{(k)}\right)\right]S \\ &= \left[G_1E^{(k)} + \{G_2 + E_1G_1\}E^{(k)^2} + \{E_1G_2 + E_2G_1 + G_3\}E^{(k)^3}\right] \\ &\quad + O\left(E^{(k)^4}\right). \end{aligned} \tag{58}$$

After replacing the value of the above equation in second part of (40), we get

$$\begin{aligned} E^{(k+1)} &= \{I - G_1\}E^{(k)} - \{G_2 + E_1G_1\}E^{(k)^2} \\ &\quad - \{E_1G_2 + E_2G_1 + G_3\}E^{(k)^3} + O\left(E^{(k)^4}\right). \end{aligned} \tag{59}$$

The final error equation of method (40) is given by

$$E^{(k+1)} = -\left(\frac{C_2^2}{3}\right)E^{(k)^3} + O\left(E^{(k)^4}\right). \tag{60}$$

Thus, we end the proof of Theorem 7. □

Table 1: Functions and their roots.

$f(x)$	α
$f_1(x) = [\sin(x)]^2 + x$	$\alpha_1 = 0$
$f_2(x) = [\sin(x)]^2 - x^2 + 1$	$\alpha_2 \approx 1.40449164821534122603508681778 6$
$f_3(x) = e^{-x} + \sin(x) - 1$	$\alpha_3 \approx 2.07683127453311261307004424475 0$
$f_4(x) = x^2 + \sin(x) + x$	$\alpha_4 = 0$
$f_5(x) = \sin[2\cos(x)] - 1 - x^2 + e^{\sin(x^3)}$	$\alpha_5 \approx 1.30617520184682782501484290906 6$
$f_6(x) = x^6 - 10x^3 + x^2 - x + 3$	$\alpha_6 \approx 0.65860484711814043676386001471 0$
$f_7(x) = x^4 - x^3 + 11x - 7$	$\alpha_7 \approx 0.80351119911077768897813766029 3$
$f_8(x) = x^3 - \cos(x) + 2$	$\alpha_8 \approx -1.17257796475397001267333271486 8$
$f_9(x) = \sqrt{x} - \cos(x)$	$\alpha_9 \approx 0.64171437087288265839856530031 6$
$f_{10}(x) = \log(x) - x^3 + 2\sin(x)$	$\alpha_{10} \approx 1.29799774328037184716447923828 6$

The multivariate case of (33) is given by

$$\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}\left[\left[F'\left(X^{(k)}\right)\right]^{-1} F\left(X^{(k)}\right)\right], \\ X^{(k+1)} &= X^{(k)} - \left[\frac{13}{16} I + \frac{3}{8}\left(\left[F'\left(X^{(k)}\right)\right]^{-1} F'\left(Y^{(k)}\right)\right)\right. \\ &\qquad \left. - \frac{3}{16}\left(\left[F'\left(X^{(k)}\right)\right]^{-1} F'\left(Y^{(k)}\right)\right)^2\right] \\ &\qquad \cdot 4\left[7F'\left(Y^{(k)}\right) - 3F'\left(X^{(k)}\right)\right]^{-1} F'\left(Y^{(k)}\right) \\ &\qquad \cdot \left[\left[F'\left(X^{(k)}\right)\right]^{-1} F\left(X^{(k)}\right)\right]. \end{aligned} \tag{61}$$

The following theorem shows that this method has fourth-order convergence.

Theorem 8. *Let $F : D \subseteq \mathfrak{R}^n \to \mathfrak{R}^n$ be sufficiently Frechet differentiable in a convex set D, containing a root ξ of $F(X) = 0$. Let one suppose that $F'(X)$ is continuous and nonsingular in D and $X^{(0)}$ is close to ξ. Then the sequence $\{X^{(k)}\}_{k\geq 0}$ obtained by the iterative expression* (61) *converges to ξ with order four.*

