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Research Article

On the Torsion Units of Integral Adjacency Algebras of Finite Association Schemes

Allen Herman and Gurmail Singh

Department of Mathematics and Statistics, University of Regina, Regina, SK, Canada S4S 0A2

Correspondence should be addressed to Allen Herman; allen.herman@uregina.ca

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Torsion units of group rings have been studied extensively since the 1960s. As association schemes are generalization of groups, it is natural to ask about torsion units of association scheme rings. In this paper we establish some results about torsion units of association scheme rings analogous to basic results for torsion units of group rings.

1. Introduction

In this paper we will consider torsion units of rings generated by finite association schemes, which we now define. Let X be a finite set of size $n > 0$. Let S be a partition of $X \times X$ such that every relation in S is nonempty. For a relation $s \in S$, there corresponds an adjacency matrix, denoted by σ_s, which is the $n \times n$ $(0,1)$-matrix whose (i, j) entries are 1 if $(i, j) \in s$ and 0 otherwise. (X, S) is an *association scheme* if

(i) S is a partition of $X \times X$ consisting of nonempty sets,

(ii) S contains the identity relation $1_X := \{(x, x) : x \in X\}$,

(iii) for all s in S the adjoint relation $s^* := \{(y, x) \in X \times X : (x, y) \in s\}$ also belongs to S,

(iv) for all s, t, and u in S there exists a nonnegative integer structure constant a_{stu} such that $\sigma_s\sigma_t = \sum_{u\in S} a_{stu}\sigma_u$.

A finite association scheme (X, S) is said to have order $n = |X|$ and rank $r = |S|$. For notation and background on association schemes, see [1].

The structure constants of the scheme (X, S) make the integer span of its adjacency matrices into a natural $\mathbb{Z}$-algebra $\mathbb{Z}S := \oplus_{s\in S}\mathbb{Z}\sigma_s$. This is known as the *integral adjacency algebra* of the scheme (X, S), which we will simply refer to as the *integral scheme ring*. Note that the multiplicative identity of $\mathbb{Z}S$ is the $n \times n$ identity matrix, which is the adjacency matrix $\sigma_{1_X} := \sigma_1$. Similarly we can define the R-algebra RS for any commutative ring R with identity, which is known as the adjacency algebra of the scheme over R.

The complex adjacency algebra $\mathbb{C}S$ is a semisimple algebra with involution defined by $u^* = \sum_s \overline{u_s}\sigma_{s^*}$. This involution is an antiautomorphism of the algebra $\mathbb{C}S$. The natural inclusion $\mathbb{C}S \hookrightarrow M_n(\mathbb{C})$ is the *standard representation* of $\mathbb{C}S$ (or (X, S)). Its character ρ satisfies $\rho(\sigma_1) = n = |X|$ and $\rho(\sigma_s) = 0$ for all $1 \neq s \in S$. Clearly the degree of the standard representation is $n = |X|$.

It is easy to show using the definition of a scheme that the structure constant $a_{st1} \neq 0$ if and only if $t = s^*$. We write n_s instead of a_{ss^*1} and call n_s the *valency* of s. The linear extension of the valency map defines a degree one algebra representation $\mathbb{C}S \to \mathbb{C}$ by

$$u = \sum_{s\in S} u_s\sigma_s \longrightarrow n_u = \sum_s u_s n_s. \tag{1}$$

We say that $s \in S$ is a *thin* element of S when $n_s = 1$. The *thin radical* $\mathcal{O}_\vartheta(S)$ of S is the subset consisting of the thin elements of S. It follows from the fact that the valency map is a ring homomorphism that $\{\sigma_t : t \in \mathcal{O}_\vartheta(S)\}$ is a group.

If R is a ring with identity, then $U(R)$ denotes the group of units of R and $U(R)^{\text{tor}}$ denotes its subset consisting of torsion units (i.e., units with finite multiplicative order). The subgroup of $U(\mathbb{Z}S)$ consisting of units with valency 1 is denoted by $V(\mathbb{Z}S)$. Its subset $V(\mathbb{Z}S)^{\text{tor}}$ consists of normalized torsion units.

