Research Article

Disjointness Preserving and Functional Type Disjointness Preserving Operators

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A new concept called functional type disjointness preserving operators is introduced and structure of disjointness preserving and functional type disjointness preserving operators on some function spaces are analysed.

1. Introduction

There are many articles for construction of unbounded disjointness preserving operators. The paper [1] of Abramovich and Lipecki develops techniques to construct unbounded disjointness preserving linear functionals on any infinite dimensional F -lattice, and this is also mentioned in Remark 2.7 of the paper [2]. This paper [2] is devoted to construct unbounded disjointness preserving linear functionals on function spaces $C_0(X)$, when X is a locally compact Hausdorff space. The paper [3] of Jeang and Wong presents a simplified procedure to construct unbounded disjointness preserving functionals on function spaces $C_0(X)$. Jarosz [4] presents a construction of disjointness preserving operators on $C(X)$, where X is a compact Hausdorff space. Disjointness preserving operators are defined on many structures, namely, lattices, function spaces, algebras, and so forth. The present paper is restricted to study disjointness preserving operators on function spaces. A linear operator T between two function spaces is said to be disjointness preserving if $Tf \cdot Tg = 0$ whenever $f \cdot g = 0$. The second section of the paper exercises the standard techniques applicable to derive an essential structure of disjointness preserving operators. This is done on the function spaces $C_c(X)$, $C_0(X)$ for a normal space X , and on $\mathcal{D}(\mathbb{R})$, $C_0^\infty(\mathbb{R})$. Although results hold for both real and complex cases, it is assumed that the results on $\mathcal{D}(\mathbb{R})$, $C_0^\infty(\mathbb{R})$ are over real fields.

For a disjointness preserving linear functional T we have $Tf = 0$ or $Tg = 0$, whenever $f \cdot g = 0$. Jarosz [4] constructed an unbounded disjointness preserving operator T from $C(X)$ onto E , for any given infinite compact Hausdorff spaces X and Y , and for any given linear subspace E of $C(Y)$ such that $1 \leq \dim E \leq c$ (cardinality of continuum). These mappings have the following property: $f \cdot g = 0$ implies $Tf = 0$ or $Tg = 0$. So, let us give a new name to this type of mappings.

Definition 1. A linear operator T from a function space into a function space is said to be a functional type disjointness preserving (FTDP) if $f \cdot g = 0$ implies $Tf = 0$ or $Tg = 0$.

An essential structure of FTDP-mappings is derived on some function spaces in Section 3.

For a real or complex valued function f on X , we define the cozero set of f by $\text{coz}(f) = \{x \in X : f(x) \neq 0\}$, and zero set of f by $z(f) = \{x \in X : f(x) = 0\}$ and if X is a topological space, then we define the support of f by $\text{supp } f = \overline{\text{coz}(f)}$, closure of cozero of f . For a nonempty set Y , $F(Y)$ will denote the linear space of all real or complex valued functions defined on Y . To each $y \in Y$, δ_y will denote the evaluation functional defined on $F(Y)$ by $\delta_y(f) = f(y)$, for every $f \in F(Y)$. All the topological spaces to be considered are Hausdorff spaces. For a topological space X , $C(X)$ will denote the linear space of all real or complex valued bounded continuous functions; $C_c(X)$ will denote the linear space of

