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Review Article

A Review on Unique Existence Theorems in Lightlike Geometry

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This is a review paper of up-to-date research done on the existence of unique null curves, screen distributions, Levi-Civita connection, symmetric Ricci tensor, and scalar curvature for a large variety of lightlike submanifolds of semi-Riemannian (in particular, Lorentzian) manifolds, supported by examples and an extensive bibliography. We also propose some open problems.

1. Introduction

The theory of Riemannian and semi-Riemannian manifolds (M, g) and their submanifold is one of the most interesting areas of research in differential geometry. Most of the work on the Riemannian, semi-Riemannian, and Lorentzian manifolds has been described in the standard books by Chen [1], Beem and Ehrlich [2], and O'Neill [3]. Berger's book [4] includes the major developments of Riemannian geometry since 1950, covering the works of differential geometers of that time and many cited therein. In general, an inner product g on a vector space $\mathbf{V}$ is of type (r, ℓ, m), where $r = \dim\{u \in \mathbf{V} \mid g(u, v) = 0 \text{ for all } v \in \mathbf{V}\}$, $\ell = \sup\{\dim W \mid W \subset \mathbf{V} \text{ with } g(w, w) < 0 \text{ for all nonzero } w \in W\}$, and $m = \sup\{\dim W \mid W \subset \mathbf{V} \text{ with } g(w, w) > 0 \text{ for all nonzero } w \in W\}$. Kupeli [5] called a manifold (M, g) of this type a *singular semi-Riemannian* manifold if M admits a Koszul derivative; that is, g is Lie parallel along the degenerate vector fields on M. Based on this, he studied the intrinsic geometry of such degenerate manifolds. On the other hand, a degenerate submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ may not be studied intrinsically since due to the degenerate tensor field g on M one cannot use, in general, the geometry of $\overline{M}$. To overcome this difficulty, Kupeli used the quotient space $TM^* = TM/\mathrm{Rad}(TM)$ and the canonical projection $P : TM \to TM^*$ for the study of intrinsic geometry of M, where $\mathrm{Rad}(TM)$ is its radical distribution.

For a general study of extrinsic geometry of degenerate submanifolds (popularly known as *lightlike submanifolds*) of a semi-Riemannian manifold, we refer to three books [6–8] published in 1996, 2007, and 2010, respectively. A submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called lightlike submanifold if it is a lightlike manifold with respect to the degenerate metric g induced from $\overline{g}$ and $S(TM)$ is a nondegenerate screen distribution which is complementary of the radical distribution $\mathrm{Rad}(TM)$ in TM; that is,

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM), \tag{1}$$

where $\oplus_{\mathrm{orth}}$ is a symbol for orthogonal direct sum. The technique of using a nondegenerate $S(TM)$ was first introduced by Bejancu [9] for null curves and then by Bejancu and Duggal [10] for hypersurfaces to study the induced geometry of lightlike submanifolds. Unfortunately, (i) the induced objects on M depend on $S(TM)$ which, in general, is not unique. This raises the question of the existence of unique or canonical null curves and screen distributions in lightlike geometry. (ii) The induced connection ∇ on M is not a unique metric (Levi-Civita) connection and depends on both the induced metric g and the choice of a screen, which creates a problem in justifying that the induced objects on M are geometrically stable. (iii) The induced Ricci tensor of M is not a symmetric tensor so, in general, it does not have a geometric or physical meaning similar to the Riemannian Ricci tensor, and (iv) since the inverse of degenerate metric g does not exist, one fails to have well-defined concept of a scalar curvature by contracting Ricci tensor. At the time of the 1996 book [6], nothing much on the above anomalies was available. In 2007, I published a report with limited information available on

how to deal with this nonuniqueness problem for null curves and hypersurfaces [11]. Since then considerable further work has been done on these issues, in particular reference to all types of null curves and submanifolds, which has provided strong foundation for the lightlike geometry.

The objective of this second report is to review up-to-date results on canonical or unique existence of all types of null curves and screen distributions and, then, find those lightlike submanifolds which also admit a unique metric connection, a symmetric Ricci tensor, and how to recover the induced scalar curvature, subject to some reasonable geometric conditions. We also propose open problems. Our approach is to give brief information on the motivation for dealing with each anomaly, chronological development of the main results and a sketch of their proofs with examples. In order to include a large number of results in one paper, we provide a good bibliography with the aim to encourage those wishing to pursue this subject further. More details on these and related works may be seen in Bibliography of Lightlike Geometry prepared by Sahin [12].

2. Canonical or Unique Nongeodesic Null Curves

Let C be a smooth curve immersed in an $(m+2)$-dimensional proper semi-Riemannian manifold $(M = M_q^{m+2}, g)$ of a constant index $q \geq 1$. By proper we mean that q is *nonzero*. With respect to a local coordinate neighborhood $\mathcal{U}$ on C and a parameter t, C is given by

$$\begin{aligned} &x^i = x^i(t), \quad i \in \{0, \ldots, m+1\}, \\ &\operatorname{rank}\left(dx_t^0 \cdots dx_t^{m+1}\right) = 1, \quad \forall t \in I, \end{aligned} \tag{2}$$

where I is an open interval of a real line and we denote each dx^i/dt by dx_t^i. The nonzero tangent vector field on $\mathcal{U}$ is given by $d_t \equiv (dx_t^0, \ldots, dx_t^{m+1}) \equiv \xi$.

Suppose the curve C is a null curve which preserves its causal character. Then, all its tangent vectors are null. Thus, C is a null curve if and only if at each point x of C we have $g(\xi, \xi) = 0$. The normal bundle of TC is given by

$$\begin{aligned} &TC^\perp = \{X \in \Gamma(TM) : g(X, \xi) = 0\}, \\ &\dim (TC^\perp)_x = m+1. \end{aligned} \tag{3}$$

However, null curves behave differently compared to the nonnull curves as follows:

(1) $TC^\perp$ is also a null bundle subspace of TM,

(2) $TC \cap TC^\perp = TC \rightarrow TC \oplus TC^\perp \neq TM$.

Thus, contrary to the case of nonnull curves, since the normal bundle $TC^\perp$ contains the tangent bundle TC of C, the sum of these two bundles is not the whole of the tangent bundle TM. In other words, a vector of T_xM cannot be decomposed uniquely into a component tangent to C and a component perpendicular to C. Moreover, since the length of any arc of a null curve is zero, arc-length parameter makes no sense for null curves. For these reasons, in general, a Frenet frame (constructed by Bejancu [9] in 1994) on a Lorentzian manifold M along a null curve C depends on the choice of a pseudoparameter on C and a complementary (but not orthogonal) vector bundle $S(TC^\perp)$ to C in $TC^\perp$, calling its screen distribution. In the following, we review how one can generate a canonical or unique set of Frenet equations subject to reasonable geometric conditions. We discuss this in two subsections of null curves in Lorentzian and semi-Riemannian manifolds (of index $q > 1$), respectively.

2.1. Null Curves in Lorentzian Manifolds ($q = 1$). The main idea (first used by Cartan [13] in 1937 followed by Bonnor [14] in 1969) is to choose minimum number of curvature functions in the Frenet equations. We need the following two geometric conditions on a curve $C(p)$:

(a) $C(p)$ is nongeodesic with respect to a pseudo-arc-parameter p,

(b) choose $C(p)$ such that its first curvature function is of unit length.

Let $C(p)$ be a nongeodesic null curve of a Minkowski spacetime $(M = \mathbf{R}_1^3, g)$ with a Frenet frame $\{\xi, N, W\}$ where p is a pseudo-arc-parameter and

$$g(\xi, N) = 1 = g(W, W), \qquad g(N, N) = g(\xi, \xi) = 0. \tag{4}$$

To deal with the problem of nonuniqueness, Cartan constructed the following, called Cartan Frenet frame, for $C(p)$ by using the above two conditions:

$$\nabla_\xi \xi = W, \qquad \nabla_\xi N = \tau W, \qquad \nabla_\xi W = -\tau\xi - N, \tag{5}$$

where the torsion function τ is invariant up to a sign, under Lorentzian transformations. The above frame is now called the *null Cartan frame* and the corresponding curve is the *null Cartan curve* (also, see Bonnor [14] for the 4-dimensional case using Cartan method). In 1994 Bejancu [9] proved the following fundamental existence and uniqueness theorem for null curves of an $(m+2)$-dimensional Minkowski space (R_1^{m+2}, g). First we define in R_1^{m+2} a quasiorthonormal basis

$$\begin{aligned} &\widehat{W}_o = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right), \\ &\widehat{W}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right), \\ &\widehat{W}_2 = (0, 0, 1, 0, , \ldots, 0), \ldots, \\ &\widehat{W}_{m+1} = (0, 0, \ldots, 0, 1), \end{aligned} \tag{6}$$

where $\{\widehat{W}_o, \widehat{W}_1\}$ are null vectors such that $g(\widehat{W}_o, \widehat{W}_1) = 1$ and $\{\widehat{W}_2, \ldots, \widehat{W}_{m+1}\}$ are orthonormal spacelike vectors. It is easy to see that

$$\widehat{W}_o^i \widehat{W}_1^j + \widehat{W}_o^j \widehat{W}_1^i + \sum_{\alpha=2}^{m+1} \widehat{W}_\alpha^i \widehat{W}_\alpha^j = h^{ij} \tag{7}$$

for any $i, j, \in \{0, \ldots, m+1\}$, where we put

$$h^{ij} = \begin{cases} -1, & i = j = 0 \\ 1, & i = j \neq 0 \\ 0, & i \neq j. \end{cases} \tag{8}$$

Theorem 1 (see [9]). *Let $k_1, \ldots, k_{2m} : [-\epsilon, \epsilon] \to R$ be everywhere continuous functions, let $x_o = (x_o^i)$ be a fixed point of R_1^{m+2}, let and $\{\widehat{W}_o, \ldots, \widehat{W}_{m+1}\}$ be the quasiorthonormal basis in* (6). *Then, there exists a unique null curve C of R_1^{m+2} given by the equations $x^i = x^i(p)$, $p \in [-\epsilon, \epsilon]$, where p is a distinguished parameter on C, such that $x_o^i = x^i(0)$ and $k_1, \ldots, k_{2m}$ are the curvature functions of C with respect to a Frenet frame $F = \{d/dt, N, W_2, \ldots, W_{m+1}\}$ that satisfies*

$$\begin{aligned} &\frac{d}{dp}(0) = \widehat{W}_0, \qquad N(0) = \widehat{W}_1, \\ &W_\alpha(0) = \widehat{W}_\alpha, \quad \alpha \in \{2, \ldots, m+1\}. \end{aligned} \tag{9}$$

Proof. We denote $\nabla_\xi X$ by X'. Using the general Frenet equations [6, page 55], consider the system of differential equations

$$\begin{aligned} W_o'(p) &= k_1 W_2 \\ W_1'(p) &= k_2 W_2 + k_3 W_3 \\ W_2'(p) &= -k_2 W_o - k_1 W_1 + k_4 W_3 + k_5 W_4 \\ W_3'(p) &= -k_3 W_o - k_4 W_2 + k_6 W_4 + k_7 W_5 \\ &\vdots \\ W_{m-1}'(p) &= -k_{2m-5} W_{m-3} - k_{2m-4} W_{m-2} \\ &\quad + k_{2m-3} W_m + k_{2m-1} W_{m+1} \\ W_m'(p) &= -k_{2m-4} W_{m-2} - k_{2m-2} W_{m-1} + k_{2m} W_{m+1} \\ W_{m+1}'(p) &= k_{2m-1} W_{m-1} - k_{2m} W_m. \end{aligned} \tag{10}$$

Then there exists a unique solution $\{W_o, \ldots, W_{m+1}\}$ satisfying the initial conditions $W_i(0) = \widehat{W}_i, i \in \{0, \ldots, m+1\}$. Furthermore, $\{W_o(p), \ldots, W_{m+1}(p)\}$ is a quasiorthonormal basis such that $\{W_o(p), W_1(p)\}$ and $\{W_2(p), \ldots, W_{m+1}(p)\}$ are lightlike and spacelike, respectively, for each $p \in [-\epsilon, \epsilon]$. Following Bonnor [14], it is proved that

$$F = \left\{ \frac{d}{dp} = W_o(p), N(p) = W_1(p), \right. \\ \left. W_2(p), \ldots, W_{m+1}(p) \right\} \tag{11}$$

is a Frenet frame for C with curvature functions $\{k_1, \ldots, k_{2m}\}$ and p is the distinguished parameter on C, which completes the proof. □

In year 2001, the above theorem was generalized by Ferrández et al. [15] by constructing a canonical representation for nongeodesic null Cartan curves in a general $(m+2)$-dimensional Lorentzian manifold as follows.

