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Research Article

Global Dynamics of a Mathematical Model on Smoking

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We derive and analyze a mathematical model of smoking in which the population is divided into four classes: potential smokers, smokers, temporary quitters, and permanent quitters. In this model we study the effect of smokers on temporary quitters. Two equilibria of the model are found: one of them is the smoking-free equilibrium and the other corresponds to the presence of smoking. We examine the local and global stability of both equilibria and we support our results by using numerical simulations.

1. Introduction

Tobacco epidemic is one of the biggest public health threats the world has ever faced. It kills up to half of its users. Nearly, each year six million people die from smoking of whom more than 5 million are users and ex-users and more than 600,000 are nonsmokers exposed to second-hand smoke. Tobacco users who die prematurely deprive their families of income, raise the cost of health care, and hinder economic development. In smoking death statistics, there is a death caused by tobacco every 8 seconds; 10% of the adult population who smoke die of tobacco related diseases. The World Health Organization predicts that, by 2030, 10 million people will die every year due to tobacco related illnesses. This makes smoking the biggest killer globally. In Saudi Arabia, the prevalence of current smoking ranges from 2.4 to 52.3% (median = 17.5%) depending on the age group. The results of a Saudi modern study predicted an increase of smokers number in the country to 10 million smokers by 2020. The current number of smokers in Saudi Arabia is approximately 6 million, and they spend around 21 billion Saudi Riyal on smoking annually. Clearly smoking is a prevalent problem among Saudis that requires intervention for eradication. Persistent education of the health hazards related to smoking is recommended particularly at early ages in order to prevent initiation of smoking [1, 2].

In 1997, Castillo-Garsow et al. [3] proposed a simple mathematical model for giving up smoking. They considered a system described by the simplified PSQ model with a total constant population which is divided into three classes: potential smokers, that is, people who do not smoke yet but might become smokers in the future (P), smokers (S), and people (former smokers) who have quit smoking permanently (Q). Later, this mathematical model was developed by Sharomi and Gumel (2008) [4]; they introduced a new class Q_t of smokers who temporarily quit smoking and they described the dynamics of smoking by the following four nonlinear differential equations:

$$\begin{aligned} \frac{dP}{dt} &= \mu - \mu P - \beta PS, \\ \frac{dS}{dt} &= -(\mu+\gamma)S + \beta PS + \alpha Q_t, \\ \frac{dQ_t}{dt} &= -(\mu+\alpha)Q_t + \gamma(1-\sigma)S, \\ \frac{dQ_p}{dt} &= -\mu Q_p + \sigma\gamma S. \end{aligned} \tag{1}$$

They concluded that the smoking-free equilibrium is globally asymptotically stable whenever a certain threshold, known as the smokers generation number, is less than unity and unstable if this threshold is greater than unity. The public health implication of this result is that the number of smokers in the community will be effectively controlled (or eliminated) at steady state if the threshold is made to be less than unity.

Such a control is not feasible if the threshold exceeds unity (a global stability result for the smoking-present equilibrium is provided for a special case). In 2011, Lahrouz et al. [5] proved the global stability of the unique smoking-present equilibrium state of the mathematical model developed by Sharomi and Gumel. Also in 2011, Zaman [6] derived and analyzed a smoking model taking into account the occasional smokers compartment, and later [7] he extended the model to consider the possibility of quitters becoming smokers again. In 2012, Ertürk et al. [8] introduced fractional derivatives into the model and studied it numerically.

In this paper, we adopt the model developed and studied in [4, 5] and examine the effect of peer pressure on temporary quitters Q_t. By this we mean the effect of smokers on temporary quitters which may be considered as one of the main causes of their relapse. This is done by taking $\alpha = \alpha S$ as a parameter dependent on the pressure of smokers. The aim of this work is to analyze the model by using stability theory of nonlinear differential equations and supporting the results with numerical simulation. The structure of this paper is organized as follows. In Section 2, we calculate the smoking generation number, find the equilibria, and investigate their stability. In Section 3, we present numerical simulations to support our results. Finally, we end the paper with a conclusion in Section 4.