all real or complex valued continuous functions with compact support in X ; $C_0(X)$ will denote the linear space $C_0(X) = \{f : X \rightarrow \mathbb{R} \text{ or } \mathbb{C} : f \text{ is continuous on } X, \text{ and for every } \epsilon > 0 \text{ there is a compact subset } K \text{ of } X \text{ such that } |f(x)| < \epsilon, \text{ for every } x \in X \setminus K\}$. Let us use the usual notation $C^\infty(\mathbb{R})$ to denote the linear space of all real valued functions defined on the real line $\mathbb{R}$ which are infinitely many times differentiable at every point in $\mathbb{R}$. $\mathcal{D}(\mathbb{R})$ will denote the linear space of all test functions on $\mathbb{R}$ (see [5]). That is, $\mathcal{D}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f \text{ has compact support in } \mathbb{R}\}$. Let us use the notation $C_0^\infty(\mathbb{R})$ to denote the linear space: $C_0^\infty(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \text{for every } \epsilon > 0, \text{ there is a compact subset } K \text{ of } \mathbb{R} \text{ such that } |f(x)| < \epsilon, \text{ for all } x \in \mathbb{R} \setminus K\}$. We consider the linear spaces $C(X), C_c(X), C_0(X), \mathcal{D}(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ as normed spaces with the supremum norms on them. If X is normal, then $C_c(X)$ is dense in $C_0(X)$. For if $f \in C_0(X)$ and $\epsilon > 0$, then there is a $g \in C_c(X)$ such that $\|g\| = 1, 0 \leq g \leq 1, g$ has value 1 on $\{x \in X : |f(x)| \geq \epsilon\}$, g has value 0 on $\{x \in X : |f(x)| \leq \epsilon/2\}$, $f \cdot g \in C_c(X)$, and $\|f \cdot g - f\| \leq \epsilon$. Similarly, $\mathcal{D}(\mathbb{R})$ is also dense subspace of $C_0^\infty(\mathbb{R})$, which follows from Lemma 1 of Section 1.8 in [6].

2. Disjointness Preserving Mappings

The following theorem is a variation of the results obtained by Jarosz [4] and by Jeang and Wong [3].

Theorem 2 (see [3, 4]). *Let T be a linear disjointness preserving mapping from $C(X)$ (or $C_0(X)$) to $F(Y)$ where X is a compact Hausdorff space (or locally compact Hausdorff space X) and Y is a nonempty set. Let*

$$\begin{aligned} Y_1 &= \{y \in Y : \\ &\quad \delta_y \circ T \text{ is a nonzero continuous linear functional}\}, \\ Y_2 &= \{y \in Y : \delta_y \circ T \text{ is discontinuous}\}, \\ Y_3 &= \{y \in Y : \delta_y \circ T = 0\}. \end{aligned} \quad (1)$$

Then there is a function $\phi : Y_1 \cup Y_2 \rightarrow X$ (or $\phi : Y_1 \cup Y_2 \rightarrow X_\infty, X_\infty = X \cup \{\infty\}$ is the one point compactification of X) such that $Tf(y) = \alpha_y f(\phi(y))$ for all $f \in C(X)$ (or $C_0(X)$), for some fixed scalar α_y , and for all $y \in Y_1$ such that $\delta_y \circ T = 0$ on $\{f \in C(X) \text{ (or } C_0(X)) : \phi(y) \notin \text{supp } f\}$, for all $y \in Y_1 \cup Y_2$. Moreover, if every function in the range of T is a bounded function on Y , then $\phi(Y_2)$ is a finite subset of X (or X_∞ , resp., and $\phi(y) \in X = X_\infty \setminus \{\infty\}$ for $y \in Y_1$).

These results depend on the Urysohn lemma and existence of partitions of unity. Let us also use them in deriving the following theorem.

Theorem 3. *Let X be a normal space and Y be a nonempty set. Let T be a linear disjointness preserving mapping from $C_c(X)$ into $F(Y)$. Let Y_1, Y_2 , and Y_3 be defined as in Theorem 2. Then there is a function $\phi : Y_1 \cup Y_2 \rightarrow X$ such that $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $C_c(X)$ for some fixed scalar α_y , and for all $y \in Y_1$ such that $\delta_y \circ T = 0$ on $\{f \in C(X) : \phi(y) \notin \text{supp } f\}$, for all $y \in Y_1 \cup Y_2$.*