Theorem 2 (see [15]). *Let $C(p)$ be a null curve of an orientable Lorentzian manifold (M_1^{m+2}, g) with a pseudo-arc-parameter p such that a basis of $T_{C(p)}M$ for all p is given by $\{C'(p), C''(p), \ldots, C^{(m+2)}(p)\}$. Then there exists exactly one Frenet frame $F = \{\xi, N, W_1, \ldots, W_m\}$, satisfying*

$$\begin{aligned} \nabla_\xi \xi &= W_1, \\ \nabla_\xi N &= \kappa_1 W_1 + \kappa_2 W_2, \\ \nabla_\xi W_1 &= -\kappa_1 \xi - N, \\ \nabla_\xi W_2 &= -\kappa_2 \xi + \kappa_3 W_3, \\ &\vdots \\ \nabla_\xi W_i &= -\kappa_i W_{i-1} + \kappa_{i+1} W_{i+1}, \quad i \in \{3, \ldots, m-1\}, \\ \nabla_\xi W_m &= -\kappa_m W_{m-1} \end{aligned} \tag{12}$$

and fulfilling the following two conditions:

(i) *$\{C'(p), C''(p), \ldots, C^{(i+2)}(p)\}$ and $\{\xi, N, W_1, \ldots, W_i\}$ have the same orientation for $2 \leq i \leq m-1$,*

(ii) *$\{\xi, N, W_1, \ldots, W_m\}$ is positively oriented and $\kappa_i > 0$, for all $i \geq 2$.*

The above Frenet frame of the equations, its curvature functions, and the corresponding curve C are called the Cartan frame, the Cartan curvatures, and the null Cartan curve, respectively.

Corollary 3. *The Cartan curvatures of a curve C in M_1^{m+2} are invariant under Lorentzian transformations.*

Example 4. Let $C(p) = (1/\sqrt{3})(\sqrt{2}\sinh p, \sqrt{2}\cosh p, \sin p, \cos p, p)$ be a null curve of R_1^5 with vector fields of its Frenet frame F given by

$$\begin{aligned} \xi &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\cosh p, \sqrt{2}\sinh p, \cos p, -\sin p, 1\right), \\ N &= -\frac{1}{6\sqrt{3}}\left(5\sqrt{2}\cosh p, 5\sqrt{2}\sinh p, -7\cos p, 7\sin p, -1\right), \\ W_1 &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\sinh p, \sqrt{2}\cosh p, -\sin p, -\cos p, 0\right), \\ W_2 &= \frac{1}{\sqrt{6}}\left(\sqrt{2}\sinh p, \sqrt{2}\cosh p, 2\sin p, 2\cos p, 0\right), \\ W_3 &= -\frac{1}{3\sqrt{2}}\left(\sqrt{2}\cosh p, \sqrt{2}\sinh p, -2\cos p, 2\sin p, 4\right). \end{aligned} \tag{13}$$

Then, it is easy to obtain the following Frenet equation:

$$\begin{aligned}
&\nabla_\xi \xi = W_1, \qquad \nabla_\xi N = -\frac{1}{6}W_1 - \frac{4}{3\sqrt{2}}W_2,\\
&\nabla_\xi W_1 = \frac{1}{6}\xi - N, \qquad \nabla_\xi W_2 = \frac{4}{3\sqrt{2}}\xi + \frac{\sqrt{2}}{3}W_3,\\
&\nabla_\xi W_3 = -\frac{\sqrt{2}}{3}W_2.
\end{aligned} \tag{14}$$

Remark 5. Each null Cartan curve is a canonical representation for nongeodesic null curves. For a collection of papers on the use of null Cartan curves, soliton solutions [16], null Cartan helices, and relativistic particles involving the curvature of 3 and 4 dimensional null curves and their geometric/physical applications, we refer to website of Lucas [17] and a Duggal and Jin book [7].

2.2. Null Bertrand Curves. A curve C in $\mathbf{R}^3$, parameterized by the arclength, is a Bertrand curve [18, page 41] if and only if C is a plane curve or its curvature and torsion are in a linear relation, with constant coefficients. In 2003, Honda and Inoguchi [19] studied a pair of null curves $(C(p), \overline{C}(\overline{p}))$, called a null Bertrand pair which is defined as follows. Let $\{\xi, N, W_1\}$ and $\{\overline{\xi}, \overline{N}, \overline{W}_1\}$ be the Frenet frame of $C(p)$ and $\overline{C}(\overline{p})$, where p and $\overline{p}$ are their respective pseudo-arc-parameters. This pair $(C, \overline{C})$ is said to be a null Bertrand pair (with $\overline{C}$ being a Bertrand mate of C and vice versa) if W_1 and $\overline{W}_1$ are linearly dependent. They established a relation of Bertrand pairs with null helices in $\mathbf{R}^3_1$. Recently, Inoguchi and Lee have published the following result on the existence of a Bertrand mate.

Theorem 6 (see [20]). *Let $C(p)$ be a null Cartan curve in $\mathbf{R}^3_1$, where p is a pseudo-arc-parameter. Then C admits a Bertrand mate $\overline{C}$ if and only if C and $\overline{C}$ have same nonzero constant curvatures. Moreover, $\overline{C}$ is congruent to C.*

Example 7. Let $C(p) = (p, \cos p, \sin p)$ be a null helix with Frenet frame

$$\begin{aligned}
F = \Big\{ &C', N = \frac{1}{2}(-1, -\sin p, -\cos p),\\
&W = (0, -\cos p, -\sin p) \Big\}.
\end{aligned} \tag{15}$$

It is easy to see that its torsion $\tau = 1/2$. Define $\overline{C} = C + 2W$ with $\overline{p} = p$. Then, $\overline{C} = (p, -\cos p, -\sin p)$. Therefore, $\overline{C}$ is congruent to the original curve C.

In 2005, Çöken and Çiftçi [21] generalized the $\mathbf{R}^3_1$ case of Bertrand curves for the Minkowski space $\mathbf{R}^4_1$ using the following Frenet equations (see (24)):

$$\begin{aligned}
\nabla_\xi \xi &= W_1,\\
\nabla_\xi N &= \kappa_1 W_1 + \kappa_2 W_2,\\
\nabla_\xi W_1 &= -\kappa_1 \xi - N,\\
\nabla_\xi W_2 &= -\kappa_2 \xi
\end{aligned} \tag{16}$$

for a Frenet frame $\{\xi, N, W_1, W_2\}$. Following is their characterization theorem.

Theorem 8 (see [21]). *A null Cartan curve in $\mathbf{R}^4_1$ is a Bertrand null curve if and only if κ_1 is nonzero constant and κ_2 is zero.*

In a recent paper, Mehmet and Sadik [22] have shown that a null Cartan curve in $\mathbf{R}^4_1$ is not a Bertrand curve if the derivative vectors $\{C', C'', C^{(3)}, C^{(4)}\}$ (see Theorem 2) of the curve are linearly independent.

Remark 9. In Theorem 2, Ferrández et al. [15] assumed the linear independence of the derivative vectors of the curve to obtain a unique Cartan frame. However, in 2010, Sakaki [23] proved that the assumption in this Theorem 2 can be lessened for obtaining a unique Cartan frame in $\mathbf{R}^n_1$.

Open Problem. We have seen in this section that the study on Bertrand curves is focused on 3- and 4-dimensional Minkowski spaces. Theorem 2 of Ferrández et al. [15] on unique existence of null Cartan curves in a Lorentzian manifold has opened the possibility of research on null Bertrand curves and null Bertrand mates in a 3-, 4-, and also n-dimensional Lorentzian manifold and their relation with the corresponding unique null Cartan curves.

2.3. Null Curves in Semi-Riemannian Manifolds of Index $q > 1$. Let C be a null curve of a semi-Riemannian manifold (M^{m+2}_q, g) of index $q > 1$. Its screen distribution $S(TC^\perp)$ is semi-Riemannian of index $q - 1$. Therefore, contrary to the case of $q = 1$, any of its base vector $\{W_1, \ldots, W_m\}$ might change its causal character on $\mathcal{U} \subset C$. This opens the possibility of more than one type of Frenet equations. To deal with this possibility, in 1999 Duggal and Jin [24] studied the following two types of Frenet frames for $q = 2$.

Type 1 Frenet Frames for $q = 2$. For a null curve C of M^{m+2}_2, any of its screen distributions is Lorentzian. Denote its general Frenet frame by

$$F_1 = \{\xi, N, W_1, \ldots, W_m\}, \tag{17}$$

when one of W_i is timelike. Call F_1 a Frenet frame of Type 1. Similar to the case of $q = 1$, we have the following general Frenet equations of Type 1:

$$\begin{aligned}
\nabla_\xi \xi &= h\xi + \kappa_1 W_1,\\
\nabla_\xi N &= -hN + \kappa_2 W_1 + \kappa_3 W_2,\\
\epsilon_1 \nabla_\xi W_1 &= -\kappa_2 \xi - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3,
\end{aligned}$$

$$\begin{aligned}
\epsilon_2\nabla_\xi W_2 &= -\kappa_3\xi - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4,\\
\epsilon_3\nabla_\xi W_3 &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4 + \kappa_9 W_5,\\
&\vdots\\
\epsilon_{m-1}\nabla_\xi W_{m-1} &= -\kappa_{2m-3}W_{m-3}\\
&\quad -\kappa_{2m-2}W_{m-2} + \kappa_{2m}W_m,\\
\epsilon_m\nabla_\xi W_m &= -\kappa_{2m-1}W_{m-2} - \kappa_{2m}W_{m-1},
\end{aligned} \tag{18}$$

where h and $\{\kappa_1,\dots,\kappa_{2m}\}$ are smooth functions on $\mathcal{U}$, $\{W_1,\dots,W_m\}$ is an orthonormal basis of $\Gamma(S(TC^\perp)_{|\mathcal{U}})$, and $(\epsilon_1,\dots,\epsilon_m)$ is the signature of the manifold M_2^{m+2} such that $\epsilon_i\delta_{ij} = g(W_i,W_j)$. The functions $\{\kappa_1,\dots,\kappa_{2m}\}$ are called curvature functions of C with respect to F_1.