The results of Section 2 show that $V(\mathbb{Z}S)^{\text{tor}}$ is often equal to the thin radical of S when S is a commutative finite scheme. In particular this holds for symmetric association schemes or if the valency of any element of S is divisible by a prime p. In Section 3 we establish a "Lagrange-type" theorem for finite subgroups of $V(\mathbb{Z}S)^{\text{tor}}$, by showing that the order of any finite subgroup of $V(\mathbb{Z}S)^{\text{tor}}$ divides the order of S and is bounded by the rank of S. In Section 4 this result is directly applied to Schur rings and Hecke algebras.

Throughout the paper ζ_k will denote a complex primitive kth root of unity for a given positive integer k. When $u \in \mathbb{C}S$, we will consistently use the notation $u = \sum_s u_s\sigma_s$ with $u_s \in \mathbb{C}$ for all $s \in S$.

2. The Support of Normalized Torsion Units of $\mathbb{Z}S$

Our first lemma is an analogue of Berman-Higman's proposition on torsion units of group rings (see [2, 3]).

Lemma 1. *Let (X, S) be a finite association scheme. Suppose $u \in V(\mathbb{Z}S)^{tor}$. Then $u_1 \neq 0 \Rightarrow u = \sigma_1$.*

Proof. Let $\Gamma : \mathbb{C}S \to M_n(\mathbb{C})$ be the standard representation of (X, S) of degree $n = |X|$. Let ρ be the standard character; so

$$\rho(\sigma_s) = \begin{cases} n & \text{if } s = 1_X \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

$\Gamma(u)$ is diagonalizable since $\Gamma(u)^k = I$ for some integer k. If spec$\Gamma(u)$ denotes the list of eigenvalues of $\Gamma(u)$ (including multiplicities), then spec$\Gamma(u) = \{\zeta_k^{f_i}\}_{i=1}^n$ consists of kth roots of 1. Now $\rho(u) = \sum_{i=1}^n \zeta_k^{f_i} = u_1 \cdot n$. Then $|u_1|n = |\sum_{i=1}^n \zeta_k^{f_i}| \leq n$, and $|u_1|n = n \Leftrightarrow$ all $\zeta_k^{f_i}$'s are equal to $\zeta_k^{f_1}$. Thus $u_1 \in \{-1, 0, 1\}$, and $u_1 \neq 0 \Rightarrow \Gamma(u) = \zeta_k^{f_1} \cdot I \Rightarrow u = \zeta_k^{f_1}\sigma_1 = u_1\sigma_1$. As $u \in V(\mathbb{Z}S)^{\text{tor}}$, $u_1 \neq -1$. Therefore $u = \sigma_1$. □

Let (X, S) be an association scheme and let R be a commutative ring with identity. Let $u = \sum_{s\in S} u_s\sigma_s \in RS$; then s in S belongs to the *support* of u (briefly supp(u)) if and only if $u_s \neq 0$. We will say that $u \in U(RS)$ is a *trivial unit* if u is a unit of RS for which $u = u_s\sigma_s$ for some $u_s \in U(R)$ and a unique element s in the support of u, which is necessarily a thin element. Trivial units of $\mathbb{Z}S$ are permutation matrices with possibly negative sign in the standard representation.

Proposition 2. *Let u be a unit of $\mathbb{Z}S$ with $uu^* = \sigma_1$. Then u is a trivial unit.*

Proof. Consider $uu^* = \sigma_1 \Rightarrow 1 = (uu^*)_1 = \sum_s u_s\overline{u_s}n_s = \sum_s |u_s|^2 n_s$. Since $u \in \mathbb{Z}S$, it follows that $u_s = 0$ except for exactly one $s \in S$ with $n_s = 1$ and $u_s = \pm 1$. □

Proposition 3. *Let $u \in V(\mathbb{Z}S)^{tor}$. If $s \in \mathcal{O}_\vartheta(S) \cap$ supp (u) and σ_s commutes with u, then $u = \sigma_s$ is a trivial unit of $\mathbb{Z}\mathcal{O}_\vartheta(S)$.*

Proof. Let $u \in V(\mathbb{Z}S)^{\text{tor}}$. Let $s \in \mathcal{O}_\vartheta(S) \cap \text{supp}(u)$ for which σ_s commutes with u. Then σ_s is a unit of $\mathbb{Z}S$ and $u_s \neq 0$. Let $u' = \sigma_s^{-1}u$. Since σ_s commutes with u, u' has finite order. Since $u'_1 = u_s \neq 0$, we must have $u' = \sigma_1$ by Lemma 1. Therefore $u = \sigma_s$. □

The *center* of the finite association scheme (X, S) is defined to be $Z(S) = \{t \in S : \sigma_t\sigma_s = \sigma_s\sigma_t, \text{ for all } s \in S\}$. The scheme (X, S) is a *commutative* scheme if $Z(S) = S$. The next two corollaries are immediate from Proposition 3.