Proof. To each $y \in Y$, define $\Phi(y) = \{x \in X : \text{for every open neighbourhood } U \text{ of } x, \text{ there is an } f \in C_c(X) \text{ such that } \text{coz}(f) \subset U \text{ and } Tf(y) \neq 0\}$. First, we claim that to each $y \in Y$, $\Phi(y)$ contains at most one point. On the contrary, suppose $x_1, x_2 \in \Phi(y)$ for some $y \in Y$, with $x_1 \neq x_2$. Let U_1 and U_2 be two disjoint open neighbourhoods of x_1 and x_2 respectively. Then there are $f_1, f_2 \in C_c(X)$ such that $Tf_i(y) \neq 0$ and $\text{coz}(f_i) \subset U_i, i = 1, 2$. This is a contradiction to the assumption that T is disjointness linear preserving. So, $\Phi(y)$ contains at most one point, for every $y \in Y$. We next claim that $\Phi(y)$ is nonempty for every $y \in Y_1 \cup Y_2$. On the contrary we assume that $\Phi(y)$ is empty for some $y \in Y_1 \cup Y_2$. Note that $\delta_y \circ T \neq 0$. Fix $f \in C_c(X)$. Let $K = \text{supp } f$. To each $x \in K$, let U_x be an open neighbourhood of x such that $Tg(y) = 0$ whenever $g \in C_c(X)$ and $\text{coz}(g) \subset U_x$. Then we can find a finite subfamily $\{U_1, U_2, \dots, U_n\}$ of $\{U_x : x \in X\}$ such that the subfamily covers K . Let $\{g_1, g_2, \dots, g_n\} \subset C(K)$ be a continuous decomposition of the identity subordinate to $\{U_1 \cap K, U_2 \cap K, \dots, U_n \cap K\}$. Then, by the Tietze extension theorem, there are functions $f_1, f_2, \dots, f_n$ in $C(X)$ such that $f_{i|K} = g_i$ and $0 \leq f_i \leq 1$, for every i . Then $f \cdot f_i \in C_c(X)$ and $\text{coz}(f \cdot f_i) \subset U_i$ and hence $T(f \cdot f_i)(y) = 0$, for every i . So, $Tf(y) = T(\sum_{i=1}^n f \cdot f_i)(y) = \sum_{i=1}^n T(f \cdot f_i)(y) = 0$. Thus $\delta_y \circ T = 0$ on $C_c(X)$, which is a contradiction. Hence $\Phi(y)$ is a singleton set for each $y \in Y_1 \cup Y_2$. Define a function $\phi : Y_1 \cup Y_2 \rightarrow X$ by $\{\phi(y)\} = \Phi(y)$. To each $y \in Y_1 \cup Y_2$, write $J_y = \{f \in C_c(X) : \phi(y) \notin \text{supp } f\}$, and $K_y = \{f \in C_c(X) : f(\phi(y)) = 0\}$. If $y \in Y_1 \cup Y_2$ and $f \in J_y$, then $\phi(y) \notin \text{supp } f$ and there is a $g \in C_c(X)$ with $\text{coz}(g) \subset X \setminus \text{supp } f$ such that $Tg(y) \neq 0$. Since T is disjointness preserving, $Tf(y) = 0$. Thus $J_y \subset \ker(\delta_y \circ T)$. Since X is normal, if $f \in K_y, y \in Y_1 \cup Y_2$, and $\epsilon > 0$, then there is a $g \in C_c(X)$ such that $\|g\| = 1, 0 \leq g \leq 1, g$ has value 1 on $\{x \in X : |f(x)| \geq \epsilon\}$, and g has value 0 on $\{x \in X : |f(x)| \leq \epsilon/2\}$, $f \cdot g \in C_c(X)$, $f \cdot g \in J_y$, and $\|f \cdot g - f\| \leq \epsilon$. Thus J_y is dense in K_y . If $y \in Y_1$, then $\ker(\delta_y \circ T)$ is closed and hence $K_y \subset \ker(\delta_y \circ T)$. So $K_y = \ker(\delta_y \circ T)$ and $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $C_c(X)$, for some α_y . This proves the theorem. $\square$