Proof. For the convenience of calculation we replace $2/3$ by β and put $a_1 = 13/16$, $a_2 = 3/8$, and $a_3 = -3/16$ in (61). From (46) and (50), we have

$$\begin{aligned} t &= \left[F'\left(X^{(k)}\right)\right]^{-1} F'\left(Y^{(k)}\right) \\ &= I - 2\beta C_2 E^{(k)} + \left\{6\beta C_2^2 + 3C_3\left(\beta^2 - 2\beta\right)\right\} E^{(k)^2} \\ &\quad + \left\{-16\beta C_2^3 + \left(-6\beta^2 + 16\beta\right) C_2 C_3\right. \\ &\qquad \left. + 6\beta\left(2 - \beta\right) C_3 C_2 + \left(4\left(1-\beta\right)^3 - 4\right) C_4\right\} E^{(k)^3} \\ &\quad + O\left(E^{(k)^4}\right). \end{aligned} \tag{62}$$

From the above equation we have

$$\begin{aligned} t^2 &= \left(\left[F'\left(X^{(k)}\right)\right]^{-1} F'\left(Y^{(k)}\right)\right)^2 \\ &= I - 4\beta C_2 E^{(k)} + \left\{\left(12\beta + 4\beta^2\right) C_2^2 + 6\left(\beta^2 - 2\beta\right) C_3\right\} E^{(k)^2} \\ &\quad + \left\{\left(-32\beta - 24\beta^2\right) C_2^3 + \left(-6\beta^3 + 32\beta\right) C_2 C_3\right. \\ &\qquad \left. + \left(-6\beta^3 + 24\beta\right) C_3 C_2 + 2\left(4\left(1-\beta\right)^3 - 4\right)\right\} E^{(k)^3} \\ &\quad + O\left(E^{(k)^4}\right). \end{aligned} \tag{63}$$

With the help of (62) and (63), we can obtain

$$\begin{aligned} &a_1 I + a_2 t + a_3 t^2 \\ &\quad = \left(a_1 + a_2 + a_3\right) I \\ &\qquad + \left(-2\beta a_2 - 4\beta a_3\right) C_2 E^{(k)} \\ &\qquad + \left\{\left(3\left(\beta^2 - 2\beta\right) a_2 + 6\left(\beta^2 - 2\beta\right) a_3\right) C_3\right. \\ &\qquad\quad \left. + \left(6\beta a_2 + \left(12\beta + 4\beta^2\right) a_3\right) C_2^2\right\} E^{(k)^2} \\ &\qquad + \left\{\left(-16\beta a_2 + \left(-32\beta - 24\beta^2\right) a_3\right) C_2^3\right. \\ &\qquad\quad + \left(\left(-6\beta^2 + 16\beta\right) a_2 + \left(-6\beta^3 + 32\beta\right) a_3\right) C_2 C_3 \\ &\qquad\quad + \left(6\beta\left(2 - \beta\right) a_2 + \left(-6\beta^3 + 24\beta\right) a_3\right) C_3 C_2 \\ &\qquad\quad \left. + \left(\left(4\left(1-\beta\right)^3 - 4\right) a_2 + 2\left(4\left(1-\beta\right)^3 - 4\right) a_3\right) C_4\right\} \\ &\qquad \times E^{(k)^3} + O\left(E^{(k)^4}\right). \end{aligned} \tag{64}$$

By multiplying (64) to (58), we have

$$\begin{aligned} &\left(a_1 I + a_2 t + a_3 t^2\right) 4\left[7F'\left(Y^{(k)}\right) - 3F'\left(X^{(k)}\right)\right]^{-1} \left[F'\left(Y^{(k)}\right)\right] S \\ &\quad = \left(T_1 E^{(k)} + T_2 E^{(k)^2} + T_3 E^{(k)^3} + T_4 E^{(k)^4} + O\left(E^{(k)^5}\right)\right), \end{aligned} \tag{65}$$

Table 2: Comparison of absolute value of the functions by different methods after fourth iteration (TNFE-12).