Corollary 4. *Let (X, S) be a finite association scheme. Suppose $u \in V(\mathbb{Z}S)^{tor}$ is a nontrivial unit. If $s \in$ supp (u), then either $n_s \geq 2$ or $s \notin Z(S)$.*

Corollary 5. *Let (X, S) be a finite commutative association scheme. Suppose $u \in V(\mathbb{Z}S)^{tor}$ is a nontrivial unit. If $s \in$ supp (u), then $n_s \geq 2$.*

If G is a finite group, then it is well known that central torsion units of $\mathbb{Z}G$ are trivial [4, Theorem 2.1]. We are able to extend this result to finite association schemes whose nonthin elements have valencies divisible by a single prime.

Theorem 6. *Suppose (X, S) is a finite association scheme. Suppose there is a prime integer p that divides n_s for every $s \in S$ with $n_s > 1$. Then every normalized central torsion unit of $\mathbb{Z}S$ is a trivial unit.*

Proof. Let $u = \sum_{s\in S} u_s\sigma_s \in V(\mathbb{Z}S)^{\text{tor}}$ be a central element of $\mathbb{Z}S$ with multiplicative order k. Suppose u is not trivial. By Proposition 3, every $s \in \text{supp}(u)$ has $n_s > 1$. Our assumption then implies that p divides n_s, for every $s \in \text{supp}(u)$.

Then $1 = (u^k)_1 \in \text{span}_{\mathbb{Z}}\{(\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_k})_1 : s_i \in \text{supp}(u), \text{ for } 1 \leq i \leq k\}$. If $(\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_k})_1 \neq 0$, then it is divisible by $(\sigma_{s_1}\sigma_{s_1^*})_1 = n_{s_1}$, hence divisible by p. This contradicts $1 = (u^k)_1$; hence the result follows. □

A finite association scheme (X, S) is *p-valenced* for some prime integer p if n_s is a power of p for all $s \in S$. We know that, for a finite abelian group G, every torsion unit of the integral group ring $\mathbb{Z}G$ is a trivial unit. The next corollary generalizes this result to p-valenced commutative schemes.

Corollary 7. *If (X, S) is a finite p-valenced commutative association scheme, then every normalized torsion unit of $\mathbb{Z}S$ is a trivial unit.*

Proof. Since (X, S) is a commutative association scheme, the adjacency algebra $\mathbb{Z}S$ is a commutative ring. Therefore, every unit of $\mathbb{Z}S$ is central. By Theorem 6, every $u \in V(\mathbb{Z}S)^{\text{tor}}$ must be a trivial unit; that is, $u = \sigma_s$, for some $s \in S$ with $n_s = 1$. □

An association scheme (X, S) is *symmetric* if all of the adjacency matrices σ_s for $s \in S$ are symmetric matrices, or, equivalently, $s^* = s$ for all $s \in S$. It is easy to show that symmetric association schemes are commutative.

Theorem 8. *Let (X, S) be a finite symmetric association scheme. If $u \in V(\mathbb{Z}S)^{tor}$, then $u = \sigma_s$, for some $s \in S$ and $\sigma_s^2 = \sigma_1$. In particular, torsion units of $\mathbb{Z}S$ are trivial with order 2 at most.*

Proof. Suppose $u \in V(\mathbb{Z}S)$ has multiplicative order k. Since every element of $\mathbb{Z}S$ is a symmetric matrix, the eigenvalues of u are totally real algebraic integers. Since u has finite multiplicative order, the eigenvalues of u must also be roots of unity. Therefore, the only possibilities for eigenvalues of u are ± 1, and the order of u can only be 1 or 2.