Corollary 4. *Let X be a normal space and Y a nonempty set. Let T be a linear disjointness preserving mapping from $C_0(X)$ to $F(Y)$. Suppose $\delta_y \circ T$ is continuous on $C_0(X)$, for every $y \in Y$. Let*

$$\begin{aligned} Y_1 &= \{y \in Y : \\ &\quad \delta_y \circ T \text{ is a nonzero continuous linear functional}\}, \\ Y_2 &= \{y \in Y : \delta_y \circ T = 0\}. \end{aligned} \quad (2)$$

Then there is a function $\phi : Y_1 \rightarrow X$ such that $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $C_0(X)$ for some fixed scalar α_y and for all $y \in Y_1$.

Proof. Consider the restriction of T to $C_c(X)$. Then we conclude that $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $C_c(X)$, for some fixed scalar α_y , because $\delta_y \circ T = 0$ on $C_c(X)$ if and only if $\delta_y \circ T = 0$ on $C_0(X)$. Continuity assumption on $\delta_y \circ T$ also implies that the relation $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ is true on $C_0(X)$ also because $C_c(X)$ is dense in $C_0(X)$.

The classical Urysohn lemma and the result on existence of partitions of unity have their version in $\mathcal{D}(\mathbb{R})$ (see: [6, Section 1.8 Lemma 1] and [5, Theorem 6.20]). So we have the following version of the previous theorem to the space $\mathcal{D}(\mathbb{R})$. $\square$

Theorem 5. *Let Y be a nonempty set. Let T be a linear disjointness preserving mapping from $\mathcal{D}(\mathbb{R})$ to $F(Y)$. Let Y_1, Y_2 , and Y_3 be defined as in Theorem 2. Then there is a function $\phi : Y_1 \cup Y_2 \rightarrow \mathbb{R}$ such that $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $\mathcal{D}(\mathbb{R})$ for some fixed scalar α_y and for all $y \in Y_1$ and $\delta_y \circ T = 0$ on $\{f \in \mathcal{D}(\mathbb{R}) : \phi(y) \notin \text{supp } f\}$, for all $y \in Y_1 \cup Y_2$.*

Proof. To each $y \in Y$, define $\Phi(y) = \{x \in \mathbb{R} : \text{for every open neighbourhood } U \text{ of } x, \text{ there is an } f \in \mathcal{D}(\mathbb{R}) \text{ such that } \text{coz}(f) \subset U \text{ and } Tf(y) \neq 0\}$. It is easy to verify that $\Phi(y)$ contains at most one point, for every $y \in Y$. We next claim that $\Phi(y)$ is nonempty for every $y \in Y_1 \cup Y_2$. On the contrary we assume that $\Phi(y)$ is empty for some $y \in Y_1 \cup Y_2$. Note that $\delta_y \circ T \neq 0$. Fix $f \in \mathcal{D}(\mathbb{R})$. Let $K = \text{supp } f$. To each $x \in K$, let U_x be an open neighbourhood of x such that $Tg(y) = 0$ whenever $g \in \mathcal{D}(\mathbb{R})$ and $\text{coz}(g) \subset U_x$. Then we can find a finite subfamily $\{U_1, U_2, \dots, U_n\}$ of $\{U_x : x \in K\}$ such that the subfamily covers K . By [5, Theorem 6.20] there are functions $f_1, f_2, \dots, f_n$ in $\mathcal{D}(\mathbb{R})$ such that $0 \leq f_i \leq 1$, for every i , $f_1 + f_2 + \dots + f_n = 1$ on K and $\text{supp } f_i \subset U_i$, for all i . Then $f \cdot f_i \in \mathcal{D}(\mathbb{R})$ and $\text{coz}(f \cdot f_i) \subset U_i$ and hence $T(f \cdot f_i)(y) = 0$, for every i . So, $Tf(y) = T(\sum_{i=1}^n f \cdot f_i)(y) = \sum_{i=1}^n T(f \cdot f_i)(y) = 0$. Thus $\delta_y \circ T = 0$ on $\mathcal{D}(\mathbb{R})$, which is a contradiction. Hence $\Phi(y)$ is a singleton set for each $y \in Y_1 \cup Y_2$. Define a function $\phi : Y_1 \cup Y_2 \rightarrow \mathbb{R}$ by $\{\phi(y)\} = \Phi(y)$. To each $y \in Y_1 \cup Y_2$, write $J_y = \{f \in \mathcal{D}(\mathbb{R}) : \phi(y) \notin \text{supp } f\}$, and $K_y = \{f \in \mathcal{D}(\mathbb{R}) : f(\phi(y)) = 0\}$. If $y \in Y_1 \cup Y_2$ and $f \in J_y$, then $\phi(y) \notin \text{supp } f$ and there is a $g \in \mathcal{D}(\mathbb{R})$ with $\text{coz}(g) \subset X \setminus \text{supp } f$ such that $Tg(y) \neq 0$. Since T is disjointness preserving, $Tf(y) = 0$. Thus $J_y \subset \ker(\delta_y \circ T)$. If $f \in K_y$, $y \in Y_1 \cup Y_2$, and $\epsilon > 0$, then by [6, Lemma 1 of Section 1.8] there is a $g \in \mathcal{D}(\mathbb{R})$ such that $\|g\| = 1$, $0 \leq g \leq 1$, g has value 1 on $\{x \in \mathbb{R} : |f(x)| \geq \epsilon\}$, and g has value 0 on $\{x \in \mathbb{R} : |f(x)| \leq \epsilon/2\}$, $f \cdot g \in J_y$, and $\|f \cdot g - f\| \leq \epsilon$. Thus J_y is dense in K_y . If $y \in Y_1$, then $\ker(\delta_y \circ T)$ is closed and hence $K_y \subset \ker(\delta_y \circ T)$. So $K_y = \ker(\delta_y \circ T)$ and $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $\mathcal{D}(\mathbb{R})$, for some α_y . This proves the theorem. $\square$