2. Formulation of the Model

Let the total population size at time t be denoted by $N(t)$. We divide the population $N(t)$ into four subclasses: potential smokers (nonsmoker) $P(t)$, smokers $S(t)$, smokers who temporarily quit smoking $Q_t(t)$, and smokers who permanently quit smoking $Q_p(t)$, such that $N(t) = P(t) + S(t) + Q_t(t) + Q_p(t)$. We describe the dynamics of smoking by the following four nonlinear differential equations:

$$\begin{aligned} \frac{dP}{dt} &= \mu - \mu P - \beta PS, \\ \frac{dS}{dt} &= -(\mu+\gamma)S + \beta PS + \alpha SQ_t, \\ \frac{dQ_t}{dt} &= -\mu Q_t - \alpha SQ_t + \gamma(1-\sigma)S, \\ \frac{dQ_p}{dt} &= -\mu Q_p + \sigma\gamma S. \end{aligned} \tag{2}$$

Here, β is the contact rate between potential smokers and smokers, μ is the rate of natural death, α is the contact rate between smokers and temporary quitters who revert back to smoking, γ is the rate of quitting smoking, $(1-\sigma)$ is the fraction of smokers who temporarily quit smoking (at a rate γ), and σ is the remaining fraction of smokers who permanently quit smoking (at a rate γ).

We assume that the class of potential smokers is increased by the recruitment of individuals at a rate μ. In model 2, the total population is constant and $P(t)$, $S(t)$, $Q_t(t)$, and $Q_p(t)$ are, respectively, the proportions of potential smokers, smokers, temporarily quitters, and permanent quitters where $P(t)+S(t)+Q_t(t)+Q_p(t) = 1$. Since the variable Q_p of model 2 does not appear in the first three equations, we will only consider the subsystem:

$$\begin{aligned} \frac{dP}{dt} &= \mu - \mu P - \beta PS, \\ \frac{dS}{dt} &= -(\mu+\gamma)S + \beta PS + \alpha SQ_t, \\ \frac{dQ_t}{dt} &= -\mu Q_t - \alpha SQ_t + \gamma(1-\sigma)S. \end{aligned} \tag{3}$$

From model 3 we find that

$$\frac{dP}{dt} + \frac{dS}{dt} + \frac{dQ_t}{dt} \leqslant \mu - \mu(P+S+Q_t). \tag{4}$$

Let $N_1 = P + S + Q_t$; then $N_1' \leqslant \mu - \mu N_1$. The initial value problem $\Phi' = \mu - \mu\Phi$, with $\Phi(0) = N_1(0)$, has the solution $\Phi(t) = 1 - ce^{-\mu t}$, and $\lim_{t\to\infty}\Phi(t) = 1$. Therefore, $N_1(t) \leqslant \Phi(t)$, which implies that $\lim_{t\to\infty}\sup N_1(t) \leqslant 1$. Thus, the considered region for model 3 is

$$\Gamma = \{(P,S,Q_t) : P+S+Q_t \leqslant 1,\ P>0,\ S\geqslant 0,\ Q_t \geqslant 0\}. \tag{5}$$

All solutions of model 3 are bounded and enter the region Γ. Hence, Γ is positively invariant. That is, every solution of model 3, with initial conditions in Γ, remains there for all $t > 0$.

3. Equilibria and Smokers Generation Number

Setting the right-hand side of the equations in model 3 to zero, we find two equilibria of the model: the smoking-free equilibrium $E_0 = (1,0,0)$ and smoking-present equilibrium $E^* = (P^*, S^*, Q_t^*)$, where

$$\begin{aligned} P^* &= \frac{\mu}{\mu+\beta S^*}, \\ Q_t^* &= \frac{\gamma(1-\sigma)S^*}{\mu+\alpha S^*}, \end{aligned} \tag{6}$$

and S^* satisfies the equation

$$\begin{aligned} S^{*2} &+ \left(\frac{\mu(\mu\beta+\gamma\beta+\alpha(\mu-\beta)+\alpha\gamma\sigma)}{\beta\alpha(\mu+\gamma\sigma)}\right)S^* \\ &+ \frac{\mu^2(\mu+\gamma-\beta)}{\beta\alpha(\mu+\gamma\sigma)} = 0. \end{aligned} \tag{7}$$

The smokers generation number R_0 is found by the method of next generation matrix [9, 10]. Let $X = (S, Q_t, P)$; then model 3 can be rewritten as $X' = \mathfrak{F}(X) - \mathfrak{V}(X)$ such that

$$\begin{aligned} \mathfrak{F}(X) &= \begin{bmatrix} \beta PS \\ 0 \\ 0 \end{bmatrix}, \\ \mathfrak{V}(X) &= \begin{bmatrix} (\mu+\gamma)S - \alpha SQ_t \\ (\mu+\alpha)SQ_t - \gamma(1-\sigma)S \\ -\mu + \mu P + \beta PS \end{bmatrix}. \end{aligned} \tag{8}$$

By calculating the Jacobian matrices at E_0, we find that

$$D(\mathfrak{F}(E_0)) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}, \qquad D(\mathfrak{V}(E_0)) = \begin{bmatrix} V & 0 \\ J_1 & J_2 \end{bmatrix}, \tag{9}$$

where

$$F = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} (\mu+\gamma) & 0 \\ -\gamma(1-\sigma) & \mu \end{bmatrix}. \tag{10}$$

Thus, the next generation matrix is $FV^{-1} = \left[\begin{smallmatrix} \beta/(\mu+\gamma) & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and the smokers generation number R_0 is the spectral radius $\rho(FV^{-1}) = \beta/(\mu+\gamma)$.