Suppose $u \in V(\mathbb{Z}S)$ is a nontrivial torsion unit whose order is 2. Then $u \neq \sigma_s$, $\forall s \in S$, but $u^2 = \sigma_1$. Also, $u^2 = (\sum_s u_s^2 n_s)\sigma_1$. By Corollary 5, $n_s \geq 2$ for all $s \in \operatorname{supp}(u)$, so it follows that $(u^2)_1 \geq 2$, a contradiction. □

Theorem 8 has the following immediate consequence.

Corollary 9. *Let (X, S) be a symmetric association scheme. If T is a finite subgroup of $V(\mathbb{Z}S)$, then T is an elementary abelian 2-group.*

3. Lagrange's Theorem for Normalized Torsion Units of $\mathbb{Z}S$

The next proposition extends a result concerning idempotents of group algebras over fields of characteristic 0 to adjacency algebras of finite association schemes over fields of characteristic 0.

Proposition 10. *Let K be a field of characteristic 0 and let (X, S) be a finite association scheme of order n. Let $e = \sum_{s\in S} e_s\sigma_s \neq 0, 1$ be a nontrivial idempotent of KS. Then $e_1 = m/n \in \mathbb{Q}$, $0 < e_1 < 1$, where $n = |X|$ and m is the rank of e as the matrix in the standard representation.*

Proof. Let Γ be the standard representation and let ρ be the standard character of KS. As e is an idempotent, we know that $\operatorname{spec}(\Gamma(e)) = [1^{(m)}, 0^{(n-m)}]$ as a multiset, where $m = \operatorname{rank}(\Gamma(e))$. Thus

$$\rho(e) = \sum_{s\in S} e_s \rho(\sigma_s) = e_1 \cdot n = m. \tag{3}$$

Therefore, $e_1 = m/n \in \{0, 1/n, \ldots, (n-1)/n, 1\}$, and we have $e_1 = 0 = 0/n \Leftrightarrow e = 0$ and $e_1 = 1 = n/n \Leftrightarrow e = \sigma_1$. □

Corollary 11. *Let (X, S) be a finite association scheme. Then the only idempotents of $\mathbb{Z}S$ are 0 and σ_1.*

Proof. Let $e \in \mathbb{Z}S$ be an idempotent. Then $e_1 \in \mathbb{Z}$. By Proposition 10, this implies $e_1 = 0$ or 1, and by considering the rank of $\Gamma(e)$ in these respective cases we have $e = 0$ or σ_1. □

Here we give a glance on the fact that the association scheme concept generalizes the group concept. For more details, see [1, Section 5.5]. Let (X, S) be a finite association scheme of order n for which every relation in S is thin, that is, a thin association scheme. Then using the valency map it follows that $\{\sigma_s : s \in S\}$ is a group of n distinct permutation matrices. Conversely, let G be a group. For each g in G, let g^τ denote the set of all pairs $(e, f) \in G \times G$ satisfying $eg = f$. Let G^τ denote the set of all sets g^τ with g in G. Then (G, G^τ) becomes a thin association scheme. So there is correspondence between thin association schemes and groups, called the group correspondence. In this correspondence, the augmentation map of the integral group ring $\mathbb{Z}G$ agrees with the valency map of the integral scheme ring $\mathbb{Z}[G^\tau]$.

If $u = \sum_{g\in G} u(g)g \in \mathbb{Z}G$, then augmentation of u is $\sum_{g\in G} u(g) \in \mathbb{Z}$. We know any finite subgroup $H \subseteq V(\mathbb{Z}G)$ is a linearly independent set (cf. [5, Lemma (37.1)]). One can ask what happens in the case of scheme rings. The next lemma gives an answer to this question.

Lemma 12. *Let (X, S) be a finite association scheme. Then any finite group of units of valency 1 in $\mathbb{Z}S$ is a set of linearly independent elements.*

Proof. Let $T = \{u_1 = \sigma_1, u_2, \ldots, u_\ell\}$ be a finite group of units contained in $V(\mathbb{Z}S)$. Suppose $c_1 u_{i_1} + \cdots + c_m u_{i_m} = 0$ is an expression of minimal length, where u_{i_j} are elements of T and the coefficients $c_j \in \mathbb{Z}$ are not all 0. Since T is a group, we can assume without loss of generality that $u_{i_1} = \sigma_1$. Expressing the u_{i_j} for $j = 2, \ldots, m$, as $u_{i_j} = \sum_s u_{i_j,s}\sigma_s$, we have by Lemma 1 that $u_{i_j,1} = 0$ for $j = 2, \ldots, m$. It follows that

$$0 = \left(c_1\sigma_1 + c_2 u_{i_2} + \cdots + c_m u_{i_m}\right)_1 = c_1, \tag{4}$$

contradicting the minimal length assumption. Therefore, T is a linearly independent set. □

For a finite group G, the order of any finite subgroup H of $V(\mathbb{Z}G)$ divides the order of G [5, Lemma (37.3)]. Our main theorem shows this also holds for schemes.