$\lvert f\rvert$	Guess	HN3	M3	SL4	JM4	M4
	0.3	$0.1e-57$	$0.3e-93$	$0.2e-172$	$0.4e-162$	$0.5e-199$
	0.2	$0.5e-69$	$0.1e-91$	$0.1e-186$	$0.2e-198$	$0.1e-245$
$\lvert f_1\rvert$	0.1	$0.1e-90$	$0.6e-107$	$0.5e-241$	$0.2e-266$	$0.6e-339$
	−0.1	$0.2e-85$	$0.1e-93$	$0.1e-198$	$0.1e-247$	$0.7e-278$
	−0.2	$0.3e-59$	$0.9e-64$	$0.1e-99$	$0.8e-161$	$0.5e-165$
	1.3	$0.1e-92$	$0.3e-102$	$0.1e-244$	$0.4e-278$	$0.6e-297$
	1.2	$0.1e-67$	$0.7e-75$	$0.2e-152$	$0.2e-197$	$0.7e-200$
$\lvert f_2\rvert$	1.1	$0.8e-53$	$0.1e-57$	$0.4e-94$	$0.5e-147$	$0.2e-132$
	1.4	$0.6e-205$	$0.7e-217$	$0.1e-613$	$0.5e-634$	$0.1e-672$
	1.5	$0.3e-99$	$0.9e-114$	$0.1e-296$	$0.2e-300$	$0.7e-374$
	2.0	$0.1e-112$	$0.2e-122$	$0.8e-362$	$0.1e-325$	$0.1e-418$
	2.3	$0.1e-81$	$0.1e-102$	$0.1e-215$	$0.1e-229$	$0.5e-275$
$\lvert f_3\rvert$	2.1	$0.7e-157$	$0.1e-169$	$0.6e-493$	$0.1e-466$	$0.3e-543$
	2.2	$0.3e-100$	$0.1e-116$	$0.4e-288$	$0.2e-288$	$0.1e-343$
	1.9	$0.1e-81$	$0.5e-88$	$0.1e-223$	$0.1e-224$	$0.3e-272$
	0.3	$0.4e-78$	$0.9e-101$	$0.4e-157$	$0.3e-219$	$0.2e-257$
	0.2	$0.1e-90$	$0.3e-109$	$0.5e-201$	$0.4e-258$	$0.2e-301$
$\lvert f_4\rvert$	0.1	$0.2e-113$	$0.2e-128$	$0.2e-279$	$0.1e-328$	$0.1e-379$
	−0.2	$0.9e-84$	$0.4e-90$	$0.7e-223$	$0.2e-229$	$0.1e-286$
	−0.1	$0.8e-110$	$0.5e-119$	$0.1e-285$	$0.3e-314$	$0.2e-483$
	1.35	$0.1e-101$	$0.6e-112$	$0.3e-252$	$0.1e-312$	$0.2e-320$
	1.31	$0.1e-77$	$0.1e-86$	$0.3e-170$	$0.1e-236$	$0.1e-233$
$\lvert f_5\rvert$	1.29	$0.1e-69$	$0.4e-78$	$0.1e-141$	$0.5e-211$	$0.5e-203$
	1.15	$0.8e-39$	$0.2e-42$	$0.7e-28$	$0.8e-107$	$0.1e-510$
	1.20	$0.1e-46$	$0.4e-52$	$0.2e-54$	$0.3e-135$	$0.1e-101$
	0.7	$0.7e-109$	$0.1e-122$	$0.1e-288$	$0.2e-334$	$0.1e-380$
	0.6	$0.4e-94$	$0.7e-104$	$0.9e-229$	$0.3e-286$	$0.8e-300$
$\lvert f_6\rvert$	0.5	$0.3e-57$	$0.2e-63$	$0.3e-95$	$0.4e-166$	$0.2e-154$
	0.8	$0.1e-68$	$0.2e-87$	$0.2e-171$	$0.3e-207$	$0.2e-282$
	1.2	$0.6e-36$	$0.6e-52$	$0.1e-151$	$0.2e-97$	$0.1e-112$
	0.65	$0.2e-294$	$0.1e-306$	$0.2e-588$	$0.2e-807$	$0.8e-810$
	0.75	$0.2e-177$	$0.8e-187$	$0.5e-250$	$0.4e-462$	$0.1e-457$
$\lvert f_7\rvert$	0.95	$0.3e-129$	$0.5e-130$	$0.1e-134$	$0.2e-295$	$0.2e-290$
	0.90	$0.1e-137$	$0.3e-140$	$0.2e-153$	$0.3e-322$	$0.9e-318$
	0.80	$0.2e-160$	$0.3e-167$	$0.3e-207$	$0.9e-399$	$0.6e-395$
	−1.0	$0.2e-64$	$0.4e-72$	$0.1e-112$	$0.1e-193$	$0.5e-182$
	−1.1	$0.3e-96$	$0.2e-106$	$0.6e-224$	$0.8e-297$	$0.8e-301$
$\lvert f_8\rvert$	−1.2	$0.6e-132$	$0.1e-144$	$0.3e-345$	$0.8e-412$	$0.6e-431$
	−1.5	$0.5e-50$	$0.6e-71$	$0.5e-100$	$0.2e-155$	$0.5e-275$
	−0.9	$0.1e-47$	$0.2e-52$	$0.5e-46$	$0.1e-135$	$0.6e-105$
	0.9	$0.1e-127$	$0.6e-133$	$0.1e-152$	$0.5e-315$	$0.2e-307$
	0.7	$0.2e-178$	$0.1e-186$	$0.1e-315$	$0.5e-455$	$0.4e-451$
$\lvert f_9\rvert$	0.6	$0.2e-189$	$0.1e-206$	$0.1e-351$	$0.6e-482$	$0.2e-479$
	0.8	$0.6e-144$	$0.1e-149$	$0.5e-206$	$0.9e-356$	$0.3e-350$
	1.0	$0.2e-117$	$0.2e-123$	$0.3e-117$	$0.7e-295$	$0.1e-284$
	1.2	$0.4e-74$	$0.2e-81$	$0.3e-154$	$0.1e-213$	$0.8e-229$
	2.0	$0.7e-26$	$0.1e-52$	$0.2e-75$	$0.5e-76$	$0.1e-107$
$\lvert f_{10}\rvert$	1.5	$0.2e-57$	$0.1e-79$	$0.3e-139$	$0.1e-170$	$0.2e-229$
	1.3	$0.9e-214$	$0.7e-226$	$0.1e-612$	$0.4e-660$	$0.7e-708$
	1.8	$0.3e-33$	$0.2e-76$	$0.1e-83$	$0.1e-97$	$0.7e-134$

