Research Article

Parameter Estimation and Hybrid Lag Synchronization in Hyperchaotic Lü Systems

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The antiphase and complete lag synchronization of hyperchaotic Lü systems with unknown parameters is investigated. Based on the Lyapunov stability theory, the sufficient conditions for achieving hybrid lag synchronization are derived. The optimized parameter observers are approached analytically via adaptive control approach. Numerical simulation results are presented to verify the effectiveness of the proposed scheme.

1. Introduction

Chaos has been thoroughly studied over the past two decades for its “random” behavior and sensitive dependence on the initial conditions. Despite the complexity and unpredictability of chaotic behavior [1], it can be controlled and two chaotic systems can be synchronized [2]. Since Pecora and Carroll introduced a method to synchronize two identical chaotic dynamical systems [3], the synchronization of chaotic dynamical systems attracted much attention due to its theoretical challenge and potential application in secure communications, chemical reactions, biomedical science, social science, and many other fields [4–6]. Various synchronizations have been presented, such as complete synchronization (CS), phase synchronization (PS), lag synchronization (LS) or anticipated synchronization (AS), and generalized synchronization (GS) [7–16]. Subsequently, many effective synchronization methods have been proposed, such as linear or nonlinear feedback synchronization, adaptive synchronization, lag synchronization, Q-S synchronization, and anticipated synchronization [17–26].

It is an interesting problem that part of the states of the interactive chaotic system are synchronized in one type of synchronization while other states synchronized in another type of synchronization. This phenomenon is taken as mixed synchronization. Due to the potential applications of it, some

types of mixed synchronization are introduced recently. In [27], some variables may converge into synchronization while other variables are in antisynchronization state in Chen-Lee chaotic systems. In [28], using a scalar coupling, some of the state variables may be in complete synchronization while others may be in antisynchronization state in two unidirectionally coupled chaotic oscillators. It is found that lag synchronization has important technological implications. Generally, lag synchronization can be trivially accomplished by coupling the response system to a past state of the drive system or by mismatching of the system parameters. Therefore, inspired by [29, 30], it is invited to investigate the coexistence lag synchronization of chaotic systems via linear input both for theoretical research and practical applications. Mixed lag synchronization in the hyperchaotic systems could also be a useful topic, particularly for its potential application in secure communication due to its safety against attack and unmasking, the secure keys and carrier wave could be generated in more complex but safe way.

In this paper, the coexistence lag synchronization of two identical hyperchaotic Lü systems will be achieved only by linear controllers. Based on the Lyapunov stability theory, the sufficient conditions for achieving hybrid lag synchronization are derived. Numerical simulation results are presented to demonstrate the effectiveness of the proposed scheme.

FIGURE 1: The hyperchaotic attractors of system (1): (a) (x, y, z) ; (b) (x, y, w) ; (c) (x, z, w) .

2. System Description and Problem Formulation

2.1. Hyperchaotic Lü System. In this paper, we consider the following hyperchaotic Lü system:

$$\begin{aligned}\dot{x} &= a(y - x) \\ \dot{y} &= cy + w - xz \\ \dot{z} &= xy - bz \\ \dot{w} &= -r_1x - r_2y.\end{aligned}\tag{1}$$

The dynamical behavior and its control have been studied in [31]. When $a = 36$, $b = 3$, $c = 20$, $r_1 = r_2 = 2$, system (1) has two positive Lyapunov exponents, that is, $\lambda_1 = 1.4106$, $\lambda_2 = 0.1232$, and the hyperchaotic attractors of system (1) are shown in Figure 1 (3D overview).

2.2. Hybrid Lag Synchronization Formulation. In this section, the hybrid lag synchronization of hyperchaotic Lü system

will be investigated via linear feedback control method. For convenience, the drive system is denoted as the following:

$$\begin{aligned}\dot{x}_m &= a(y_m - x_m) \\ \dot{y}_m &= cy_m + w_m - x_mz_m \\ \dot{z}_m &= x_my_m - bz_m \\ \dot{w}_m &= -r_1x_m - r_2y_m.\end{aligned}\tag{2}$$

The controlled response system is written as

$$\begin{aligned}\dot{x}_s &= \hat{a}(y_s - x_s) + u_1 \\ \dot{y}_s &= \hat{c}y_s + w_s - x_sz_s + u_2 \\ \dot{z}_s &= x_sy_s - \hat{b}z_s + u_3 \\ \dot{w}_s &= -\hat{r}_1x_s - \hat{r}_2y_s + u_4,\end{aligned}\tag{3}$$

where u_1, u_2, u_3, u_4 are controllers to be constructed.