Observe that $\mathcal{D}(\mathbb{R})$ is dense in $C_0^\infty(\mathbb{R})$ in view of [6, Lemma 1 in Section 1.8]. So, we have the following corollary.

Corollary 6. *Let Y be a nonempty set. Let T be a linear disjointness preserving mapping from $C_0^\infty(\mathbb{R})$ to $F(Y)$. Suppose $\delta_y \circ T$ is continuous on $C_0^\infty(\mathbb{R})$, for every $y \in Y$. Let Y_1, Y_2 be defined as in Corollary 4. Then there is a function $\phi : Y_1 \rightarrow X$ such that $\delta_y \circ T = \alpha_y \delta_{\phi(y)}$ on $C_0^\infty(\mathbb{R})$ for some fixed scalar α_y , and for all $y \in Y_1$.*

Remark 7. One may change $\mathbb{R}$ into any open region in $\mathbb{R}^n$ in Theorem 5 and Corollary 6. $C_0^\infty(\mathbb{R})$ may be replaced in Corollary 6 by any of its linear subspace containing $\mathcal{D}(\mathbb{R})$. $C_0(X)$ may be replaced in Corollary 4 by any of its linear subspace containing $C_c(X)$.

3. FTDP-Mappings

Theorem 8. *Let X be a compact Hausdorff space and Y a nonempty set. Let $T : C(X) \rightarrow F(Y)$ be a nonzero disjointness preserving mappings. Let ϕ, Y_1, Y_2 , and Y_3 be defined as in Theorem 2. Consider the following statements.*

- (i) T is a FTDP-mapping.
- (ii) There is a unique $x_0 \in X$ such that, for every open neighbourhood U of x_0 , there is a $f \in C(X)$ with $\text{coz}(f) \subset U$ such that $Tf \neq 0$ in $F(Y)$.
- (iii) $\phi(Y_1 \cup Y_2)$ is a singleton subset of X .
- (iv) $\cap\{\text{supp } f : f \in C(X), Tf \neq 0 \text{ in } F(Y)\}$ is a singleton set.