Example 10. Let C be a null curve in $\mathbf{R}_2^6$ given by

$$C:\left(\cosh t, \frac{1}{27}t^3 + 2t, \frac{1}{3}t^2, \frac{1}{27}t^3, \sqrt{3}t, \sinh t\right), \quad t\in\mathbf{R}. \tag{19}$$

Choose the following general Type 1 frame $F = \{\xi, N, W_1, W_2, W_3, W_4\}$:

$$\begin{aligned}
\xi &= \left(\sinh t, \frac{t^2}{9} + 2, \frac{2}{3}t, \frac{t^2}{9}, \sqrt{3}, \cosh t\right),\\
N &= \frac{1}{2}\left(\sinh t, -\frac{t^2}{9} - 2, -\frac{2}{3}t, -\frac{t^2}{9}, -\sqrt{3}, \cosh t\right),\\
W_1 &= -\frac{3}{\sqrt{5}}\left(\cosh t, \frac{2}{9}t, \frac{2}{3}, \frac{2}{9}t, 0, \sinh t\right),\\
W_2 &= -\frac{1}{\sqrt{5}}\left(2\cosh t, t, 3, t, 0, 2\sinh t\right),\\
W_3 &= -\left(0, \frac{t^2}{9} + \frac{3}{2}, \frac{2}{3}t, \frac{t^2}{9} - \frac{1}{2}, \sqrt{3}, 0\right),\\
W_4 &= \left(0, \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, 1, 0\right).
\end{aligned} \tag{20}$$

Using the general Frenet equations, we obtain

$$\begin{gathered}
\nabla_\xi\xi = -\frac{\sqrt{5}}{3}W_1, \qquad \nabla_\xi N = -\frac{13}{6\sqrt{5}}W_1 + \frac{2}{\sqrt{5}}W_2,\\
\epsilon_1\nabla_\xi W_1 = \frac{13}{6\sqrt{5}}\xi + \frac{\sqrt{5}}{3}N + \frac{4}{3\sqrt{5}}W_3,\\
\epsilon_2\nabla_\xi W_2 = -\frac{2}{\sqrt{5}}\xi - \frac{2}{\sqrt{5}}W_3,\\
\epsilon_3\nabla_\xi W_3 = -\frac{4}{3\sqrt{5}}W_1 + \frac{2}{\sqrt{5}}W_2,\\
\epsilon_3\nabla_\xi W_4 = 0, \qquad h = 0.\\
\kappa_1 = -\frac{\sqrt{5}}{3}, \qquad \kappa_2 = -\frac{13}{6\sqrt{5}},\\
\kappa_3 = \frac{2}{\sqrt{5}}, \qquad \kappa_4 = 0, \qquad \kappa_5 = \frac{4}{3\sqrt{5}},\\
\kappa_6 = -\frac{2}{\sqrt{5}}, \qquad \kappa_7 = 0, \qquad \kappa_8 = 0.
\end{gathered} \tag{21}$$

Type 2 Frenet Frames for $q = 2$. Construct a quasiorthonormal basis consisting of the two null vector fields ξ and N and another two null vector fields L_i and L_{i+1} such that

$$\begin{gathered}
L_i = \frac{W_i + W_{i+1}}{\sqrt{2}}, \qquad L_{i+1} = \frac{W_{i+1} - W_i}{\sqrt{2}},\\
g\left(L_i, L_{i+1}\right) = 1,
\end{gathered} \tag{22}$$

where W_i and W_{i+1} are timelike and spacelike, respectively, all taken from F_1. The remaining $(m-2)$ subset $\{W_\alpha\}$ of F_1 has all spacelike vector fields. There are $(m-1)$ choices for L_i for a Frenet frame of the form

$$F_2 = \{\xi, N, W_1, \dots, L_i, L_{i+1}, \dots, W_m\}. \tag{23}$$

Denote F_2 Frame by Type 2. Following exactly as in the previous case, we have the following general Frenet equations of Type 2:

$$\begin{aligned}
\nabla_\xi\xi &= h\xi + \kappa_1 W_1 + \tau_1 W_2,\\
\nabla_\xi N &= -hN + \kappa_2 W_1 + \kappa_3 W_2 + \tau_2 W_3,\\
\nabla_\xi W_1 &= \kappa_2\xi + \kappa_1 N - \kappa_4 W_2 - \kappa_5 W_3 - \tau_3 W_4,\\
\nabla_\xi W_2 &= -\kappa_3\xi - \tau_1 N - \kappa_4 W_1\\
&\quad + \kappa_6 W_3 + \kappa_7 W_4 + \tau_4 W_5,\\
\nabla_\xi W_3 &= -\tau_2\xi - \kappa_5 W_1 - \kappa_6 W_2\\
&\quad + \kappa_8 W_4 + \kappa_9 W_5 + \tau_5 W_6,\\
\nabla_\xi W_4 &= -\tau_3 W_1 - \kappa_7 W_2 - \kappa_8 W_3\\
&\quad + \kappa_{10} W_5 + \kappa_{11} W_6 + \tau_6 W_7\\
&\vdots\\
\nabla_\xi W_{m-1} &= -\tau_{m-2}W_{m-4} - k_{2m-3}W_{m-3}\\
&\quad - k_{2m-2}W_{m-2} + k_{2m}W_m,\\
\nabla_\xi W_m &= -\tau_{m-1}W_{m-3} - k_{2m-1}W_{m-2}\\
&\quad - k_{2m}W_{m-1}.
\end{aligned} \tag{24}$$

Example 11. Let C be a null curve in $\mathbf{R}_2^6$ given by

$$C : \left(\cosh t, \frac{1}{12}t^3 + 2t, \frac{1}{2}t^2, \frac{1}{12}t^3, \sqrt{3}t, \sinh t\right), \quad t \in R. \tag{25}$$

Choose a Frenet frame $F_2 = \{\xi, N, L_1, L_2, W_3, W_4\}$ of Type 2 as follows:

$$\begin{aligned}
\xi &= \left(\sinh t, \frac{t^2}{4} + 2, t, \frac{t^2}{4}, \sqrt{3}, \cosh t\right),\\
N &= \frac{1}{2}\left(\sinh t, -\frac{t^2}{4} - 2, -t, -\frac{t^2}{4}, -\sqrt{3}, \cosh t\right),\\
L_1 &= \frac{1}{\sqrt{2}}\left(\cosh t, \frac{t}{2}, 1, \frac{t}{2}, 0, \sinh t\right),\\
L_2 &= \frac{1}{\sqrt{2}}\left(-\cosh t, \frac{t}{2}, 1, \frac{t}{2}, 0, -\sinh t\right),\\
W_3 &= -\left(0, \frac{t^2}{4} + \frac{3}{2}, t, \frac{t^2}{4} - \frac{1}{2}, \sqrt{3}, 0\right),\\
W_4 &= \left(0, \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, 1, 0\right).
\end{aligned} \tag{26}$$

It is easy to obtain the following Frenet equations for the above frame F_2:

$$\begin{aligned}
&\nabla_\xi \xi = \sqrt{2}L_1, \qquad h = 0, \qquad \nabla_\xi N = -\frac{1}{\sqrt{2}}L_2,\\
&\nabla_\xi L_1 = -\frac{1}{\sqrt{2}}\xi + \frac{1}{\sqrt{2}}L_1,\\
&\nabla_\xi L_2 = -\sqrt{2}N + \frac{1}{\sqrt{2}}W_3,\\
&\nabla_\xi W_3 = -\frac{1}{\sqrt{2}}L_1 - \frac{1}{\sqrt{2}}L_2,\\
&\nabla_\xi W_4 = 0.
\end{aligned} \tag{27}$$

Remark 12. Note that there are m and $m-1$ different choices of constructing Frenet frames and their Frenet equations of Type 1 and Type 2, respectively. Moreover, Type 2 is preferable as it is invariant with respect to the change of its causal character on $\mathcal{U} \subset C$. Also, see [25] on null curves of R_2^{m+2}.

Frenet Frames of Type $q(\geq 3)$. Using the above procedure, we first construct Frenet frames of null curves C in M_3^{m+2}. Their screen distribution $S(TC^\perp)$ is of index 2. Therefore, we have 2 timelike vector fields in $\{W_1, \ldots, W_m\}$. To understand this, take a case when $\{W_1, W_2\}$ are timelike. The construction of Type 1 and Type 2 frames is exactly the same as that in the case $q = 2$ so we give details for Type 3. Transform the Frenet frame F_1 of Type 1 into another frame which consists of two null vector fields ξ and N and additional four null vector fields L_1, L_2, L_3, and L_4 such that

$$\begin{aligned}
L_1 &= \frac{W_1 + W_2}{\sqrt{2}}, \qquad L_2 = \frac{W_2 - W_1}{\sqrt{2}}, \qquad g(L_1, L_2) = 1,\\
L_3 &= \frac{W_3 + W_4}{\sqrt{2}}, \qquad L_4 = \frac{W_4 - W_3}{\sqrt{2}}, \qquad g(L_3, L_4) = 1.
\end{aligned} \tag{28}$$

The remaining $(m-4)$ vector fields of subset $\{W_\alpha\}$ of F_1 are all spacelike. In this case, we have a Frenet frame of the form

$$F_3 = \{\xi, N, L_1, L_2, L_3, L_4, W_5, \ldots, W_m\}. \tag{29}$$

Denote F_3 frame by Type 3 which is preferable choice as any of its vector fields will not change its causal character on $\mathcal{U} \subset C$. In this way one can use all possible choices of two timelike vector fields from F_1 and construct corresponding forms of Frenet frames of Type 3.

The above procedure can be easily generalized to show that the null curves of M_q^{m+2} have F_q Frenet frames of Type 1, Type 2, …, Type q. Also, there are a variety of each of such type and their corresponding Frenet equations.

Precisely, if $q = 1$, then M_1^{m+2} has Type 1 of Frenet frames; if $q = 2$, then M_2^{m+2} has two types of Frenet frames, labeled Type 1 and Type 2, up to the signs of W_i, and if $q = n$, then M_n^{m+2} have n-types, labeled Type 1, Type 2, …, Type n, up to the signs of W_i. However, for each $q = n > 1$, only one frame of Type n will be a preferable frame as it is invariant with respect to the change of its causal character on $\mathcal{U} \subset C$.