Next we explore the existence of positive smoking-present equilibrium E^* of model 3. Equation (7) has solutions of the form

$$S^*_{1,2} = -\frac{A_1}{2} \pm \frac{1}{2}\sqrt{A_1^2 - 4A_2}, \tag{11}$$

where A_1 and A_2 are

$$\begin{aligned} A_1 &= \frac{\mu(\mu\beta + \gamma\beta + \alpha(\mu-\beta) + \alpha\gamma\sigma)}{\beta\alpha(\mu+\gamma\sigma)}, \\ A_2 &= \frac{\mu^2(\mu+\gamma-\beta)}{\beta\alpha(\mu+\gamma\sigma)} = \frac{\mu^2(\mu+\gamma)(1-R_0)}{\beta\alpha(\mu+\gamma\sigma)}. \end{aligned} \tag{12}$$

If $R_0 > 1$, then $A_2 < 0$ and $\sqrt{A_1^2 - 4A_2} > A_1$. So we have only one positive solution:

$$S^* = \frac{1}{2}\left(-A_1 + \sqrt{A_1^2 - 4A_2}\right). \tag{13}$$

If $R_0 = 1$, then $A_2 = 0$ and there are no positive solutions. If $R_0 < 1$ and $\mu \geqslant \beta$, then $A_1 > 0$ and $A_2 > 0$ and there are no positive solutions. These results are summarized below.

Theorem 1. *Model 3 always has the smoking-free equilibrium point $E_0 = (1, 0, 0)$. As for the existence of a smoking-present equilibrium point E^*, one has three cases:*

(i) *if $R_0 < 1$ and $\mu \geqslant \beta$, one has no positive equilibrium point;*

(ii) *if $R_0 > 1$, one has one positive equilibrium point E^*;*

(iii) *if $R_0 = 1$, one has no positive equilibrium point.*

4. Stability of the Equilibria

4.1. Local Stability. First, we investigate the local stability of E_0 and state the following theorem.

Theorem 2 (local stability of E_0). *If $R_0 < 1$, the smoking-free equilibrium point E_0 of model 3 is locally asymptotically stable. If $R_0 = 1$, E_0 is locally stable. If $R_0 > 1$, E_0 is unstable.*

Proof. Evaluating the Jacobian matrix of model 3 at E_0 (using linearization method [11]) gives

$$J(E_0) = \begin{bmatrix} -\mu & -\beta & 0 \\ 0 & -(\mu+\gamma)+\beta & 0 \\ 0 & \gamma(1-\sigma) & -\mu \end{bmatrix}. \tag{14}$$

Thus, the eigenvalues of $J(E_0)$ are $\lambda_1 = \lambda_2 = -\mu < 0$, $\lambda_3 = -(\mu+\gamma)+\beta$. Clearly λ_3 is negative if $R_0 < 1$. So all eigenvalues are negative if $R_0 < 1$, and hence E_0 is locally asymptotically stable. If $R_0 = 1$, then $\lambda_3 = 0$ and E_0 is locally stable. If $R_0 > 1$, then $\lambda_3 > 0$ which means that there exists a positive eigenvalue. So E_0 is unstable. □

Next, we investigate the local stability of the positive equilibrium E^* by using the following lemma.

Lemma 3 (see [12–14]). *Let M be a 3×3 real matrix. If $\mathrm{tr}(M)$, $\det(M)$, and $\det(M^{[2]})$ are all negative, then all of the eigenvalues of M have negative real part.*

$M^{[2]}$ in the previous lemma is known as the second additive compound matrix with the following definition.