Table 3: Comparison of CPU time (in seconds) between some existing methods and our proposed methods.

Function	CPU time					
	Guess	HN3	M3	SL4	JM4	M4
f_1	0.3	0.2867	0.2644	0.3060	0.2449	0.2449
f_2	1.5	0.2943	0.2510	0.3049	0.2682	0.3043
f_3	2.3	0.3019	0.3658	0.3457	0.3562	0.3483
f_4	0.3	0.3091	0.2850	0.2832	0.2399	0.2428
f_5	1.35	0.3399	0.3694	0.3938	0.4149	0.3940
f_6	0.7	0.2896	0.2708	0.2388	0.2613	0.2550
f_7	0.65	0.2517	0.2356	0.2938	0.2644	0.2880
f_8	−1.00	0.2697	0.2279	0.2739	0.2934	0.2900

where

$$\begin{aligned}
T_1 &= G_1\left(a_1+a_2+a_3\right),\\
T_2 &= G_1\left(-2\beta a_2-4\beta a_3\right)C_2+\left(a_1+a_2+a_3\right)\left(G_2+E_1G_1\right),\\
T_3 &= \left(a_1+a_2+a_3\right)\left(E_1G_2+E_2G_1+G_3\right)\\
&\quad+\left(-2\beta a_2-4\beta a_3\right)C_2\left(G_2+E_1G_1\right)\\
&\quad+G_1\left[\left(3\left(\beta^2-2\beta\right)a_2+6\left(\beta^2-2\beta\right)a_3\right)C_3\right.\\
&\qquad\left.+\left(6\beta a_2+\left(12\beta+4\beta^2\right)a_3\right)C_2^2\right],\\
T_4 &= \left(G_2+E_1G_1\right)\left(\left[\left(3\left(\beta^2-2\beta\right)a_2+6\left(\beta^2-2\beta\right)a_3\right)C_3\right]\right.\\
&\qquad\left.+\left[\left(6\beta a_2+\left(12\beta+4\beta^2\right)a_3\right)C_2^2\right]\right)\\
&\quad+\left(E_1G_2+E_2G_1+G_3\right)\left[\left(-2\beta a_2-4\beta a_3\right)C_2\right]\\
&\quad+\left(a_1+a_2+a_3\right)\left(E_2G_2+E_3G_1+E_1G_3+G_4\right)\\
&\quad+G_1\left(\left\{-16\beta a_2+\left(-32\beta-24\beta^2\right)a_3\right\}C_2^3\right.\\
&\qquad+\left\{\left(-6\beta^2+16\beta\right)a_2+\left(-6\beta^3+32\beta\right)a_3\right\}C_2C_3\\
&\qquad+\left\{6\beta\left(2-\beta\right)a_2+\left(-6\beta^3+24\beta\right)a_3\right\}C_3C_2\\
&\qquad\left.+\left\{\left(4(1-\beta)^3-4\right)a_2+2\left(4(1-\beta)^3-4\right)a_3\right\}C_4\right).
\end{aligned} \tag{66}$$