Example 13. Let C be a null curve in $\mathbf{R}_3^6$ given by

$$C : (\cos t, \sin t, \sinh t, \cosh t, t, t) \tag{30}$$

with Type 3 Frenet frame $F_3 = \{\xi, N, L_1, L_2, L_3, L_4\}$ given as follows:

$$\begin{aligned}
\xi &= (-\sin t, \cos t, \cosh t, \sinh t, 1, 1),\\
N &= \frac{1}{2}(\sin t, -\cos t, -\cosh t, -\sinh t, 1, -1),\\
L_1 &= (-\cos t, -\sin t, \sinh t, \cosh t, 0, 0),\\
L_2 &= \frac{1}{2}(\cos t, \sin t, \sinh t, \cosh t, 0, 0),\\
L_3 &= (-\sin t, \cos t, 0, 0, 0, 1),\\
L_4 &= (0, 0, \cosh t, \sinh t, 0, 1).
\end{aligned} \tag{31}$$

Then one can calculate its following Frenet equations:

$$\begin{aligned}
&\nabla_\xi \xi = L_1, \qquad \nabla_\xi N = -\frac{1}{2}L_1,\\
&\nabla_\xi L_1 = -L_3 + L_4,\\
&\nabla_\xi L_2 = \frac{1}{2}\xi - N - \frac{1}{2}(L_3 + L_4),\\
&\nabla_\xi L_3 = \frac{1}{2}L_1 - L_2, \qquad \nabla_\xi L_4 = \frac{1}{2}L_1.
\end{aligned} \tag{32}$$

Related to the focus of this paper, we now discuss the issue of unique existence of null curves in a semi-Euclidean space (R_q^{m+2}, g).

Fundamental Theorems of Unique Null Curves in R_q^{m+2}. Suppose C is a null curve in (R_q^{m+2}, g) locally given by

$$x^A = x^A(t), \quad t \in \mathbf{I} \subset R, \ A \in \{0, 1, \ldots, (m+1)\} \tag{33}$$

with a semi-Euclidean metric

$$g(x, y) = -\left(\sum_{i=0}^{q-1} x^i y^i\right) + \left(\sum_{\alpha=q}^{m+1} x^\alpha y^\alpha\right). \tag{34}$$

Let $F_1 = \{\xi, N, W_1, \ldots, W_m\}$ be its general Frenet frame of Type 1. Take $\{W_1, \ldots, W_{q-1}\}$ timelike and the rest $(m - q - 1)$ vector fields $\{W_q, \ldots, W_m\}$ spacelike. There is one fundamental theorem for each of q-types. We give details for the last Type q whose Frenet frame is given by

$$\begin{aligned} F_q = \{&\xi = L_1, N = L_1^*, L_2, \ldots, \\ &L_q, L_2^*, \ldots, L_q^*, W_{q+1}, \ldots, W_m\}, \end{aligned} \tag{35}$$

where its vector fields are defined by

$$\begin{aligned}
L_1 &= \left(\frac{1}{\sqrt{2}}, 0, 0, 0, \ldots, 0, 0, \ldots, 0, 0, \frac{1}{\sqrt{2}}\right), \\
L_1^* &= \left(-\frac{1}{\sqrt{2}}, 0, 0, 0, \ldots, 0, 0, \ldots, 0, 0, \frac{1}{\sqrt{2}}\right), \\
L_2 &= \left(0, \frac{1}{\sqrt{2}}, 0, 0, \ldots, 0, 0, \ldots, 0, \frac{1}{\sqrt{2}}, 0\right), \\
L_2^* &= \left(0, -\frac{1}{\sqrt{2}}, 0, 0, \ldots, 0, 0, \ldots, 0, \frac{1}{\sqrt{2}}, 0\right), \\
L_3 &= \left(0, 0, \frac{1}{\sqrt{2}}, 0, \ldots, 0, 0, \ldots, 0, \frac{1}{\sqrt{2}}, 0, 0\right), \\
L_3^* &= \left(0, 0, -\frac{1}{\sqrt{2}}, 0, \ldots, 0, 0, \ldots, 0, \frac{1}{\sqrt{2}}, 0, 0\right), \\
&\vdots \\
L_q &= \left(0, \ldots, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right), \\
L_q^* &= \left(0, \ldots, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right),
\end{aligned} \tag{36}$$

where $\{L_1, \ldots, L_q, L_1^*, \ldots, L_q^*\}$ are null vector fields such that

$$\begin{aligned} &g(L_i, L_j^*) = \delta_{ij}, \qquad g(L_i, L_j) = 0, \\ &g(L_i^*, L_j^*) = 0. \end{aligned} \tag{37}$$

In this case, we find

$$\sum_{i=1}^{q-1} \left(L_i^A L_i^{*B} + L_i^B L_i^{*A}\right) = h^{AB}, \tag{38}$$

for any $A, B \in \{0, \ldots, (m+1)\}$, where we put

$$h^{AB} = \begin{cases} -1, & A = B \in \{0, \ldots, q-1\}; \\ 1, & A = B \in \{q, \ldots, (m+1)\}; \\ 0, & A \neq B. \end{cases} \tag{39}$$

Now we state the following fundamental existence and uniqueness theorem for null curves of R_q^{m+2} (which also includes Theorem 1 for the case $q = 1$).

Theorem 14 (see [7]). *Let $\kappa_i, \tau_i, \mu_i, \nu_i, \eta_i, \ldots : [-\varepsilon, \varepsilon] \to R$ be everywhere continuous functions, $x_o = (x_o^i)$ a fixed point of R_q^{m+2}, and $\{L_i, L_i^*, W_\alpha\}$, $1 \le i \le q$; $q + 1 \le \alpha \le m$ the quasiorthonormal basis of a Frenet frame F_q as displayed above. Then there exists a unique null curve $C : [-\varepsilon, \varepsilon] \to R_q^{m+2}$ such that $x^i = x^i(t)$, $C(0) = x_o$, and $\{\kappa_i, \tau_i, \mu_i, \nu_i, \eta_i, \ldots\}$ are curvature and torsion functions with respect to this Frenet frame F_q of Type q satisfying*

$$\begin{aligned} &\xi = \frac{d}{dp} = L_1, \qquad N(0) = L_1^*, \\ &W_\alpha(0) = W_\alpha, \quad q + 1 \le \alpha \le m. \end{aligned} \tag{40}$$

The construction of the Frenet equations is similar to Frenet equations (10) of Theorem 1 and the rest of the proof easily follows.

We refer to [7, Chapters 2–4] for proofs of the fundamental theorems, the geometry of all possible types of null curves in M_q^{m+2}, and many examples.

Open Problem. In previous presentation, we have seen that, contrary to the nondegenerate case, the uniqueness of any type of general Frenet equations cannot be assured even if one chooses a pseudo-arc-parameter. Each type depends on the parameter of C and the choice of a screen distribution. However, for a null curve in a Lorentzian manifold, using the natural Frenet equations we found a unique Cartan Frenet frame whose Frenet equations have a minimum number of curvature functions which are invariant under Lorentzian transformations. This raises the following question: *is there exist any unique Frenet frame for null curves in a general semi-Riemannian manifold* $(\mathbf{M}_q^{m+2}, g)$? We, therefore, invite the readers to work on the following research problem.

Find condition(s) for the existence of unique Frenet frames of nongeodesic null curves in a semi-Riemannian manifold of index q, where $q > 2$.

3. Unique or Canonical Theorems in Lightlike Hypersurfaces

Let (M, g) be a hypersurface of a prope $(m + 2)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant index

$q \in \{1, \ldots, m+1\}$. Suppose g is degenerate on M. Then, there exists a vector field $\xi \neq 0$ on M such that $g(\xi, X) = 0$, for all $X \in \Gamma(TM)$. The radical subspace $\mathrm{Rad}T_xM$ of T_xM, at each point $x \in M$, is defined by

$$\mathrm{Rad}T_xM = \{\xi \in T_xM : g_x(\xi, X) = 0\} = T_xM^{\perp}, \quad \forall X \in T_xM, \tag{41}$$

$\dim(\mathrm{Rad}T_xM) = 1$ and (M, g) is called a *lightlike hypersurface* of $(\overline{M}, \overline{g})$. We call $\mathrm{Rad}TM$ a radical distribution of M. Since $TM \supset TM^{\perp}$, contrary to the nondegenerate case, their sum is not the whole of tangent bundle space $T\overline{M}$. In other words, a vector of $T_x\overline{M}$ cannot be decomposed uniquely into a component tangent to T_xM and a component of $T_xM^{\perp}$. Therefore, the standard text-book definition of the second fundamental form and the Gauss-Weingarten formulas do not work for the lightlike case. To deal with this problem, in 1991, Bejancu and Duggal [10] introduced a geometric technique by splitting the tangent bundle TM into two nonintersecting complementary (but not orthogonal) vector bundles (one null and one nonnull) as follows. Consider a complementary vector bundle $S(TM)$ of $TM^{\perp} = \mathrm{Rad}TM$ in TM. This means that

$$TM = \mathrm{Rad}TM \oplus_{\mathrm{orth}} S(TM), \tag{42}$$

where $S(TM)$ is called a screen distribution on M which is nondegenerate. Thus, along M we have the following decomposition:

$$T\overline{M}_{|M} = S(TM) \perp S(TM)^{\perp}, \quad S(TM) \cap S(TM)^{\perp} \neq \{0\}; \tag{43}$$

that is, $S(TM)^{\perp}$ is orthogonal complement to $S(TM)$ in $T\overline{M}_{|M}$ which is also nondegenerate, but it includes $\mathrm{Rad}TM$ as its sub bundle. We need the following taken from [6, Chapter 4].

There exists a unique vector bundle $\mathrm{tr}(TM)$ of rank 1 over M, such that for any nonzero section ξ of $\mathrm{Rad}TM$ on a coordinate neighborhood $\mathcal{U} \subset M$ we have a unique section N of $\mathrm{tr}(TM)$ on $\mathcal{U}$ satisfying

$$\begin{gathered} \overline{g}(N, \xi) = 1, \\ \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \Gamma\left(ST(M)|_{\mathcal{U}}\right). \end{gathered} \tag{44}$$

It follows that $\mathrm{tr}(TM)$ is lightlike such that $\mathrm{tr}(TM)_u \cap T_uM = \{0\}$ for any $u \in M$. Moreover, we have the following decompositions:

$$T\overline{M}_{|M} = S(TM) \oplus_{\mathrm{orth}} \left(TM^{\perp} \oplus \mathrm{tr}(TM)\right) = TM \oplus \mathrm{tr}(TM). \tag{45}$$

Hence for any screen distribution $S(TM)$ there is a unique $\mathrm{tr}(TM)$ which is complementary vector bundle to TM in $T\overline{M}_{|M}$, called the lightlike transversal vector bundle of M with respect to $S(TM)$. Denote by $(M, g, S(TM))$ a lightlike hypersurface of $(\overline{M}, \overline{g})$. The Gauss and Weingarten type equations are

$$\begin{gathered} \nabla_X Y = \nabla_X Y + B(X, Y) N, \\ \overline{\nabla}_X N = -A_N X + \tau(X) N, \end{gathered} \tag{46}$$

respectively, where B is the local second fundamental form of M and A_N is its shape operator. It is easy to see that $B(X, \xi) = 0$, for all $X \in \Gamma(TM_{|\mathcal{U}})$. Therefore, B is degenerate with respect to g. Moreover, the connection ∇ on M is not a metric connection and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y). \tag{47}$$

In the lightlike case, we also have another second fundamental form and its corresponding shape operator which we now explain as follows.