Definition 4 (see [15] (second additive compound matrix)). Let $A = (a_{ij})$ be an $n \times n$ real matrix. The second additive compound of A is the matrix $A^{[2]} = (b_{ij})$ defined as follows:

$$\begin{aligned} n = 2 &: A^{[2]} = a_{11} + a_{22}, \\ n = 3 &: A^{[2]} = \begin{vmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{vmatrix}. \end{aligned} \tag{15}$$

Theorem 5 (local stability of E^*). *The smoking-present equilibrium E^* of model 3 is locally asymptotically stable if $\alpha \leqslant \beta$.*

Proof. Linearizing model 3 at the equilibrium $E^*(P^*, S^*, Q_t^*)$ gives

$$J(E^*) = \begin{bmatrix} -\mu - \beta S^* & -\beta P^* & 0 \\ \beta S^* & 0 & \alpha S^* \\ 0 & -\alpha Q_t^* + \gamma(1-\sigma) & -\mu - \alpha S^* \end{bmatrix}. \tag{16}$$

The second additive compound matrix $J^{[2]}(E^*)$ is

$$J^{[2]}(E^*) = \begin{bmatrix} -\mu - \beta S^* & \alpha S^* & 0 \\ -\alpha Q_t^* + \gamma(1-\sigma) & -2\mu - \beta S^* - \alpha S^* & -\beta P^* \\ 0 & \beta S^* & -\mu - \alpha S^* \end{bmatrix}. \tag{17}$$

Here

$$\begin{aligned}
\operatorname{tr}\left(J\left(E^{*}\right)\right) &= -2\mu-\beta S^{*}-\alpha S^{*}<0,\\
\det\left(J\left(E^{*}\right)\right) &= \beta\alpha S^{*}\left(\mu Q_{t}^{*}-\beta P^{*}S^{*}\right)+\mu\left(\alpha\mu Q_{t}^{*}-\beta^{2}P^{*}S^{*}\right)\\
&\leqslant \beta\alpha S^{*}\left(\mu Q_{t}^{*}-\beta P^{*}S^{*}\right)\\
&\quad+\mu\beta\left(\mu Q_{t}^{*}-\beta P^{*}S^{*}\right),\quad \text{if } \alpha\leqslant\beta\\
&= \mu\left(\beta\alpha S^{*}+\mu\beta\right)\left(Q_{t}^{*}+P^{*}-1\right)<0,\\
&\qquad\qquad \text{since } P^{*}+Q_{t}^{*}<1,\\
\det\left(J^{[2]}\left(E^{*}\right)\right) &= -\left(\mu+\beta S^{*}\right)\left[\left(2\mu+\beta S^{*}+\alpha S^{*}\right)\left(\mu+\alpha S^{*}\right)\right.\\
&\qquad\left.+\beta^{2}P^{*}S^{*}\right]+\alpha\mu Q_{t}^{*}\left(\mu+\alpha S^{*}\right)\\
&= -\left(\mu+\beta S^{*}\right)\left[\left(2\mu+\beta S^{*}+\alpha S^{*}\right)\left(\mu+\alpha S^{*}\right)\right]\\
&\quad+\mu\left(\alpha\mu Q_{t}^{*}-\beta^{2}P^{*}S^{*}\right)\\
&\quad+S^{*}\left(\alpha^{2}\mu Q_{t}^{*}-\beta^{3}P^{*}S^{*}\right)<0,\quad \text{if } \alpha\leqslant\beta.
\end{aligned}\tag{18}$$

Hence, by previous lemma, E^{*} is locally asymptotically stable if $\alpha\leqslant\beta$. □

4.2. Global Stability. We will examine the global stability of E_0 when $\beta\leqslant\mu$. First, it should be noted that $P<1$ in Γ for all $t>0$. Consider the following Lyapunov function [16]:

$$\begin{aligned}
L &= S+Q_{t},\\
\frac{dL}{dt} &= -\left(\mu+\gamma\right)S+\beta PS+\alpha SQ_{t}\\
&\quad-\mu Q_{t}-\alpha SQ_{t}+\gamma\left(1-\sigma\right)S\\
&\leqslant\left(-\mu+\beta\right)S-\mu Q_{t}\leqslant 0.
\end{aligned}\tag{19}$$

We have $dL/dt<0$ for $\beta\leqslant\mu$ and $dL/dt=0$ only if $S=0$ and $Q_t=0$. Therefore, by LaSalle's Invariance Principle [16], every solution to the equations of the model with initial conditions in Γ approaches E_0 as $t\to\infty$. We have the following result.

Theorem 6 (global stability of E_0). *If $\beta\leqslant\mu$, then E_0 is globally asymptotically stable in Γ.*

Now we examine the global stability of E^{*} by using the following theorem [12, 17] to prove that model 3 has no periodic solutions, homoclinic loops, and oriented phase polygons inside the invariant region.