The final error equation of method (61) is given by

$$E^{(k+1)}=\left(-\frac{1}{9}C_2^3+8C_2C_3-C_3C_2+\frac{1}{9}C_4\right)E^{(k)^4}+O\left(E^{(k)^5}\right), \tag{67}$$

which confirms the theorem. □

4. Numerical Testing

4.1. Single Variate Case. In this section, ten different test functions have been considered in Table 1 for single variate case to illustrate the accuracy of the proposed iterative methods. The root of each nonlinear test function is also listed. All computations presented here have been performed in *MATHEMATICA 8*. Many streams of science and engineering require very high precision degree of scientific computations. We consider 1000 digits floating point arithmetic using "*SetAccuracy* []" command. Here we compare the performance of our proposed methods with some well-established third-order and fourth-order iterative methods. In Table 2, we have represented Huen's method by HN3, our proposed third-order method (15) by M3, fourth-order method (17) of [5] by SL4, fourth-order Jarratt's method by JM4, and proposed fourth-order method by M4. The results are listed in Table 2.

An effective way to compare the efficiency of methods is CPU time utilized in the execution of the programme. In present work, the CPU time has been computed using the command "*TimeUsed* []" in *MATHEMATICA*. It is well known that the CPU time is not unique and it depends on the specification of the computer. The computer characteristic is Microsoft Windows 8 Intel(R) Core(TM) i5-3210M CPU@ 2.50 GHz with 4.00 GB of RAM, 64-bit operating system throughout this paper. The mean CPU time is calculated by taking the mean of 10 performances of the programme. The mean CPU time (in seconds) for different methods is given in Table 3.

4.2. Multivariate Case. Further, six nonlinear systems (Examples 9–14) are considered for numerical testing of system of nonlinear equations. Here we compare our proposed third-order method (40) (MM3) with Algorithm (2.2) (NR1) and Algorithm (2.3) (NR2) of [13] and fourth-order method (61) (MM4) with (22) (SH4) of [14] and method (3.4) (BB4) of [15]. The comparison of norm of the function for different iterations is given in Table 4.

Example 9. Consider

$$\begin{aligned}
x_1^2-x_2-19&=0,\\
-x_1^2+\frac{x_2^3}{6}+x_2-17&=0,
\end{aligned} \tag{68}$$

with initial guess $X^{(0)}=(5.1,6.1)^T$, and one of its solutions is $\alpha=(5,6)^T$.

Example 10. Consider

$$\begin{aligned}
-\operatorname{Sin}\left(x_1\right)+\operatorname{Cos}\left(x_2\right)&=0,\\
-\frac{1}{x_2}+\left(x_3\right)^{x_1}&=0,\\
e^{x_1}-\left(x_3\right)^2&=0,
\end{aligned} \tag{69}$$

with initial guess $X^{(0)}=(1,0.5,1.5)^T$, and one of its solutions is $\alpha=(0.9095\cdots0.6612\cdots1.5758\cdots)^T$.

Table 4: Norm of the functions by different methods after first, second, third, and fourth iteration.