Then the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv), and (iii) $\Rightarrow$ (ii) are true. If (i) is true, then the singleton sets in (iii) and (iv) are $\{x_0\}$ for x_0 given in (ii). If (ii) is true, then $x_0 \in \overline{\phi(Y_1 \cup Y_2)}$.

Proof. (i) $\Rightarrow$ (ii): Suppose T is a FTDP-mapping. Suppose there are two distinct points $x_1, x_2 \in X$ with the property given in (ii). We can find two disjoint open neighbourhoods U_1, U_2 of x_1, x_2 , respectively. Then there are $f_1, f_2 \in C(X)$ with $\text{coz}(f_1) \subset U_1, \text{coz}(f_2) \subset U_2$ such that $Tf_1 \neq 0$ and $Tf_2 \neq 0$. This is a contradiction. So, uniqueness of x_0 in (ii) is established. If there is no x_0 in X with the property mentioned in (ii), then for each $x \in X$, there is a neighbourhood U_x of x such that $Tf = 0$ in $F(Y)$ whenever $\text{coz}(f) \subset U_x$. Then we find a subcover $\{U_1, U_2, \dots, U_n\}$ of $\{U_x : x \in X\}$ for X . Then we find a partition $g_1, g_2, \dots, g_n$ for unity in $C(X)$ such that $\text{coz}(g_i) \subset U_i, i = 1, 2, \dots, n$. Then for every $f \in C(X)$, we have $Tf = T(\sum_{i=1}^n fg_i) = \sum_{i=1}^n T(fg_i) = 0$. Thus $T = 0$ on $C(X)$, a contradiction. So (ii) is true.

Note that $\phi(Y_1 \cup Y_2)$ is nonempty, because $T \neq 0$ on $C(X)$. Moreover $y \in Y_1 \cup Y_2$ if and only if $\delta_y \circ T \neq 0$ on $C(X)$. We claim that $x_0 \in \overline{\phi(Y_1 \cup Y_2)}$. On the contrary, we assume that $x_0 \notin \overline{\phi(Y_1 \cup Y_2)}$. Find an open neighbourhood U_0 of x_0 and a function $f_0 \in C(X)$ such that $\text{coz}(f_0) \subset U_0, Tf_0 \neq 0$ in $F(Y)$, and $\overline{U_0} \cap \overline{\phi(Y_1 \cup Y_2)} = \emptyset$. Fix any $y \in Y_1 \cup Y_2$. Let $x_1 = \phi(y)$. Then there is an open neighbourhood U_1 of x_1 such that $\overline{U_0} \cap U_1 = \emptyset$. Find a function $f_1 \in C(X)$ such that $\text{coz}(f_1) \subset U_1$ and $Tf_1(y) \neq 0$ in $C(X)$. So $Tf_0(y) = 0$, because T is disjointness preserving. Thus $Tf_0(y) = 0$, for any $y \in Y_1 \cup Y_2$. So $Tf_0 = 0$, where $Tf_0 \neq 0$ in $C(X)$. Therefore, we conclude that $x_0 \in \overline{\phi(Y_1 \cup Y_2)}$, if (ii) is true.

(i) $\Rightarrow$ (iii): Suppose T is a FTDP-mapping. Since $T \neq 0, Y_1 \cup Y_2$ is nonempty. Suppose $\phi(y_1) \neq \phi(y_2)$ for some $y_1, y_2 \in Y_1 \cup Y_2$. Find two disjoint open sets U and V such that $\phi(y_1) \in U$ and $\phi(y_2) \in V$. Find $f, g \in C(X)$ such that $\text{coz}(f) \subset U, \text{coz}(g) \subset V$, and $Tf(y_1) \neq 0, Tg(y_2) \neq 0$. Then $f \cdot g = 0$ in $C(X)$, but $Tf \neq 0$ and $Tg \neq 0$ in $F(Y)$. This is a contradiction. So, $\phi(Y_1 \cup Y_2)$ is a singleton set, and $\phi(Y_1 \cup Y_2) = \overline{\phi(Y_1 \cup Y_2)}$.