Theorem 7. *Let $g(P,S,Q_t)=\{g_1(P,S,Q_t),g_2(P,S,Q_t),g_3(P,S,Q_t)\}$ be a vector field which is piecewise smooth on Γ^{*} and which satisfies the conditions $g\cdot f=0$, $(\operatorname{curl} g)\cdot\vec{n}<0$ in the interior of Γ^{*}, where $\vec{n}$ is the normal vector to Γ^{*} and $f=(f_1,f_2,f_3)$ is a Lipschitz continuous field in the interior of Γ^{*} and* $\operatorname{curl} g=(\partial g_3/\partial S-\partial g_2/\partial Q_t)\vec{i}-(\partial g_3/\partial P-\partial g_1/\partial Q_t)\vec{j}+(\partial g_2/\partial P-\partial g_1/\partial S)\vec{k}$. *Then the differential equation system $P=f_1$, $S=f_2$, and $Q_t=f_3$ has no periodic solutions, homoclinic loops, and oriented phase polygons inside Γ^{*}.*

Let $\Gamma^{*}=\{(P,S,Q_t):P+((\mu+\gamma\sigma)/\mu)S+Q_t=1,P>0,S\geqslant 0,Q_t\geqslant 0\}$. One can easily prove that $\Gamma^{*}\subset\Gamma$, Γ^{*} is positively invariant, and $E^{*}\in\Gamma^{*}$. Thus, we state the following theorem.

Theorem 8. *Model 3 has no periodic solutions, homoclinic loops, and oriented phase polygons inside the invariant region Γ^{*}.*

Proof. Let f_1, f_2, and f_3 denote the right-hand side of equations in model 3, respectively. Using $P+((\mu+\gamma\sigma)/\mu)S+Q_t=1$ to rewrite f_1, f_2, and f_3 in the equivalent forms, we have

$$\begin{aligned}
f_1\left(P,S\right) &= \mu-\mu P-\beta PS,\\
f_1\left(P,Q_t\right) &= \mu-\mu P-\beta P\left[\frac{\mu}{\mu+\gamma\sigma}\left(1-P-Q_t\right)\right],\\
f_2\left(P,S\right) &= -\left(\mu+\gamma\right)S+\beta PS+\alpha S\left[1-P-\left(\frac{\mu+\gamma\sigma}{\mu}\right)S\right],\\
f_2\left(S,Q_t\right) &= -\left(\mu+\gamma\right)S\\
&\quad+\beta\left[1-Q_t-\left(\frac{\mu+\gamma\sigma}{\mu}\right)S\right]S+\alpha SQ_t,\\
f_3\left(P,Q_t\right) &= -\mu Q_t\\
&\quad-\left[\alpha Q_t-\gamma\left(1-\sigma\right)\right]\left[\frac{\mu}{\mu+\gamma\sigma}\left(1-P-Q_t\right)\right],\\
f_3\left(S,Q_t\right) &= -\mu Q_t-\alpha SQ_t+\gamma\left(1-\sigma\right)S.
\end{aligned}\tag{20}$$

Let $g=(g_1,g_2,g_3)$ be a vector field such that

$$\begin{aligned}
g_1 &= \frac{f_3\left(P,Q_t\right)}{PQ_t}-\frac{f_2\left(P,S\right)}{PS}\\
&= -\frac{\alpha\mu}{P\left(\mu+\gamma\sigma\right)}+\frac{\alpha\mu}{\mu+\gamma\sigma}+\frac{\alpha\mu Q_t}{P\left(\mu+\gamma\sigma\right)}\\
&\quad+\frac{\mu\gamma\left(1-\sigma\right)}{PQ_t\left(\mu+\gamma\sigma\right)}-\frac{\mu\gamma\left(1-\sigma\right)}{Q_t\left(\mu+\gamma\sigma\right)}\\
&\quad-\frac{\mu\gamma\left(1-\sigma\right)}{P\left(\mu+\gamma\sigma\right)}+\frac{\gamma}{P}-\beta\\
&\quad-\frac{\alpha}{P}+\alpha\left(\frac{\mu+\gamma\sigma}{\mu}\right)\frac{S}{P}+\alpha,\\
g_2 &= \frac{f_1\left(P,S\right)}{PS}-\frac{f_3\left(S,Q_t\right)}{SQ_t}\\
&= \frac{\mu}{PS}-\beta+\alpha-\frac{\gamma\left(1-\sigma\right)}{Q_t},
\end{aligned}$$