Example	Guess	Method	$\|\|F(x^{(1)})\|\|$	$\|\|F(x^{(2)})\|\|$	$\|\|F(x^{(3)})\|\|$	$\|\|F(x^{(4)})\|\|$
Example 9	(5.1, 6.1)	NR1	$3.8774e-4$	$9.2700e-15$	$7.7652e-47$	$4.0858e-143$
		NR2	$3.8774e-4$	$9.2700e-15$	$7.7652e-47$	$4.0858e-143$
		MM3	$1.2657e-4$	$1.0705e-16$	$3.9789e-53$	$1.8320e-162$
		BB4	$2.1416e-5$	$1.1267e-24$	$4.6477e-102$	$1.2561e-411$
		SH4	$1.2923e-5$	$9.2420e-26$	$1.2710e-106$	$4.2184e-430$
		MM4	$3.0768e-6$	$3.9419e-29$	$8.7758e-121$	$2.1039e-487$
Example 10	(1, 0.5, 1.5)	NR1	$3.0006e-2$	$1.3681e-4$	$1.3174e-11$	$1.1754e-32$
		NR2	$2.9765e-2$	$1.3230e-4$	$1.1848e-11$	$8.5484e-33$
		MM3	$9.9051e-3$	$3.7473e-6$	$9.1835e-17$	$1.3035e-48$
		BB4	$2.1133e-2$	$6.9602e-6$	$7.1401e-20$	$7.7987e-76$
		SH4	$1.5676e-2$	$1.1309e-6$	$2.4814e-23$	$5.5195e-90$
		MM4	$6.1451e-3$	$8.1169e-8$	$2.3264e-28$	$6.9642e-110$
Example 11	(0.5, 0.5, 0.5, −0.2)	NR1	$2.3097e-3$	$7.5761e-10$	$5.6516e-30$	$2.8662e-91$
		NR2	$2.3097e-3$	$7.5761e-10$	$5.6516e-30$	$2.8662e-91$
		MM3	$9.4336e-4$	$1.6380e-11$	$1.7401e-35$	$2.5268e-108$
		BB4	$9.1400e-4$	$1.8627e-14$	$8.7599e-59$	$6.9713e-238$
		SH4	$5.3618e-4$	$1.4537e-15$	$2.1746e-63$	$1.7744e-256$
		MM4	$7.7084e-5$	$2.1932e-20$	$3.4979e-84$	$3.6487e-341$
Example 12	(1.0, 2.0)	NR1	$2.1427e-3$	$1.7987e-10$	$1.0504e-31$	$2.0958e-95$
		NR2	$2.1498e-3$	$1.8262e-10$	$1.1001e-31$	$2.4077e-95$
		MM3	$7.6174e-4$	$2.3592e-12$	$7.9632e-38$	$3.0435e-114$
		BB4	$5.3124e-4$	$8.2104e-16$	$3.8411e-63$	$2.8216e-252$
		SH4	$2.9895e-4$	$6.5567e-17$	$1.8332e-67$	$1.5913e-269$
		MM4	$1.0131e-4$	$2.8562e-19$	$3.4019e-77$	$2.4842e-308$
Example 13	(−0.8, 1.1, 1.1)	NR1	$2.9692e-4$	$5.5149e-14$	$3.8063e-40$	$1.2456e-118$
		NR2	$2.9718e-5$	$5.5137e-14$	$3.8044e-40$	$1.2438e-118$
		MM3	$9.8775e-6$	$6.7719e-16$	$2.3491e-46$	$9.7596e-138$
		BB4	$4.0723e-6$	$1.6873e-21$	$9.1974e-83$	$7.8287e-328$
		SH4	$2.1907e-6$	$8.6294e-23$	$3.6734e-87$	$1.1711e-349$
		MM4	$1.0838e-6$	$8.4404e-25$	$1.2327e-96$	$4.3437e-384$
Example 14	(0.5, 1.5)	NR1	$1.8661e-1$	$7.1492e-4$	$4.5647e-11$	$1.1889e-32$
		NR2	$1.7417e-1$	$5.7596e-4$	$2.3870e-11$	$1.7000e-33$
		MM3	$1.2770e-1$	$4.6794e-5$	$4.1708e-15$	$3.0222e-45$
		BB4	$9.8299e-2$	$9.2624e-6$	$8.3046e-22$	$5.3716e-86$
		SH4	$1.0359e-1$	$5.4166e-6$	$4.6302e-23$	$2.4821e-91$
		MM4	$1.4490e-1$	$1.0558e-5$	$6.7122e-22$	$1.0964e-86$

Example 11. Consider

$$\begin{aligned} x_2x_3 + x_4(x_2 + x_3) &= 0, \\ x_1x_3 + x_4(x_1 + x_3) &= 0, \\ x_1x_2 + x_4(x_1 + x_2) &= 0, \\ x_1x_2 + x_1x_3 + x_2x_3 &= 1, \end{aligned} \tag{70}$$

with initial guess $X^{(0)} = (0.5, 0.5, 0.5, -0.2)^T$, and one of its solutions is $\alpha \approx (0.577350, 0.577350, 0.577350, -0.288675)^T$.