The target of this paper is to determine the controllers, such that the state variables x_s, y_s , and w_s in response

FIGURE 2: The evolutions of the state variables x_s , y_s , z_s , w_s in response system and x_m , y_m , z_m , w_m in drive system at $k = 2.3$.

system antiphase synchronize to the x_m , y_m , and w_m in drive system with a time lag, respectively, while the third state variable z_s complete-synchronizes to z_m with a time lag. For this purpose, the errors of corresponding variables of hybrid synchronization with a time lag and parameters are often defined as

$$\begin{aligned} e_1 &= x_s + x_m(t - \tau), & e_2 &= y_s + y_m(t - \tau), \\ e_3 &= z_s - z_m(t - \tau), & e_4 &= w_s + w_m(t - \tau), \\ e_a &= a - \hat{a}, & e_b &= b - \hat{b}, & e_c &= c - \hat{c}, \\ e_{r_1} &= r_1 - \hat{r}_1, & e_{r_2} &= r_2 - \hat{r}_2. \end{aligned} \quad (4)$$

Then we can get the following theorem.

Theorem 1. *The third corresponding pair variables of systems (2) and (3) will reach complete lag synchronization, while other three pairs of corresponding variables of systems (2) and (3) will reach lag antisynchronization for any original values, when*

$$u_i = -k_i e_i, \quad (i = 1, 2, 3, 4), \quad (5)$$

if the feedback coefficients k_i ($i = 1, 2, 3, 4$) are large enough and the condition

$$\begin{aligned} \dot{e}_a &= e_1(y_s - x_s) \\ \dot{e}_b &= -e_3 z_s \\ \dot{e}_c &= e_2 y_s \\ \dot{e}_{r_1} &= -e_4 x_s \\ \dot{e}_{r_2} &= -e_4 y_s \end{aligned} \quad (6)$$

is satisfied.

Proof. According to (4), the error system can be derived as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) - e_a(y_s - x_s) + u_1 \\ \dot{e}_2 &= ce_2 + e_4 - x_m(t - \tau)z_m(t - \tau) - x_s z_s - e_c y_s + u_2 \\ \dot{e}_3 &= x_s y_s - x_m(t - \tau)y_m(t - \tau) - be_3 + e_b z_s + u_3 \\ \dot{e}_4 &= -r_1 e_1 - r_2 e_2 + e_{r_1} x_s + e_{r_2} y_s + u_4. \end{aligned} \quad (7)$$

FIGURE 3: The evolutions of the state variables $x_m(t - \tau)$, $y_m(t - \tau)$, $z_m(t - \tau)$, $w_m(t - \tau)$ and x_s , y_s , z_s , w_s .

Let Lyapunov stability function be

$$V = \frac{[e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_{r_1}^2 + e_{r_2}^2]}{2}, \quad (8)$$

then

$$\begin{aligned} \dot{V} &= e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 + e_a \dot{e}_a \\ &\quad + e_b \dot{e}_b + e_c \dot{e}_c + e_{r_1} \dot{e}_{r_1} + e_{r_2} \dot{e}_{r_2} \\ &= e_1 [a(e_2 - e_1)] + e_a [\dot{e}_a - e_1(y_s - x_s)] - k_1 e_1^2 \\ &\quad + e_2 [ce_2 + e_4 - x_m(t - \tau)e_3 - z_s e_1] - k_2 e_2^2 \\ &\quad + e_c [\dot{e}_c - e_2 y_s] + e_3 [e_1 y_s - x_m(t - \tau)e_2 - be_3] \\ &\quad + e_b [\dot{e}_b + e_3 z_s] - k_3 e_3^2 + e_4 (-r_1 e_1 - r_2 e_2) \\ &\quad + e_{r_1} [\dot{e}_{r_1} + e_4 x_s] + e_{r_2} [\dot{e}_{r_2} + e_4 y_s] - k_4 e_4^2. \end{aligned} \quad (9)$$