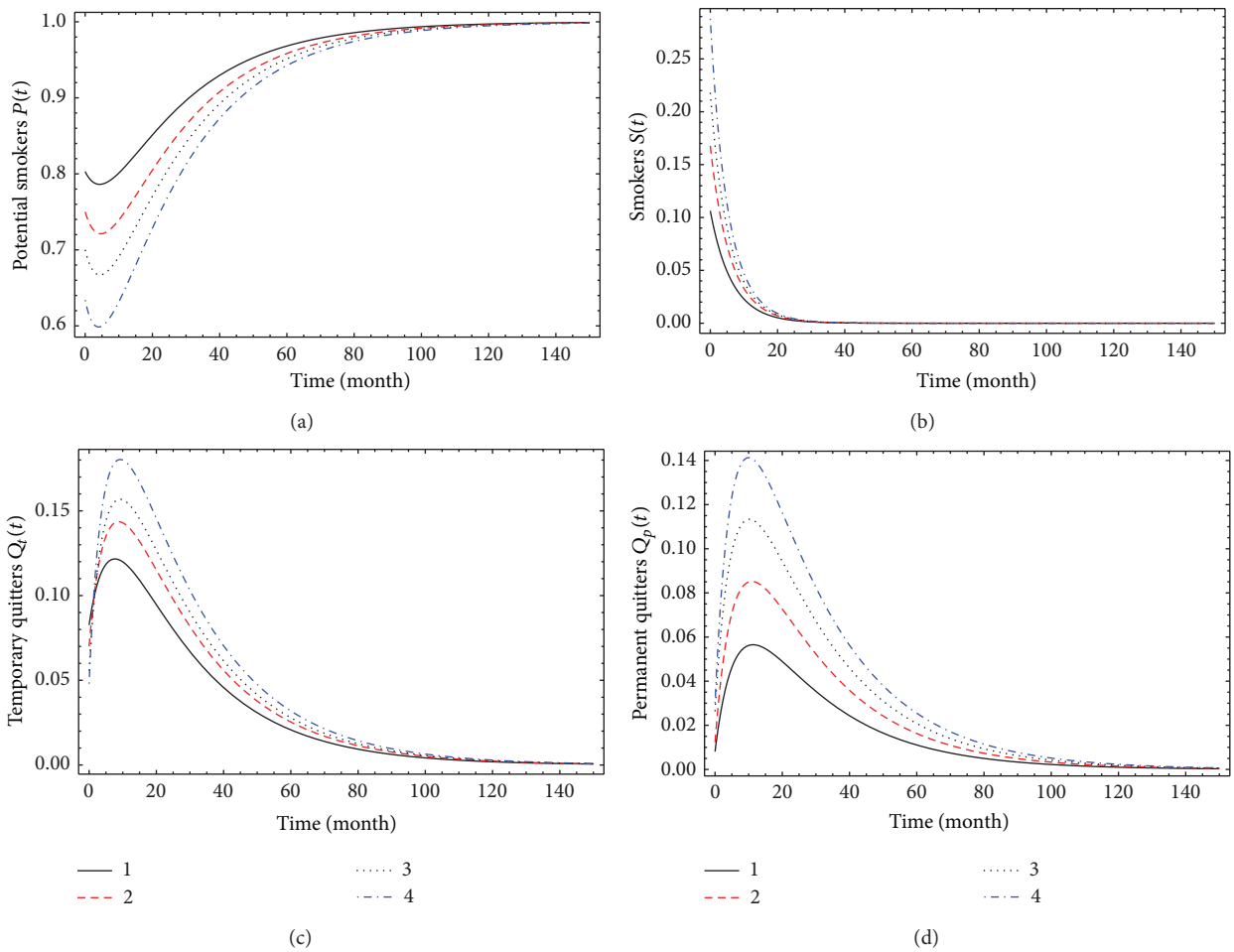

Figure 1: Time plots of model 2 with different initial conditions for $R_0 < 1$. (a) Potential smokers; (b) smokers; (c) temporary quitters; (d) permanent quitters.

$$\begin{aligned} g_3 &= \frac{f_2(S,Q_t)}{SQ_t} - \frac{f_1(P,Q_t)}{PQ_t} \\ &= -\frac{(\mu+\gamma)}{Q_t} + \frac{\beta}{Q_t} - \beta - \frac{\beta(\mu+\gamma\sigma)S}{\mu Q_t} \\ &\quad + \alpha - \frac{\mu}{PQ_t} + \frac{\mu}{Q_t} + \frac{\beta\mu}{Q_t(\mu+\gamma\sigma)} \\ &\quad - \frac{\beta\mu P}{Q_t(\mu+\gamma\sigma)} - \frac{\beta\mu}{(\mu+\gamma\sigma)}. \end{aligned} \tag{21}$$

Since the alternate forms of f_1, f_2, and f_3 are equivalent in Γ, then

$$\begin{aligned} g\cdot f &= g_1 f_1 + g_2 f_2 + g_3 f_3 \\ &= \frac{f_3 f_1}{PQ_t} - \frac{f_2 f_1}{PS} + \frac{f_1 f_2}{PS} - \frac{f_3 f_2}{SQ_t} \\ &\quad + \frac{f_2 f_3}{SQ_t} - \frac{f_1 f_3}{PQ_t} = 0. \end{aligned} \tag{22}$$

So $g\cdot f = 0$ on Γ^*.