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. We obtain

$$\begin{gathered} \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi, \\ \nabla_X \xi = -A_\xi^* X - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM), \end{gathered} \tag{48}$$

where $C(X, PY)$ is the screen fundamental form of $S(TM)$. The two second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$\begin{gathered} B(X, Y) = g\left(A_\xi^* X, Y\right), \qquad \overline{g}\left(A_\xi^* X, N\right) = 0, \\ C(X, PY) = g\left(A_N X, PY\right), \qquad \overline{g}\left(A_N Y, N\right) = 0. \end{gathered} \tag{49}$$

3.1. Unique or Canonical Screen Distributions. Unfortunately, the induced objects (second fundamental forms, induced connection, structure equations, etc.) depend on the choice of a screen $S(TM)$ which, in general, is not unique. This raises the question of finding those lightlike hypersurfaces which admit a unique or canonical $S(TM)$, needed in the lightlike geometry. Although a positive answer to this question for an arbitrary M is not possible, due to the degenerate induced metric, considerable progress has been made to heal this essential anomaly for specific classes.

There are several approaches in dealing with this nonuniqueness problem. Some authors have used specific methods suitable for their problems. For example, Akivis and Goldberg [26–28]; Bonnor [29]; Leistner [30]; Bolós [31] are samples of many more authors in the literature. Also, Kupeli [5] has shown that $S(TM)$ is canonically isometric to the factor vector bundle $TM^* = TM/TM^{\perp}$ and used canonical projection $\pi : TM \to TM^*$ in studying the intrinsic geometry of degenerate semi-Riemannian manifolds, where our review in this paper is focused on the extrinsic geometry which is in line with the classical theory of submanifolds [1]. Consequently, although specific techniques are suitable for good applicable results, nevertheless, for the fundamental deeper study of extrinsic geometry of lightlike spaces, one must look for a canonical or a unique screen distribution. For this purpose, we first start with a chronological history of some isolated results and then quote two main theorems.

In 1993, Bejancu [32] constructed a canonical $S(TM)$ for lightlike hypersurfaces M of semi-Euclidean spaces R_q^{n+1}. Then, in [10, Chapter 4] it was proved that such a canonical screen distribution is integrable on any lightlike hypersurface of R_1^n and on any lightlike cone $\wedge_{q-1}^n$ of R_q^{n+1}. This information was used by Bejancu et al. [33] in showing some interested

geometric results. Later on, Akivis and Goldberg [28] pointed out that such a canonical construction was neither invariant nor intrinsically connected with the geometry of M. Therefore, in the same paper [28], they constructed invariant normalizations intrinsically connected with the geometry of M and investigated induced linear connections by these normalizations, using relative and absolute invariant defined by the first and second fundamental forms of M.

Let $F = \{\xi, N, W_a\}$, $a \in \{1, \ldots, m\}$ be a quasiorthonormal basis of $\overline{M}$ along M, where $\{\xi\}$, $\{N\}$, and $\{W_a\}$ are null basis of $\Gamma(\mathrm{Rad}TM_{|\mathcal{U}})$, $\Gamma(\mathrm{tr}(TM)_{|\mathcal{U}})$, and orthonormal basis of $\Gamma(S(TM)_{|\mathcal{U}})$, respectively. For the same ξ, consider other quasiorthonormal frames fields $F' = \{\xi, N', W'_a\}$ induced on $\mathcal{U} \subset M$ by $\{S'(TM), (\mathrm{tr})'(TM)\}$. It is easy to obtain

$$\begin{aligned} W'_a &= \sum_{b=1}^{m} W_a^b \left(W_b - \epsilon_b \mathbf{f}_b \xi\right), \\ N' &= N + \mathbf{f}\xi + \sum_{a=1}^{m} \mathbf{f}_a W_a, \end{aligned} \tag{50}$$

where $\{\epsilon_a\}$ are signatures of orthonormal basis $\{W_a\}$ and W_a^b, $\mathbf{f}$, and $\mathbf{f}_a$ are smooth functions on $\mathcal{U}$ such that $[W_a^b]$ is $m \times m$ semiorthogonal matrices. Computing $\overline{g}(N', N') = 0$ and $\overline{g}(W_a, W_a) = 1$ we get $2\mathbf{f} + \sum_{a=1}^{m} \epsilon_a (\mathbf{f}_a)^2 = 0$. Using this in the second relation of the above two equations, we get

$$\begin{aligned} W'_a &= \sum_{b=1}^{m} W_a^b \left(W_b - \epsilon_b \mathbf{f}_b \xi\right), \\ N' &= N - \frac{1}{2}\left\{\sum_{a=1}^{m} \epsilon_a (\mathbf{f}_a)^2\right\} \xi \\ &\quad + \sum_{a=1}^{m} \mathbf{f}_a W_a. \end{aligned} \tag{51}$$

The above two relations are used to investigate the transformation of the induced objects when the pair $\{S(TM), \mathrm{tr}(TM)\}$ changes with respect to a change in the basis. To look for a condition so that a chosen screen is invariant with respect to a change in the basis, in 2004 Atindogbe and Duggal observed that a nondegenerate hypersurface has only one fundamental form where as a lightlike hypersurface admits an additional fundamental form of its screen distribution and their two respective shape operators. Moreover, we know [1] that the fundamental form and its shape operator of a nondegenerate hypersurface are related by the metric tensor. Contrary to this, we see from the two equations of (49) that in the lightlike case there are interrelations between its two second fundamental forms. Because of the above differences, Atindogbe and Duggal were motivated to connect the two shape operators by a conformal factor as follows.

Definition 15 (see [34]). A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is called screen locally conformal if the shape operators A_N and A_ξ^* of M and $S(TM)$, respectively, are related by

$$A_N = \varphi A_\xi^*, \tag{52}$$

where φ is a nonvanishing smooth function on a neighborhood $\mathcal{U}$ in M.

To avoid trivial ambiguities, we take $\mathcal{U}$ connected and maximal in the sense that there is no larger domain $\mathcal{U}' \supset \mathcal{U}$ on which the above relation holds. It is easy to show that two second fundamental forms B and C of a screen conformal lightlike hypersurface M and its $S(TM)$, respectively, are related by

$$C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma\left(TM_{|\mathcal{U}}\right). \tag{53}$$

Denote by $\mathcal{S}^1$ the first derivative of $S(TM)$ given by

$$\mathcal{S}^1(x) = \mathrm{span}\left\{[X, Y]|_x, X_x, Y_x \in S\left(T_x M\right)\right\}, \quad \forall x \in M. \tag{54}$$

Let $S(TM)$ and $S(TM)'$ be two screen distributions on M, B, and B' their second fundamental forms with respect to $\mathrm{tr}(TM)$ and $\mathrm{tr}(TM)'$, respectively, for the same $\xi \in \Gamma(TM^\perp|_{\mathcal{U}})$. Denote by ω the dual 1-form of the vector field $W = \sum_{a=1}^{m} \mathbf{f}_a W_a$ with respect to g. Following is a unique existence theorem.

Theorem 16 (see [8], page 61). *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, with $\mathcal{S}^1$ the first derivative of $S(TM)$ given by* (54). *Then,*

(1) *a choice of the screen $S(TM)$ of M satisfying* (52) *is integrable;*

(2) *the one form ω vanishes identically on $\mathcal{S}^1$;*

(3) *if $\mathcal{S}^1$ coincides with $S(TM)$, then M can admit a unique screen distribution up to an orthogonal transformation and a unique lightlike transversal vector bundle. Moreover, for this class of hypersurfaces, the screen second fundamental form C is independent of its choice.*

Proof. It follows from the screen conformal condition (52) that the shape operator A_ξ^* of $S(TM)$ is symmetric with respect to g. Therefore, a result [6, page 89] says that a choice of screen distribution of a screen conformal lightlike hypersurface M is integrable, which proves (1).

As $S(TM)$ is integrable, $\mathcal{S}^1$ is its subbundle. Assume $\mathcal{S}^1 = S(TM)$. Then, it is easy to show that ω vanishes on $S(TM)$, which implies that the functions $\mathbf{f}_a$ of the transformation equations vanish. Thus, the transformation equation (51) becomes $W'_a = \sum_{b=1}^{m} W_a^b W_b$, $(1 \le a \le m)$ and $N' = N$, where (W_a^b) is an orthogonal matrix of $S(T_x M)$ at any point x of M, which proves the first part of (3). Then independence of C follows which completes the proof. □

Remark 17. Based on the above theorem, one may ask the following converse question. Does the existence of a canonical or a unique distribution $S(TM)$ of a lightlike hypersurface imply that $S(TM)$ is integrable? Unfortunately, the answer, in general, is negative, which we support by recalling the following known results from [6, pages 114–117].

There exists a canonical screen distribution for any lightlike hypersurface of a semi-Euclidean space R_q^{m+2}; however, only the canonical screen distribution on any lightlike hypersurface of R_1^{m+2} is integrable. Therefore, although any screen conformal lightlike hypersurface admits an integrable screen distribution, the above results say that not every such integrable screen coincides with the corresponding canonical screen; that is, there are cases for which $\mathcal{S}^1 \neq S(TM)$.

Now, one may ask whether there is a class of semi-Riemannian manifolds which admit screen conformal lightlike hypersurfaces and, therefore, can admit a unique screen distribution. This question has been answered as follows.

Theorem 18 (see [11]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}_q^{m+2}, \overline{g})$, with E a complementary vector bundle of $TM^{\perp}$ in $S(TM)^{\perp}$ such that E admits a covariant constant timelike vector field. Then, with respect to a section ξ of $RadTM$, M is screen conformal. Thus, M can admit a unique screen distribution.*

To get a better idea of the proof of this theorem, we give the following example.

Example 19 (see [8], page 62). Consider a smooth function $F : \Omega \to R$, where Ω is an open set of R^{m+1}. Then

$$M = \left\{\left(x^0, \ldots, x^{m+1}\right) \in R_q^{m+2} : x^0 = F\left(x^1, \ldots, x^{m+1}\right)\right\} \tag{55}$$

is a Monge hypersurface. The natural parameterization on M is

$$x^0 = F\left(v^0, \ldots, v^m\right); \qquad x^{\alpha+1} = v^{\alpha}, \quad \alpha \in \{0, \ldots, m\}. \tag{56}$$

Hence, the natural frames field on M is globally defined by

$$\partial_{v^{\alpha}} = F'_{x^{\alpha+1}} \partial_{x^0} + \partial_{x^{\alpha+1}}, \quad \alpha \in \{0, \ldots, m\}. \tag{57}$$

Then

$$\xi = \partial_{x^0} - \sum_{s=1}^{q-1} F'_{x^s} \partial_{x^s} + \sum_{a=q}^{m+1} F'_{x^a} \partial_{x^a} \tag{58}$$

spans $TM^{\perp}$. Therefore, M is lightlike (i.e., $TM^{\perp} = \mathrm{Rad}TM$), if and only if the global vector field ξ is spanned by $\mathrm{Rad}TM$ which means, if and only if, F is a solution of the partial differential equation

$$1 + \sum_{s=1}^{q-1} \left(F'_{x^s}\right)^2 = \sum_{a=q}^{m+1} \left(F'_{x^a}\right)^2. \tag{59}$$

Along M consider the constant timelike section $V = \partial_{x^0}$ of $\Gamma(TR_q^{m+2})$. Then $\overline{g}(V, \xi) = -1$ implies that V is not tangent to M. Therefore, the vector bundle $E = \mathrm{Span}\{V, \xi\}$ is nondegenerate on M. The complementary orthogonal vector bundle $S(TM)$ to E in TR_q^{m+2} is a nondegenerate distribution on M and is complementary to $\mathrm{Rad}TM$. Thus $S(TM)$ is a screen distribution on M. The transversal bundle $tr(TM)$ is spanned by $N = -V + (1/2)\xi$ and $\tau(X) = 0$ for any $X \in \Gamma(TM)$. Indeed, $\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi) = (1/2)\overline{g}(\overline{\nabla}_X \xi, \xi) = 0$. The Weingarten equations reduce to $\overline{\nabla}_X N = -A_N X$ and $\nabla_X \xi = -A_{\xi}^* X$, which implies

$$A_N X = \frac{1}{2} A_{\xi}^* X, \quad \forall X \in \Gamma(TM). \tag{60}$$

Hence, any lightlike Monge hypersurface of R_q^{m+2} is screen globally conformal with $\varphi(x) = 1/2$. Therefore, it can admit a unique screen distribution.