Let M be the bound positive value of hyperchaotic Lü system, that is, $|x_m|$, $|y_m|$, $|z_m|$, $|w_m|$, $|x_s|$, $|y_s|$, $|z_s|$, and $|w_s| \leq M$, which implies that

$$\begin{aligned} \dot{V} &\leq -(k_1 - a)e_1^2 - (k_2 - c)e_2^2 - (k_3 - b)e_3^2 - k_4 e_4^2 \\ &\quad + (a + M)|e_1||e_2| + M|e_1||e_3| + r_1|e_1||e_4| \\ &\quad + 2M|e_2||e_3| + (r_2 + 1)|e_2||e_4| \\ &= -(|e_1|, |e_2|, |e_3|, |e_4|)P(|e_1|, |e_2|, |e_3|, |e_4|)^T, \end{aligned} \quad (10)$$

where

$$P = \begin{pmatrix} k_1 - a & -\frac{(a + M)}{2} & -\frac{M}{2} & -\frac{r_1}{2} \\ -\frac{(a + M)}{2} & k_2 - c & -M & -\frac{(r_2 + 1)}{2} \\ -\frac{M}{2} & -M & k_3 - b & 0 \\ -\frac{r_1}{2} & -\frac{(r_2 + 1)}{2} & 0 & k_4 \end{pmatrix}. \quad (11)$$

It is obvious that, for suitable values of k_i ($i = 1, 2, 3, 4$), the matrix P is positive definite and $\dot{V}$ is negative semidefinite. So

FIGURE 4: The synchronization errors of the state variables between the master system and the slave system.

one obtains $\lim_{t \rightarrow \infty} \|e_i\| = 0$ ($i = 1, 2, 3, 4$). It means that the states of response system (3) and the states of drive system (2) are ultimately hybrid lag synchronized asymptotically. $\square$

3. Numerical Simulations

In this section, an illustrative example is presented to demonstrate the effectiveness of the proposed scheme. In the simulations, the system parameters are chosen as $a = 36$, $b = 3$, $c = 20$, $r_1 = r_2 = 2$. The time delay is $\tau = 1$. According to Theorem 1, the initial conditions can be given as any values. For simplicity, the initial conditions of the master system and slave system are set to be $(x_m(0), y_m(0), z_m(0), w_m(0)) = (-2.0, 3.0, 0.4, 0.2)$ and $(x_s(0), y_s(0), z_s(0), w_s(0)) = (1.2, 0.3, 0.5, 0.6)$, respectively, and the feedback gain coefficients are supposed as $k_i = k$ ($i = 1, 2, 3, 4$). From the theoretical analysis, it is known that conditions obtained analytically for the synchronization are only the sufficient conditions. Stronger feedback gain coefficient could induce synchronization easily, while smaller gain coefficient could not ensure the two chaotic systems reach hybrid lag synchronization. Figure 2 shows the evolutions of the variables x_m, y_m, z_m, w_m in drive system (2) and

x_s, y_s, z_s, w_s in response system (3) at $k = 2.3$. The evolutions of the state variables $x_m(t - \tau), y_m(t - \tau), z_m(t - \tau), w_m(t - \tau)$ and x_s, y_s, z_s, w_s are plotted in Figure 3. Figure 4 depicts the synchronization errors of the state variables between the master system and the slave system. Figure 5 draws the identification of unknown parameters a, b, c, r_1 , and r_2 , from which it is obvious to see that the five unknown parameters converge to their real values, respectively. From the simulation results, we know that the proposed scheme works well and the system error states are regulated to zero asymptotically. Therefore, the hybrid synchronization of systems (2) and (3) can be achieved; that is, the third corresponding pair variables of systems (2) and (3) reach complete lag synchronization, while other three pair corresponding variables of systems (2) and (3) arrive lag antisynchronization.

4. Conclusion

In this paper, mixed lag synchronization in the hyperchaotic systems is studied. Considering the unknown system parameters, an adaptive control scheme has been proposed to ensure the hybrid lag synchronization between the drive and

FIGURE 5: The identified results of the unknown parameters a , b , c , r_1 , and r_2 .

the controlled response systems. Numerical simulations have verified the effectiveness of the proposed technique.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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