Using the normal vector $\vec{n} = (1, (\mu+\gamma\sigma)/\mu, 1)$ to Γ^*, we can see that

$$\begin{aligned} (\operatorname{curl} g)\cdot\vec{n} &= -\frac{\beta\gamma\sigma}{\mu Q_t} - \frac{\gamma(1-\sigma)}{PQ_t^2} \\ &\quad - \frac{\mu+\gamma\sigma}{P^2 Q_t} - \frac{\mu}{P^2 S} - \frac{\alpha\gamma\sigma}{\mu P} < 0. \end{aligned} \tag{23}$$

Hence, model 3 has no periodic solutions, homoclinic loops, and oriented phase polygons inside the invariant region Γ^*. □

Consequently, we have the following result.

Theorem 9 (global stability of E^*). *If $R_0 > 1$, then the smoking-present equilibrium point E^* of system (3) is globally asymptotically stable.*

Proof. We know that, if $R_0 > 1$ in Γ^*, then E_0 is unstable. Also Γ^* is a positively invariant subset of Γ and the ω limit set of each solution of (3) is a single point in Γ^* since there is no periodic solutions, homoclinic loops, and oriented phase

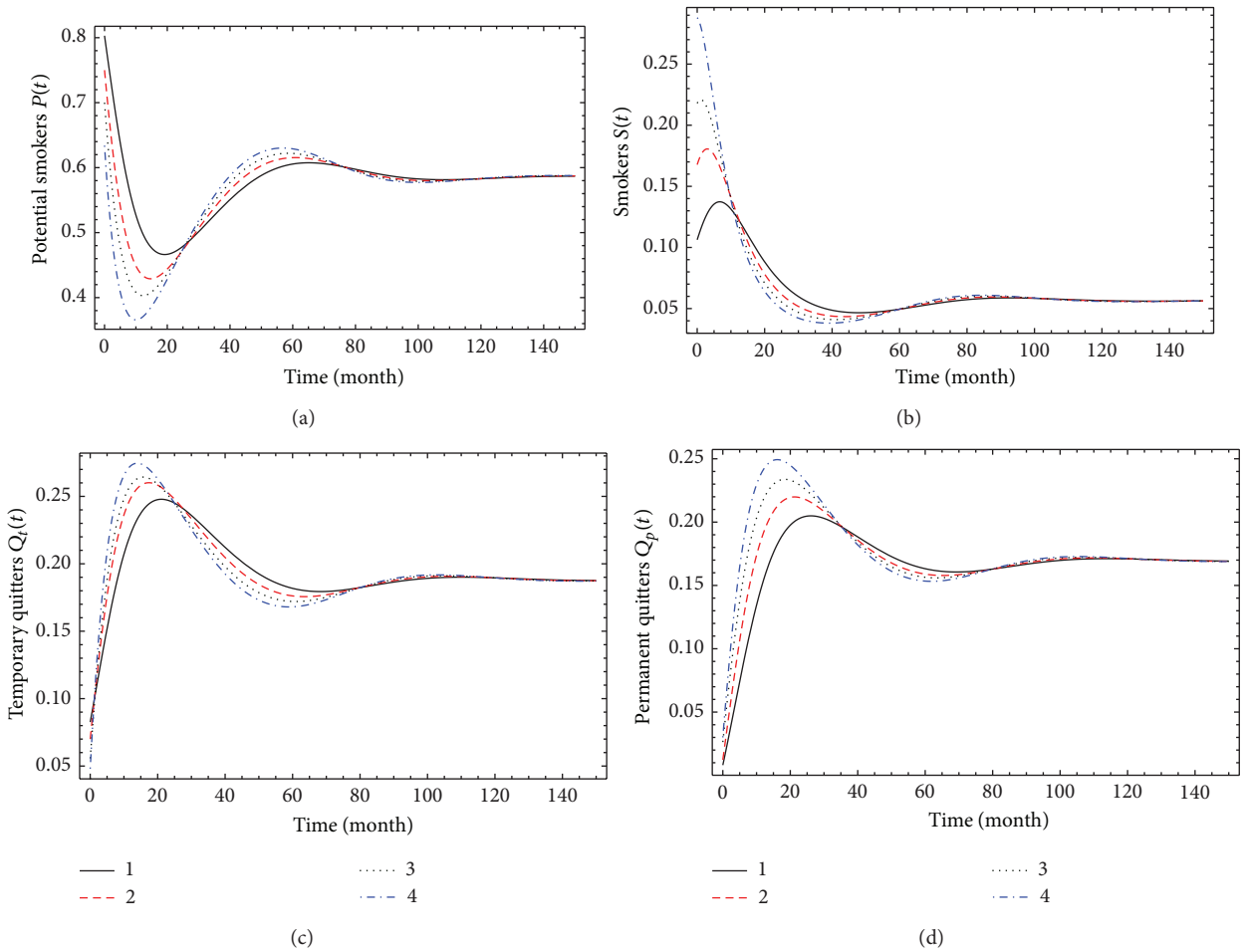

FIGURE 2: Time plots of model 2 with different initial conditions for $\beta \geqslant \alpha$. (a) Potential smokers; (b) smokers; (c) temporary quitters; (d) permanent quitters.

polygons inside Γ^*. Therefore E^* is globally asymptotically stable. □

5. Numerical Simulations

In this section, we illustrate some numerical solutions of model 2 for different values of the parameters, and we show that these solutions are in agreement with the qualitative behavior of the solutions. We use the following parameters: $\mu = 0.04$, $\gamma = 0.3$, $\alpha = 0.25$, and $\sigma = 0.4$, with two different values of the contact rate between potential smokers and smokers: for $R_0 < 1$ we use $\beta = 0.2$ and for $\beta \geqslant \alpha$ we use $\beta = 0.5$. Model 2 is simulated for the following different initial values such that $P + S + Q_t + Q_p = 1$:

(1) $P(0) = 0.80301$, $S(0) = 0.10628$, $Q_t(0) = 0.08260$, and $Q_p(0) = 0.00811$;

(2) $P(0) = 0.75000$, $S(0) = 0.16772$, $Q_t(0) = 0.07000$, and $Q_p(0) = 0.01228$;

(3) $P(0) = 0.70000$, $S(0) = 0.21800$, $Q_t(0) = 0.05566$, and $Q_p(0) = 0.02634$;

(4) $P(0) = 0.63400$, $S(0) = 0.28800$, $Q_t(0) = 0.04800$, and $Q_p(0) = 0.03000$.

For $R_0 = 0.588235 < 1$, Figure 1(a) shows that the number of potential smokers increases and approaches the total population 1. Figure 1(b) shows that the number of the smokers decreases and approaches zero. In Figures 1(c) and 1(d), temporary quitters and permanent quitters increase at first; after that they decrease and approach zero. We see from Figure 1 that, for any initial value, the solution curves tend to the equilibrium E_0, when $R_0 < 1$. Hence, model 2 is locally asymptotically stable about E_0 for the previous set of parameters.