3.2. Unique Metric Connection and Symmetric Ricci Tensor. We know from (47) that the induced connection ∇ on a lightlike submanifold (M, g) is a metric (Levi-Civita) connection if and only if the second fundamental form B vanishes on M. The issue is to find conditions on the induced objects of a lightlike hypersurface which admit such a unique Levi-Civita connection. First, we recall the following definitions.

In case any geodesic of M with respect to an induced connection ∇ is a geodesic of $\overline{M}$ with respect to $\overline{\nabla}$, we say that M is a totally geodesic lightlike hypersurface of $\overline{M}$. Also, note that a vector field X on a lightlike manifold (M, g) is said to be a Killing vector field if $\pounds_X g = 0$. A distribution D on M is called a Killing distribution if each vector field of D is Killing.

Now we quote the following theorem on the existence of a unique metric connection on M, which also shows, from the Gauss equation, that the definition of totally geodesic M does not depend on the choice of a screen.

Theorem 20 (see [6]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the following assertions are equivalent:*

(a) *M is totally geodesic in $\overline{M}$;*

(b) *B vanishes identically on M;*

(c) *A_{ξ}^* vanish on $\Gamma(TM)$ for any $\xi \in \Gamma(RadTM)$;*

(d) *There exists a unique torsion-free metric connection ∇ on M;*

(e) *Rad TM is a Killing distribution;*

(f) *Rad TM is a parallel distribution with respect to ∇.*

On the issue of obtaining an induced symmetric Ricci tensor of M, we proceed as follows. Consider a type $(0, 2)$ induced tensor on M given by

$$R(X, Y) = \mathrm{trace}\{Z \longrightarrow R(X, Z)Y\}, \quad \forall X, Y \in \Gamma(TM). \tag{61}$$

Let $\{\xi; W_a\}$ be an induced quasiorthonormal frame on M, where $\mathrm{Rad}\ TM = \mathrm{Span}\{\xi\}$ and $S(TM) = \mathrm{Span}\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frames field on $\overline{M}$. Then, we obtain

$$R(X, Y) = \sum_{a=1}^{m} \epsilon_a g\left(R(X, W_a)Y, W_a\right) + \overline{g}\left(R(X, \xi)Y, N\right), \tag{62}$$

where ϵ_a denotes the causal character (± 1) of respective vector field W_a. Using Gauss-Codazzi equations, we obtain

$$\begin{aligned} g(R(X,W_a)Y,W_a) &= \overline{g}\left(\overline{R}(X,W_a)Y,W_a\right) \\ &\quad + B(X,Y)C(W_a,W_a) \\ &\quad - B(W_a,Y)C(X,W_a). \end{aligned} \tag{63}$$

Substituting this into the previous equation and using the relations (49), we obtain

$$\begin{aligned} R(X,Y) &= \overline{\mathrm{Ric}}(X,Y) + B(X,Y)\,trA_N \\ &\quad - g\left(A_N X, A_\xi^* Y\right) - \overline{g}(R(\xi,Y)X,N), \\ &\qquad \forall X,Y \in \Gamma(TM), \end{aligned} \tag{64}$$

where $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}$. This shows that $R(X,Y)$ is not symmetric. Therefore, in general, it has no geometric or physical meaning similar to the symmetric Ricci tensor of $\overline{M}$. Thus, this $R(X,Y)$ can be called an induced Ricci tensor of M only if it is symmetric. Thus, one may ask the following question: are there any lightlike hypersurfaces with symmetric Ricci tensor? The answer is affirmative for which we quote the following result.

Theorem 21 (see [34]). *Let $(M,g,S(TM))$ be a locally (or globally) screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}(c),\overline{g})$ of constant sectional curvature c. Then, M admits an induced symmetric Ricci tensor.*

Proof. Using the curvature identity $\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y$ and the equation in (64), we obtain

$$\begin{aligned} R(X,Y) &= \overline{\mathrm{Ric}}(X,Y) + B(X,Y)\,\mathrm{tr}\,A_N \\ &\quad - g\left(A_N X, A_\xi^* Y\right) - c\overline{g}(X,Y). \end{aligned} \tag{65}$$

Then using the screen conformal relation (52) mentioned above it is easy to show that $R(X,Y)$ is symmetric and, therefore, it is an induced Ricci tensor of M.

In particular, if M is totally geodesic in $\overline{M}$, then using the curvature identity and proceeding similarly to what is mentioned above one can show that

$$R(X,Y) = \overline{\mathrm{Ric}}(X,Y) - c\overline{g}(X,Y). \tag{66}$$

Since $\overline{\mathrm{Ric}}$ and $\overline{g}$ are symmetric we conclude that *any totally geodesic lightlike hypersurface of $\overline{M}(c)$ admits an induced symmetric Ricci tensor.* □

Finally, we quote a general result on the induced symmetric Ricci tensor.

Theorem 22 (see [6]). *Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$. Then the tensor $R(X,Y)$, defined in (61), of the induced connection ∇ is a symmetric Ricci tensor, if and only if each 1-form τ induced by $S(TM)$ is closed; that is, $d\tau = 0$, on any $\mathcal{U} \subset M$.*

Remark 23. The symmetry property of the Ricci tensor on a manifold M equipped with an *affine connection* has also been studied by Nomizu-Sasaki. In fact, we quote the following result (Proposition 3.1, Chapter 1) in their 1994 book.

Proposition 24 (see [35]). *Let (M,∇) be a smooth manifold equipped with a torsion-free affine connection. Then the Ricci tensor is symmetric if and only if there exists a volume element ω satisfying $\nabla\omega = 0$.*

Open Problem. Give an interpretation of Theorem 22 in terms of affine geometry.

3.3. Induced Scalar Curvature. To introduce a concept of induced scalar curvature for a lightlike hypersurface M we observe that, in general, the nonuniqueness of screen distribution $S(TM)$ and its nondegenerate causal structure rule out the possibility of a definition for an arbitrary M of a semi-Riemannian manifold. Although now there are many cases of a canonical or unique screen and canonical transversal vector bundle, the problem of scalar curvature must be classified subject to the causal structure of a screen. For this reason, work has been done on lightlike hypersurfaces M of a Lorentzian manifold $(\overline{M},\overline{g})$ for which we know that any choice of its screen $S(TM)$ is Riemannian. This case is also physically useful. To calculate an induced scalar function r by setting $X = Y = \xi$ and then $X = Y = W_a$ in (62) and using Gauss-Codazzi equations, we obtain

$$\begin{aligned} r &= R(\xi,\xi) + \sum_{a=1}^{m} R(W_a,W_a) \\ &= \sum_{a=1}^{m}\left\{\sum_{b=1}^{m} g(R(W_a,W_b)W_a,W_b)\right\} \\ &\quad + \sum_{a=1}^{m}\{g(R(\xi,W_a)\xi,W_a) \\ &\qquad + \overline{g}(R(W_a,\xi)W_a,N)\}. \end{aligned} \tag{67}$$

In general, r given by the above expression cannot be called a scalar curvature of M since it has been calculated from a tensor quantity $R(X,Y)$. It can only have a geometric meaning if it is symmetric and its value is independent of the screen, its transversal vector bundle, and the null section ξ. Thus to recover a scalar curvature, we recall the following conditions [36] on M.

A lightlike hypersurface M (labeled by $M^{\mathbf{0}}$) of a Lorentzian manifold $(\overline{M},\overline{g})$ is of *genus* zero with screen $S(TM)^{\mathbf{0}}$ if

(a) M admits a canonical or unique screen distribution $S(TM)$ that induces a canonical or unique lightlike transversal vector bundle N;

(b) M admits an induced symmetric Ricci tensor, denoted by Ric.

Denote by $\mathscr{C}[M]^{\mathbf{0}} = [(M,g,S(TM))]^{\mathbf{0}}$ a class of lightlike hypersurfaces which satisfy the above two conditions.

Definition 25. Let (M^0, g^0, ξ^0, N^0) belong to $\mathscr{C}[M]^{\mathbf{0}}$. Then, the scalar r, given by (67), is called its induced scalar curvature of genus zero.

It follows from (a) that $S(TM)^0$ and N^0 are either canonical or unique. For the stability of r with respect to a choice of the second fundamental form B and the 1-form τ, it is easy to show that with canonical or unique N^0, both B and τ are independent of the choice of ξ^0, except for a nonzero constant factor. Finally, we know [8, page 70] that the Ricci tensor does not depend on the choice of ξ^0. Thus, r is a well-defined induced scalar curvature of a class of lightlike hypersurfaces of genus zero. The following result shows that there exists a variety of Lorentzian manifolds which admit hypersurfaces of class $\mathscr{C}[M]^{\mathbf{0}}$.

Theorem 26 (see [36]). *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian space form $(\overline{M}(c), \overline{g})$, with $\mathcal{S}^1$ the first derivative of $S(TM)$ given by* (54). *If $\mathcal{S}^1 = S(TM)$, then M belongs to $\mathscr{C}[M]^{\mathbf{0}}$. Consequently, this class of lightlike hypersurfaces admits induced scalar curvature of genus zero.*

Since $\mathcal{S}^1 = S(TM)$, it follows from Theorem 16 that M admits a unique screen distribution $S(TM)$ that induces a unique lightlike transversal vector bundle, which satisfies the condition (a). The condition (b) also holds from Theorem 21.

Moreover, consider a class of Lorentzian manifolds $(\overline{M}, \overline{g})$ which admit at least one covariant constant timelike vector field. Then, Theorem 18 says that M belongs to $\mathscr{C}[M]^{\mathbf{0}}$ and, therefore, it admits an induced scalar curvature. Also see [37, 38] for a followup on scalar curvature.

Remark 27. We know from Duggal and Bejancu's book [6, Page 111] that any lightlike surface M of a 3-dimensional Lorentz manifold is either totally umbilical or totally geodesic. Moreover, there exists a canonical screen distribution $S(TM)$ for a *lightlike Monge surface* $(M, g, S(TM))$ of R_1^3 for which the induced linear connection is flat (see Proposition 7.1 on page 126 in [6]). In particular, the *null cone* of R_1^3 has flat induced connection. Readers may get more information on pages 123–138 on lightlike hypersurfaces of R_1^3, R_1^4, and R_2^4 in [6]. In general, see Section 9.1 of Chapter 9 in a book by Duggal and Sahin [8] on *null surfaces of spacetimes*.