Theorem 13. *Let (X, S) be a finite association scheme of order n and rank r. Then the order of any finite subgroup T of $V(\mathbb{Z}S)$ divides n and is at most r. Symbolically, $|T|$ divides $|X|$ and $|T| \leq |S|$.*

Proof. Since $\mathbb{Z}S$ is a free module with basis S, any basis of $\mathbb{Z}S$ must have $|S|$ elements. By Lemma 12, T is a linearly independent subset. Therefore $|T| \leq |S|$.

Now let $e = \sum_{t\in T}(t/|T|) = \widehat{T}/|T| = e^2$, where $\sum_{t\in T} t = \widehat{T}$. Let Γ be the standard representation and let ρ be the standard character of (X, S). Since $e^2 = e \neq 0$, $\operatorname{spec}(\Gamma(e)) = [1^{(m)}, 0^{(n-m)}]$, where m is the rank of the matrix $\Gamma(e)$. Therefore, $\rho(e) = m \in \mathbb{Z}^+$. Also $\rho(e) = (1/|T|)\rho(\widehat{T}) = (1/|T|)n(\widehat{T})_1 = n/|T|$, since the argument of Lemma 12 implies $(\widehat{T})_1 = 1$. Therefore, $m = n/|T|$; hence $|T|$ divides $n = |X|$, which proves the theorem. □

We have been unable to settle the question of whether or not the order of any finite subgroup of $V(\mathbb{Z}S)$ must divide the order of $\mathcal{O}_\vartheta(S)$. Related to this is a possible generalization of the Zassenhaus conjecture on torsion units to integral scheme rings, which would be that any normalized torsion unit of $\mathbb{Z}S$ should be conjugate in $\mathbb{Q}S$ to some σ_s, for an $s \in \mathcal{O}_\vartheta(S)$.

If H is a subgroup of $V(\mathbb{Z}G)$ for a finite group G with $|H| = |G|$, then $\mathbb{Z}G = \mathbb{Z}H$ (cf. [5, Lemma (37.4)]). The following lemma proves an analogous result for schemes.

Lemma 14. *Let (X, S) be a finite association scheme with rank r. If T is a finite subgroup of $V(\mathbb{Z}S)$ with $|T| = r$, then $\mathbb{Z}S = \mathbb{Z}T$.*

Proof. By Lemma 12, T is linearly independent and thus $\mathbb{Q}S = \mathbb{Q}T$. It follows that $\mathbb{Z}S \supseteq \mathbb{Z}T$ and $m\mathbb{Z}S \subset \mathbb{Z}T$ for some positive integer m.

Let $T = \{t_1 = \sigma_1, t_2, \ldots, t_r\}$ and let $s \in S$. Then

$$m\sigma_s = \sum_{i=1}^{r} c_i t_i, \quad \text{for some } c_i \in \mathbb{Z}. \tag{5}$$

We wish to show that each c_i is a multiple of m. For each $j \in \{1, \ldots, r\}$, we have

$$m\sigma_s t_j^{-1} = c_j\sigma_1 + \sum_{i \neq j} c_i \left(t_i t_j^{-1}\right). \tag{6}$$

Since, by Lemma 1, $(t_i t_j^{-1})_1 = 0$ for $i \neq j$, the coefficient of σ_1 on the right hand side is c_j whereas on the left hand side it is a multiple of m. It follows that $m \mid c_j$ for $j = 1, \ldots, r$. Therefore, $\sigma_s \in \mathbb{Z}T$ for all $s \in S$, and hence $\mathbb{Z}S = \mathbb{Z}T$. □

While thin association schemes give immediate examples where the conclusion of the preceding theorem holds, we are uncertain as to whether $\mathbb{Z}S$ can possess a finite subgroup of normalized units of order $|S|$ when (X, S) is not thin. The next example shows that it is certainly possible for the adjacency algebra $\mathbb{Q}S$ to be ring isomorphic to a group algebra when (X, S) is not thin.