In Figure 2, we use the same parameters and initial values as previously with $\beta = 0.5$. Figure 2(a) shows that the number of potential smokers decreases at first; then it increases and approaches P^*. Figure 2(b) shows that the number of smokers increases at first; then it decreases and approaches S^*. In Figures 2(c) and 2(d), temporary quitters and permanent quitters increase at first; after that they decrease and approach Q_t^* and Q_p^*. We see from Figure 2 that, for any initial value, the solution curves tend to the equilibrium E^*, when $\beta \geqslant \alpha$.

Hence, model 2 is locally asymptotically stable about E^* for the above set of parameters.

6. Discussion and Conclusions

In this paper, we presented a mathematical model to analyze the behavior of smoking dynamics in a population with peer pressure effect on temporary quitters Q_t. Local asymptotic stability for the smoking-free equilibrium state is obtained when the threshold quantity R_0 is less than 1 (i.e., when the contact rate β between potential smokers and smokers is less than the sum of natural death rate μ and the quitting rate γ). A Lyapunov function is used to show global stability of the smoking-free equilibrium when the contact rate between potential smokers and smokers is less than or equal to the natural death rate ($\beta \leq \mu$). This means that the number of smokers may be controlled by reducing the contact rate β to be less than the natural death rate μ. On the other hand, if $\beta \geqslant \alpha$ (i.e., when the contact rate β between potential smokers and smokers is greater than the contact rate α between smokers and temporary quitters who revert back to smoking), then the smoking-present equilibrium state is locally asymptotically stable. By showing that this model has no periodic solutions, homoclinic loops, and oriented phase polygons inside the invariant region Γ^*, we proved the global asymptotic stability of E^*. This means that, if $R_0 > 1$, smoking will persist. Some numerical simulations are performed to illustrate the findings of the analytical results. For $R_0 > 1$, the results show that the smokers population reaches a steady state of approximately 6% of the total population.

This model may be extended to include, for example, smokers who after quitting smoking may become potential smokers again. We may also consider the public health impact of smoking related illnesses as well.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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