4. Unique Existence Theorems in Lightlike Submanifolds of Index $q > 1$

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional proper semi-Riemannian manifold, where $m > 1$, $n > 1$ with $\overline{g}$ a semi-Riemannian metric on $\overline{M}$ of constant index $q \in \{2, \ldots, m+n-1\}$. Hence $\overline{M}$ is never a Riemannian manifold. The conditions $m > 1$ and $n > 1$ imply that M is neither a curve nor a hypersurface

$$T_pM^{\perp} = \left\{V_p \in T_p\overline{M}; \overline{g}_p\left(V_p, W_p\right) = 0, \forall W_p \in T_pM\right\}. \tag{68}$$

For a lightlike M there exists a smooth distribution such that

$$\mathrm{Rad}T_pM = T_pM \cap T_pM^{\perp} \neq \{0\}, \quad \forall p \in M. \tag{69}$$

The following are four cases of lightlike submanifolds:

(A) r-lightlike submanifold, $0 < r < \min\{m, n\}$;

(B) coisotropic submanifold, $1 < r = n < m$;

(C) isotropic submanifold, $1 < r = m < n$;

(D) totally lightlike submanifold, $1 < r = m = n$.

We follow [8] for notations. For the case (A) of r-lightlike submanifold there exist two complementary nondegenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $\mathrm{Rad}(TM)$ in TM and $TM^{\perp}$, respectively, called the screen and screen transversal distributions on M, such that

$$\begin{aligned} TM &= \mathrm{Rad}\,(TM) \oplus_{\mathrm{orth}} S\,(TM), \\ TM^{\perp} &= \mathrm{Rad}\,(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{70}$$

Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_i\}$ be a lightlike basis of $\Gamma(\mathrm{ltr}(TM)_{|_{\mathscr{U}}})$ consisting of smooth sections of $S(TM)^{\perp}_{|_{\mathscr{U}}}$, where $\mathscr{U}$ is a coordinate neighborhood of M, such that

$$\overline{g}\left(N_i, \xi_j\right) = \delta_{ij}, \qquad \overline{g}\left(N_i, N_j\right) = 0, \tag{71}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$. Then,

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}\,(TM) \\ &= \{\mathrm{Rad}\,(TM) \oplus \mathrm{tr}\,(TM)\} \oplus_{\mathrm{orth}} S\,(TM) \\ &= \{\mathrm{Rad}\,(TM) \oplus \mathrm{ltr}\,(TM)\} \oplus_{\mathrm{orth}} S\,(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{72}$$

For the case (B) of coisotropic submanifold, $\mathrm{Rad}TM = TM^{\perp}$ implies that $S(TM)^{\perp} = \{0\}$. For the cases (C) and (D), $S(TM) = \{0\}$. Thus, to review the unique results on screen $S(TM)$, we only deal with the first two cases.

r-Lightlike Submanifolds $(M, g, S(TM), S(TM)^{\perp})$. Consider the following local quasiorthonormal frame of $\overline{M}$ along M:

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m, W_{r+1}, \ldots, W_n\}, \tag{73}$$

where $\{X_{r+1}, \ldots, X_m\}$ and $\{W_{r+1}, \ldots, W_n\}$ are orthonormal basis of $\Gamma(S(TM))$ and $\Gamma(S(TM^{\perp}))$. Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ and P the projection morphism of $\Gamma(TM)$

on $\Gamma(S(TM))$. For a lightlike submanifold, the local Gauss-Weingartan formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y) N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y) W_\alpha, \quad (74)$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_\alpha, \quad (75)$$

$$\overline{\nabla}_X W_\alpha = -A_{W_\alpha} X + \sum_{i=1}^{r} \phi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \theta_{\alpha\beta}(X) W_\beta, \quad (76)$$

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h_i^*(X,PY)\xi_i, \quad (77)$$

$$\nabla_X \xi_i = -A_{\xi_i}^* X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \quad \forall X, Y \in \Gamma(TM), \quad (78)$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, h_i^{ℓ} and h_α^s, given in (74), are called the local transversal second fundamental forms and screen second fundamental forms on TM, respectively, h_i^*, given in (77), are called the local radical second fundamental forms on $S(TM)$, A_{N_i}, A_{W_α}, and $A_{\xi_i}^*$ are linear operators on $\Gamma(TM)$, and τ_{ij}, $\rho_{i\alpha}$, $\phi_{\alpha i}$, and $\theta_{\alpha\beta}$ are 1-forms on TM, as given in (75), (76), and (78). Since the connection $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free connection and both h_i^{ℓ} and h_α^s are symmetric. From the fact that $h_i^{\ell}(X,Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i)$, we know that h_i^{ℓ} are independent of the choice of a screen distribution. Note that h_i^{ℓ}, τ_{ij}, and $\rho_{i\alpha}$ depend on the sections $\xi \in \Gamma(\mathrm{Rad}(TM)|_{\mathcal{U}})$ and $d(\mathrm{tr}(\tau_{ij})) = d(\mathrm{tr}(\overline{\tau}_{ij}))$. The induced connection ∇ on TM is not metric connection as it satisfies

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r} \left\{ h_i^{\ell}(X,Y)\eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y) \right\}, \quad (79)$$

where η_i are the 1-forms $\eta_i(X) = \overline{g}(X, N_i)$, $\forall X \in \Gamma(TM)$, $i \in \{1,\ldots,r\}$.

However, the connection ∇^* on $S(TM)$ is metric. The above types of second fundamental forms of M and $S(TM)$ are related as follows:

$$h_i^{\ell}(X,Y) = g\left(A_{\xi_i}^* X, Y\right) - \sum_{j=1}^{r} h_j^{\ell}(X,\xi_i)\eta_j(Y), \quad (80)$$

$$h_i^{\ell}(X,PY) = g\left(A_{\xi_i}^* X, PY\right), \qquad \overline{g}\left(A_{\xi_i}^* X, N_j\right) = 0,$$

$$\epsilon_\alpha h_\alpha^s(X,Y) = g\left(A_{W_\alpha} X, Y\right) - \sum_{i=1}^{r} \phi_{\alpha i}(X)\eta_i(Y), \quad (81)$$

$$\epsilon_\alpha h_\alpha^s(X,PY) = g\left(A_{W_\alpha} X, PY\right),$$
$$\overline{g}\left(A_{W_\alpha} X, N_i\right) = \epsilon_\alpha \rho_{i\alpha}(X), \quad (82)$$

$$h_i^*(X,PY) = g\left(A_{N_i} X, PY\right),$$
$$\eta_j\left(A_{N_i} X\right) + \eta_i\left(A_{N_j} X\right) = 0. \quad (83)$$

Consider two quasiorthonormal frames $\{\xi_i, N_i, X_a, W_\alpha\}$ and $\{\xi_i, N_i', X_a', W_\alpha'\}$ of $\{S(TM), S(TM^\perp), F\}$ and $\{S'(TM), S'(TM^\perp), F'\}$, where F and F' are the complementary vector bundles of $\mathrm{Rad}TM$ in $S(TM^\perp)^\perp$ and $S'(TM^\perp)^\perp$, respectively. Consider the following transformation equations (see [6, pages 163–165]):

$$X_a' = \sum_{b=r+1}^{m} \left\{ X_a^b \left(X_b - \epsilon_b \sum_{i=1}^{r} \mathbf{f}_{ib}\xi_i \right) \right\}$$
$$W_\alpha' = \sum_{\beta=r+1}^{n} \left\{ W_\alpha^\beta \left(W_\beta - \epsilon_\beta \sum_{i=1}^{r} Q_{i\beta}\xi_i \right) \right\}$$
$$N_i' = N_i + \sum_{j=1}^{r} N_{ij}\xi_j + \sum_{a=r+1}^{m} \mathbf{f}_{ia} X_a + \sum_{\alpha=r+1}^{n} Q_{i\alpha} W_\alpha, \quad (84)$$

where $\{\epsilon_a\}$ and $\{\epsilon_\alpha\}$ are signatures of bases $\{X_a\}$ and $\{W_\alpha\}$, respectively, X_a^b, W_α^β, N_{ij}, $\mathbf{f}_{ia}$, and $Q_{i\alpha}$ are smooth functions on $\mathcal{U}$ such that $[X_a^b]$ and $[W_\alpha^\beta]$ are $(m-r)\times(m-r)$ and $(n-r)\times(n-r)$ semiorthogonal matrices, and

$$N_{ij} + N_{ji} + \sum_{a=r+1}^{m} \epsilon_a \mathbf{f}_{ia}\mathbf{f}_{ja} + \sum_{\alpha=r+1}^{n} \epsilon_\alpha Q_{i\alpha} Q_{j\alpha} = 0.$$

$$\nabla_X Y = \nabla_X' Y + \sum_{j=1}^{r} \left\{ \sum_{i=1}^{r} h_i^l(X,Y) N_{ij} - \sum_{\alpha,\beta=r+1}^{n} \epsilon_\beta h_\alpha^{\prime s}(X,Y) W_\alpha^\beta Q_{j\beta} \right\} \xi_j + \sum_{a=r+1}^{m} \left\{ \sum_{i=1}^{r} h_i^l(X,Y)\mathbf{f}_{ia} \right\} X_a, \quad (85)$$

$$h_\alpha^s(X,Y) = \sum_{i=1}^{r} h_i^l(X,Y) Q_{i\alpha} + \sum_{\beta=r+1}^{n} h_\beta^{\prime s}(X,Y) W_\beta^\alpha.$$

$$h_i^l(X,Y) = h_i^{\prime l}(X,Y), \quad \forall X, Y \in \Gamma(TM). \quad (86)$$

Lemma 28 (see [8]). *The second fundamental forms h^* and h'^* of screens $S(TM)$ and $S(TM)'$, respectively, in an r-lightlike submanifold M are related as follows:*

$$\begin{aligned} h_i'^*(X,PY) &= h_i^*(X,PY) + g(\nabla_X PY, \mathbf{Z}_i) \\ &\quad - \sum_{i\neq j} h_j^l(X,PY)\left[N_{ji} + g(\mathbf{Z}_i,\mathbf{Z}_j)\right] \\ &\quad + \frac{1}{2}h_i^l(X,PY)\left[\|\mathbf{W}_i\|^2 - \|\mathbf{Z}_i\|^2\right] \\ &\quad + \sum_{\alpha=r+1}^{n} \varepsilon_\alpha h_\alpha^s(X,PY)\, Q_{i\alpha}, \end{aligned} \tag{87}$$

where $\mathbf{Z}_i = \sum_{a=r+1}^{m} \mathbf{f}_{ia} X_a$ *and* $\mathbf{W}_i = \sum_{\alpha=r+1}^{n} Q_{i\alpha} W_\alpha$.