(i) $\Rightarrow$ (iv): Suppose (i) is true. Consider x_0 given in (ii). Suppose $f \in C(X), Tf \neq 0$ in $F(Y)$, and $x_0 \notin \text{supp } f$. Then find $g \in C(X)$ such that $\text{coz}(g) \subset X \setminus \text{supp } f$ and $Tg \neq 0$ in $F(Y)$. Thus $f \cdot g = 0, Tf \neq 0$ and $Tg \neq 0$, a contradiction. Thus $x_0 \in \text{supp } f$ if $Tf \neq 0$. If $x_1 \in X$ and $x_1 \neq x_0$, find two disjoint

open neighbourhoods U_0 and U_1 of x_0 and x_1 , respectively. Find $f \in C(X)$ such that $Tf \neq 0$ and $\text{coz}(f) \subset U_0$. Then $x_1 \notin \text{supp } f$. So, $\{x_0\} = \cap \{\text{supp } f : f \in C(X), Tf \neq 0 \text{ in } F(Y)\}$, where x_0 the member mentioned in (ii). This proves (iv).

(iii) $\Rightarrow$ (ii): Suppose (iii) is true, and suppose $\phi(Y_1 \cup Y_2) = \{x_0\}$. Then for every $y \in Y_1 \cup Y_2$ and for every open neighbourhood U of x_0 there is a function $f \in C(X)$ such that $\text{coz}(f) \subset U$ and $Tf(y) \neq 0$, and hence $Tf \neq 0$ in $F(Y)$. Let $x_1 \in X$ be such that $x_1 \neq x_0$. Find two disjoint neighbourhoods of U_0, U_1 of x_0, x_1 , respectively. Fix $y \in Y_1 \cup Y_2$ arbitrarily. Find a function $f \in C(X)$ such that $\text{coz}(f) \subset U_0$ and $Tf(y) \neq 0$. Then for every function $g \in C(X)$ with $\text{coz}(g) \subset U_1$, we have $f \cdot g = 0$, and hence $Tg(y) = 0$, since T is disjointness preserving. Thus for every function $g \in C(X)$ with $\text{coz}(g) \subset U_1$, we have $Tg(y) = 0$, for every $y \in Y_1 \cup Y_2$, and hence we have $Tg = 0$ in $F(Y)$. This of course proves (ii), and it is proved that x_0 is the member mentioned in (ii). $\square$

Corollary 9. *Let X be a compact Hausdorff space and Y be a nonempty set. Let $T : C(X) \rightarrow F(Y)$ a nonzero FTDP- mapping. Then the following are equivalent.*

- (i) $\delta_y \circ T$ is continuous for every $y \in Y$.
- (ii) $Tf = \delta_{x_0}(f)g$ on $C(X)$ for some fixed function $g \in F(Y)$, for some fixed $x_0 \in X$, and for all $f \in C(X)$.

Proof. Suppose (i) is true. Suppose $\phi(Y_1 \cup Y_2) = \{x_0\}$. Then for every $y \in Y_1 \cup Y_2$, we have $\{f \in C(X) : x_0 \notin \text{supp } f\} \subset \ker(\delta_y \circ T)$, and hence $\{f \in C(X) : f(x_0) = 0\} = \ker(\delta_y \circ T)$. Thus, $\ker T = \{f \in C(X) : f(x_0) = 0\}$. This proves (ii). Another implication is obvious. $\square$

Remark 10. Theorem 8 is extendable to $C_c(X)$ when X is normal and to $\mathcal{D}(\mathbb{R})$. Corollary 9 is extendable to $C_c(X)$, $C_0(X)$, $\mathcal{D}(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ when X is normal. Theorem 8 is extendable to $C_0(X)$ when X is locally compact with an additional assumption that the singleton set may be $\{\infty\}$ in X_∞ . Corollary 9 is extendable to $C_0(X)$ when X is locally compact.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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