Example 12. Consider

$$\begin{aligned} -e^{x_1} + \tan^{-1}(x_2) + 2 &= 0, \\ \tan^{-1}\left(x_1^2 + x_2^2 - 5\right) &= 0, \end{aligned} \tag{71}$$

with initial guess $X^{(0)} = (1.0, 2.0)^T$, and one of its solutions is $\alpha = (1.12906503\cdots 1.930080863\cdots)^T$.

Example 13. Consider

$$\begin{aligned} -e^{-x_1} + x_2 + x_3 &= 0, \\ -e^{-x_2} + x_1 + x_3 &= 0, \\ -e^{-x_3} + x_1 + x_2 &= 0, \end{aligned} \tag{72}$$

with initial guess $X^{(0)} = (-0.8, 1.1, 1.1)^T$, and one of its solutions is $\alpha = (-0.8320\cdots 1.1489\cdots 1.1489\cdots)^T$.

Example 14. Consider

$$\begin{aligned} \log(x_2) - x_1^2 + x_1 x_2 &= 0, \\ \log(x_1) - x_2^2 + x_1 x_2 &= 0, \end{aligned} \tag{73}$$

with initial guess $X^{(0)} = (0.5, 1.5)^T$, and one of its solutions is $\alpha = (1, 1)^T$.

5. Conclusion

In the present work, we have provided a family of third- and optimal fourth-order iterative methods which yield some existing as well as many new third-order and fourth-order iterative methods. The multivariate case of these methods has also been considered. The efficiency of our methods is supported by Table 2 and Table 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. Weerakoon and T. G. I. Fernando, "A variant of Newton's method with accelerated third-order convergence," *Applied Mathematics Letters*, vol. 13, no. 8, pp. 87–93, 2000.

[2] H. H. H. Homeier, "On Newton-type methods with cubic convergence," *Journal of Computational and Applied Mathematics*, vol. 176, no. 2, pp. 425–432, 2005.

[3] C. Chun and Y. Kim, "Several new third-order iterative methods for solving nonlinear equations," *Acta Applicandae Mathematicae*, vol. 109, no. 3, pp. 1053–1063, 2010.

[4] H. T. Kung and J. F. Traub, "Optimal order of one-point and multipoint iteration," *Journal of Computational and Applied Mathematics*, vol. 21, no. 4, pp. 643–651, 1974.

[5] F. Soleymani, "Two new classes of optimal Jarratt-type fourth-order methods," *Applied Mathematics Letters*, vol. 25, no. 5, pp. 847–853, 2012.

[6] M. Sharifi, D. K. R. Babajee, and F. Soleymani, "Finding the solution of nonlinear equations by a class of optimal methods," *Computers and Mathematics with Applications*, vol. 63, no. 4, pp. 764–774, 2012.

[7] S. K. Khattri and S. Abbasbandy, "Optimal fourth order family of iterative methods," *Matematicki Vesnik*, vol. 63, no. 1, pp. 67–72, 2011.

[8] W. Gautschi, *Numerical Analysis: An Introduction*, Birkhauser, Boston, Mass, USA, 1997.

[9] J. F. Traub, *Iterative Methods for Solution of Equations*, Chelsea Publishing, New York, NY, USA, 1997.

[10] M. Grau-Sánchez and J. L. Díaz-Barrero, "Zero-finder methods derived using Runge-Kutta techniques," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5366–5376, 2011.

[11] K. Huen, "Neue methode zur approximativen integration der differentialge-ichungen einer unabhngigen variablen," *Zeitschrift für angewandte Mathematik und Physik*, vol. 45, pp. 23–38, 1900.

[12] F. Soleymani and D. K. R. Babajee, "Computing multiple zeros using a class of quartically convergent methods," *Alexandria Engineering Journal*, vol. 52, pp. 531–541, 2013.

[13] M. A. Noor and M. Waseem, "Some iterative methods for solving a system of nonlinear equations," *Computers and Mathematics with Applications*, vol. 57, no. 1, pp. 101–106, 2009.

[14] J. R. Sharma, R. K. Guha, and R. Sharma, "An efficient fourth-order weighted-Newton method for systems of nonlinear equations," *Numerical Algorithms*, vol. 62, pp. 307–323, 2013.

[15] D. K. R. Babajee, A. Cordero, F. Soleymani, and J. R. Torregrosa, "On a novel fourth-order algorithm for solving systems of nonlinear equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 165452, 12 pages, 2012.