Let ω_i be the respective n dual 1-forms of $\mathbf{Z}_i$ given by

$$\omega_i(X) = g(X,\mathbf{Z}_i), \quad \forall X \in \Gamma(TM),\ 1 \le i \le n. \tag{88}$$

Denote by $\mathcal{S}$ the first derivative of a screen distribution $S(TM)$ given by

$$\mathcal{S}(x) = \operatorname{span}\{[X,Y]_x, X_x, Y_x \in S(TM), x \in M\}. \tag{89}$$

If $S(TM)$ is integrable, then $\mathcal{S}$ is a subbundle of $S(TM)$. At a point $x \in M$ let $\mathcal{N}^1(x)$ be the space spanned by all vectors $h^s(X,Y)$, $X_x, Y_x \in T_xM$; that is,

$$\mathcal{N}^1(x) = \operatorname{span}\{h^s(X,Y)_x; X_x, Y_x \in T_xM\}. \tag{90}$$

Call $\mathcal{N}^1$ as first screen transversal space. We see from (80) and (83) that there are interrelations between the second fundamental forms of the lightlike M and its screen distribution and their respective shape operators. This interrelation indicates that the lightlike geometry depends on a choice of screen distribution which is not unique. While we know from (86) that the fundamental forms h_i^ℓ of the lightlike M are independent of a screen, the same is not true for the fundamental forms h_i^* of $S(TM)$(see (87)), which is the root of nonuniqueness anomaly in the r-lightlike geometry. The question is how to proceed further to get the choice of a unique screen. In the case of hypersurfaces, we related the two shape operators A_N and A_ξ^* by a conformal function (see (52)). However, this condition cannot be used for the case of general submanifolds for the following reason.

The single shape operator A_N of a hypersurface is $S(TM)$-valued, but, for a submanifold, there is no guarantee that each A_{N_i} is $S(TM)$-valued. Thus, shape operators cannot be used to define screen conformal condition. For this reason, Duggal and Sahin [8] modified the screen conformal condition as follows.

Definition 29. A lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g}$ is called a screen conformal submanifold if the fundamental forms h_i^* of $S(TM)$ are conformally related to the corresponding lightlike fundamental forms h_i of M by

$$\begin{aligned} &h_i^*(X,PY) = \varphi_i h_i(X,PY), \\ &\forall X, Y, \Gamma(TM|_{\mathcal{U}}), \quad i \in \{1,\dots,r\}, \end{aligned} \tag{91}$$

where $\varphi_i s$ are smooth function on a neighborhood $\mathcal{U}$ in M.

In order to avoid trivial ambiguities, we will consider $\mathcal{U}$ to be connected and maximal in the sense that there is no larger domain $\mathcal{U}' \supset \mathcal{U}$ on which (91) holds. In case $\mathcal{U} = M$, the screen conformality is said to be global. Since this material is relatively new, for the benefit of interested readers, we give complete proof of the following unique existence theorem.

Theorem 30 (see [8]). *Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional r-lightlike screen conformal submanifold of a semi-Riemannian manifold $\overline{M}^{m+n}$. Then,*

(a) *a choice of a screen distribution of M, satisfying* (91)*, is integrable;*

(b) *all the n-forms ω_i in* (89) *vanish identically on the first derivative $\mathcal{S}$ given by* (88)*;*

(c) *if $\mathcal{S} = S(TM)$ and $\mathcal{N}^1 = S(TM^\perp)$, then there exists a set of r null sections $\{\xi_1,\dots,\xi_r\}$ of $\Gamma(RadTM)$ with respect to $S(TM)$ which is a unique screen distribution of M, up to an orthogonal transformation with a unique set $\{N_1,\dots,N_r\}$ of lightlike transversal vector bundles and the screen fundamental forms h_i^* are independent of a screen distribution.*

Proof. Substituting (91) in (75) and then using (77), we get

$$g(A_{N_i}X, PY) = \varphi_i g(A_{\xi_i}X, PY) \quad \forall X \in \Gamma(TM_{|_{\mathcal{U}}}). \tag{92}$$

Since each A_{ξ_i} is symmetric with respect to g, the above equation implies that each A_{N_i} is self-adjoint on $\Gamma(S(TM))$ with respect to g, which further follows from [6, page 161] that a choice of a screen $S(TM)$, satisfying (91), is integrable. Thus, (a) holds. Now choose an integrable screen $S(TM)$ such that $\mathcal{S} = S(TM)$. Then one can obtain that all $\mathbf{f}_{ia} = 0$. Using Lemma 28, proceeding closer to the case of hypersurfaces (see Theorem 16), each $\mathbf{W}_i$ and all $Q_{i\alpha}$ vanish, which proves the three statements of the theorem. □

There exists another class of r-lightlike submanifolds which admit integrable unique screen distributions, subject to a geometric condition (different from the screen conformal condition), which we present as follows. Consider a complementary vector bundle F of RadTM in $S(TM^\perp)^\perp$ and choose a basis $\{V_i\}$, $i \in \{1,\dots,r\}$ of $\Gamma(F_{|\mathcal{U}})$. Thus, we are looking for the following sections:

$$N_i = \sum_{k=1}^{r} \{A_{ik}\xi_k + B_{ik}V_k\}, \tag{93}$$

where A_{ik} and B_{ik} are smooth functions on $\mathcal{U}$. Then $\{N_i\}$ satisfy $\overline{g}(N_i,\xi_i) = \delta_{ij}$ if and only if $\sum_{k=1}^{r} B_{ik}\overline{g}_{jk} = \delta_{ij}$, where

$\overline{g}_{jk} = \overline{g}(\xi_j, V_k)$, $j, k \in \{1, \ldots, r\}$. Observe that $G = \det[\overline{g}_{jk}]$ is everywhere nonzero on $\mathcal{U}$; otherwise $S(TM^{\perp})^{\perp}$ would be degenerate at least at a point of $\mathcal{U}$.

Theorem 31 (see [8]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be a r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ such that the complementary vector bundle F of $RadTM$ in $S(TM^{\perp})^{\perp}$ is parallel along the tangent direction. Then, all the assertions from (a) through (c) of Theorem 30 will hold.*

Proof. Covariant derivative of (93) provides

$$\overline{\nabla}_X N_i = \sum_{k=1}^{r} \left(X(A_{ik})\xi_k + A_{ik}\overline{\nabla}_X \xi_k + X(B_{ik})V_k + B_{ik}\overline{\nabla}_X V_k \right). \tag{94}$$

Using (74), (75), and (78), we obtain

$$\begin{aligned} A_{N_i}X = {} & \nabla^l_X N_i + D^s(X, N_i) \\ & - \sum_{k=1}^{r} \Big[X(A_{ik})\xi_k + A_{ik} \\ & \times \left\{ -A^*_{\xi_k}X + \nabla^{*t}_X \xi_k + h^l(X, \xi_k) + h^s(X, \xi_k) \right\} \\ & + X(B_{ik})V_k + B_{ik}\overline{\nabla}_X V_k \Big]. \end{aligned} \tag{95}$$

Since F is parallel, $\overline{\nabla}_X V_k \in \Gamma(F)$. Thus for $Y \in \Gamma(S(TM))$, we get $g(A_{N_i}X, Y) = \sum_{k=1}^{r} A_{ik} g(A^*_{\xi_k}X, Y)$. Then, we obtain

$$\overline{g}(h^*(X, Y), N_i) = \sum_{k=1}^{r} A_{ik}\overline{g}\left(h^l(X, Y), \xi_k\right). \tag{96}$$

Since the right side of (96) is symmetric, it follows that h^* is symmetric on $S(TM)$. Thus, it follows from [6, page 161] that $S(TM)$ is integrable. The rest of the proof is similar to the proof of Theorem 30. □

Remark 32. In [8] it has been shown by an example that the screen conformal condition does not necessarily imply that F is parallel along the tangent direction. Thus, the two classes of lightlike submanifolds, with the choice of unique screen, are different from each other. Finally, one can verify that the above two theorems will also hold for a subcase of coisotropic submanifold for which $S(TM^{\perp})$ vanishes and $r = n$. Details may be seen in [39, 40].

4.1. Unique Metric Connection and Symmetric Ricci Tensor. We know from (79) that the induced connection ∇ on a lightlike submanifold (M, g) is a unique metric (Levi-Civita) connection if and only if the second fundamental forms h^{ℓ}_i vanish on M. Also, any totally geodesic M admits a Levi-Civita connection. It is important to note that, contrary to the case of hypersurfaces (see Theorem 20), an r-lightlike M, with a unique metric connection, is not necessarily totally geodesic as $S(TM^{\perp})$ need not vanish on M. However, this equivalence holds for a coisotropic M for which $S(TM^{\perp})$ vanishes. Thus, the issue is to find a variety of nontotally geodesic lightlike submanifolds which admit such a unique Levi-Civita connection. On this issue, we quote the following recent result.

Theorem 33. *Let $(M, g, S(TM))$ be an r-lightlike or coisotropic submanifold of a semi-Riemannian manifold $\overline{M}$ such that $S(TM)$ is Killing and M is irrotational. Then, the induced connection ∇ on M is a Levi-Civita connection.*

Proof. Recall that a lightlike submanifold M is irrotational if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$, where $\xi \in \Gamma(\mathrm{Rad}TM)$ [5]. This further means that $h^l(X, \xi) = 0$ and $h^s(X, \xi) = 0$ for all $\xi \in \Gamma(\mathrm{Rad}TM)$. Using this condition and $S(TM)$ Killing it is easy to show that

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \left\{ h^{\ell}_i(X, Y)\eta_i(Z) + h^{\ell}_i(X, Z)\eta_i(Y) \right\} = 0, \tag{97}$$

which implies that all h^{ℓ}_i vanish. Therefore, ∇ on M is a Levi-Civita connection. For this case coisotropic M will be totally geodesic in $\overline{M}$. □

Finally, we quote a general result on the existence of a unique metric connection.

Theorem 34 (see [6]). *Let M be an r-lightlike or a coisotropic submanifold of a semi-Riemannian manifold. Then, the induced linear connection ∇ on M is a unique metric connection if and only if one of the following holds:*

(a) *A^*_{ξ} vanish on $\Gamma(TM)$ for any $\xi \in \Gamma(RadTM)$;*

(b) *Rad TM is a Killing distribution;*

(c) *Rad TM is a parallel distribution with respect to ∇.*

On an induced symmetric Ricci tensor of M, we quote the following two results.

Theorem 35 (see [24]). *Let $(M, g, S(TM))$ be an r-lightlike or a coisotropic submanifold of a semi-Riemannian manifold. Then, the induced Ricci tensor on M is symmetric if and only if each Trace(τ_{ij}) induced by $S(TM)$ is closed, where the 1-forms τ_{ij} are as given in (75).*

Theorem 36. *Let $(M, g, S(TM^{\perp}))$ be a screen conformal r-lightlike or a coisotropic submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$ such that $S(TM^{\perp})$ is Killing. Then, M admits an induced symmetric Ricci tensor.*

Proof. Consider a class of screen conformal lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$, with Killing $S(TM^{\perp})$. Using (76) and (81) and taking the Lie derivative $\overline{\pounds}$ of $\overline{g}$ with respect to any base vector W_{α}, we get

$$\left(\overline{\pounds}_{W_{\alpha}}\overline{g}\right)(X, Y) = -2\epsilon_{\alpha}h^s_{\alpha}(X, Y). \tag{98}$$

Using the above relation, Killing $S(TM^{\perp})$, the conformal screen relation (91), and some curvature properties of the space form $\overline{M}(c)$, it is easy to prove that M admits an induced symmetric Ricci tensor, which completes the proof. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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