Example 15. Let (X, S) be the fifth association scheme of order 27 in Hanaki and Miyamoto's classification of small association schemes [6]. This is a commutative nonsymmetric scheme of order 27 and rank 3. We have $S = \{1_X, s, s^*\}$, where $n_s = n_{s^*} = 13$, and the structure constants of S are determined by $\sigma_s^2 = 6\sigma_s + 7\sigma_{s^*}$, $\sigma_s\sigma_{s^*} = 13\sigma_1 + 6\sigma_s + 6\sigma_{s^*}$, and $\sigma_{s^*}^2 = 7\sigma_s + 6\sigma_{s^*}$.

Analysis of the character table of S (see [6]) shows that $\mathbb{Q}S \cong \mathbb{Q}C_3$, where C_3 is a cyclic group of order 3. Let χ_1 be the irreducible character of $\mathbb{C}S$ corresponding to the valency map, and let $\psi, \overline{\psi}$ be the other two irreducible characters of $\mathbb{C}S$. Let $\{e_{\chi_1}, e_\psi, e_{\overline{\psi}}\}$ be the centrally primitive idempotents of $\mathbb{C}S$, the character formula for which can be found in [1, Lemma 9.1.6]. An element v of $\mathbb{C}S$ with order 3 and valency 1 is given by

$$v = e_{\chi_1} + \zeta_3 e_\psi + \zeta_3^2 e_{\overline{\psi}}, \tag{7}$$

and since v is fixed by complex conjugation, $v \in \mathbb{Q}S$. Using the character formula for centrally primitive idempotents of $\mathbb{C}S$, we find that

$$v = \frac{1}{9}\left(-4\sigma_1 - \sigma_s + 2\sigma_{s^*}\right), \tag{8}$$

and $v^2 = v^*$. So if $T = \{\sigma_1, v, v^2\}$, then T is a finite subgroup of normalized units of $\mathbb{Q}S$ for which $\mathbb{Q}S = \mathbb{Q}T$. In this case $\mathbb{Z}[1/9]S \subseteq \mathbb{Z}T \subsetneq \mathbb{Z}S$.

Proposition 16. *Let (X, S) be the association scheme of order n and rank 2. Then*

$$\left|V(\mathbb{Z}S)^{tor}\right| = \begin{cases} 1 & \text{if } n \geq 3, \\ 2 & \text{if } n = 2. \end{cases} \tag{9}$$

Proof. Let u be a normalized torsion unit of $\mathbb{Z}S$ with multiplicative order k. Our Lagrange theorem for schemes implies that k divides n and $k \leq 2$. So we are done if n is odd. Suppose $k = 2$. Since S is symmetric and $u \in \mathbb{Z}S$, $u = u^*$. Therefore, $u^2 = \sigma_1 \Rightarrow uu^* = \sigma_1$, and so, by Proposition 2, $u = \sigma_s$ for some $s \in S$ with $n_s = 1$. Such an element of the scheme of rank 2 with $s \neq 1_X$ only exists when $n = 2$. □

For symmetric schemes of rank 3, we have already seen that normalized torsion units must be trivial with order 2. Nonsymmetric association schemes of rank 3, such as the one seen in the example above, arise naturally from strongly regular directed graphs.

Proposition 17. *Let (X, S) be a finite association scheme of order $n > 2$ and rank 3 with $S = \{1_X, s, s^*\}$. If $n_s > 1$, then $|V(\mathbb{Z}S)^{tor}| = 1$.*

Proof. Suppose $u \in V(\mathbb{Z}S)$ is a normalized torsion unit with $u \neq \sigma_1$. By Lemma 1, $\operatorname{supp}(u) = \{s, s^*\}$, so $u = a\sigma_s + b\sigma_{s^*}$ for some $a, b \in \mathbb{Z}$. Since $n_u = 1$, we have $1 = an_s + bn_{s^*} = (a+b)n_s$, which is not possible as $a, b \in \mathbb{Z}$. □

4. Applications to Schur Rings and Hecke Algebras

Let G be a finite group of order n. Let $\mathbb{Z}\mathcal{F}$ be a Schur ring defined on the group G. This means that $\mathcal{F}$ is a partition of the set G into nonempty subsets for which we consider the following:

(i) $\{1_G\} \in \mathcal{F}$,

(ii) for all $U = \{g_1, \ldots g_k\} \in \mathcal{F}$, $U^* = \{g_1^{-1}, \ldots, g_k^{-1}\} \in \mathcal{F}$,

(iii) for all $U, V, W \in \mathcal{F}$, there exist nonnegative integers λ_{UVW} such that

$$\widehat{U}\widehat{V} = \sum_{W \in \mathcal{F}} \lambda_{UVW}\widehat{W}, \tag{10}$$

where $\widehat{U} = \sum_{g \in U} g$ denotes the sum of the elements of U in the group ring $\mathbb{Z}G$.

The Schur ring $\mathbb{Z}\mathcal{F}$ is defined to be the $\mathbb{Z}$-span of $\{\widehat{U} : U \in \mathcal{F}\}$, considered as a subring of $\mathbb{Z}G$. $\mathbb{Z}\mathcal{F}$ is a free $\mathbb{Z}$-module of rank $r = |\mathcal{F}|$. By extension of scalars we can consider the Schur ring $R\mathcal{F}$ for any commutative ring R. We will refer to a partition of G with the above properties as a *Schur ring partition* of G. One example of a Schur ring partition is the partition $\mathcal{F}$ of G into its conjugacy classes, in which case the complex Schur ring $\mathbb{C}\mathcal{F}$ is isomorphic to the center of the group ring $\mathbb{C}G$.

We claim that the Schur ring $\mathbb{Z}\mathcal{F}$ is isomorphic to an integral scheme ring. Given the group G and Schur ring

partition $\mathcal{F}$, let $\mathcal{F}^\tau$ be the images of subsets in $\mathcal{F}$ under the group correspondence. So, given $U \in \mathcal{F}$, we set

$$U^\tau = \{(x, y) \in G \times G : xg = y \text{ for some } g \in U\}. \quad (11)$$

Using the properties of the Schur ring partition $\mathcal{F}$, it is straightforward to show that $(G, \mathcal{F}^\tau)$ is an association scheme of order $n = |G|$ and rank $r = |\mathcal{F}|$. Furthermore, $\mathbb{Z}\mathcal{F} \simeq \mathbb{Z}[\mathcal{F}^\tau]$ as rings, where the isomorphism is produced by the restriction of the regular representation of G to $\mathbb{Z}\mathcal{F}$. The restriction of the augmentation map on the group ring to $\mathbb{Z}\mathcal{F}$ corresponds to the valency map of $\mathbb{Z}[\mathcal{F}^\tau]$ under this isomorphism. The following corollary is the application of our Lagrange theorem for scheme rings to this special case.

Corollary 18. *Let $\mathcal{F}$ be a Schur ring partition of a finite group G. Then the order of any finite subgroup of $V(\mathbb{Z}\mathcal{F})$ divides $|G|$ and is at most $|\mathcal{F}|$.*

Let H be a subgroup of a finite group G that has index n. Let G/H be the set of left cosets of H in G. Let r be the number of distinct double cosets HgH of H in G. Corresponding to each double coset HgH for $g \in G$, let

$$g^H := \{(xH, yH) : y \in xHgH\}. \quad (12)$$

Let $G/\!\!/H := \{g^H : g \in G\}$. Then $(G/H, G/\!\!/H)$ is an association scheme of order n and rank r. This type of association scheme is known as a *Schurian* scheme, and its rational adjacency algebra $\mathbb{Q}[G/\!\!/H]$ is ring isomorphic to the ordinary Hecke algebra $e_H\mathbb{Q}Ge_H$, where $e_H = (1/|H|)\sum_{h\in H} h$. (For details, see [7], and note that the argument given there for this fact does not require that the field be algebraically closed.) The application of our Lagrange theorem for scheme rings in this special case gives the next result.

Corollary 19. *Let H be a subgroup of a finite group G that has n left cosets and r double cosets. Then the order of any finite subgroup of $V(\mathbb{Z}[G/\!\!/H])$ divides n and is at most r.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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