Research Article

System of Operator Quasi Equilibrium Problems

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We consider a system of operator quasi equilibrium problems and system of generalized quasi operator equilibrium problems in topological vector spaces. Using a maximal element theorem for a family of set-valued mappings as basic tool, we derive some existence theorems for solutions to these problems with and without involving Φ -condensing mappings.

1. Introduction

In 2002, Domokos and Kolumbán [1] gave an interesting interpretation of variational inequality and vector variational inequalities (for short, VVI) in Banach space settings in terms of variational inequalities with operator solutions (for short, OVVI). The notion and viewpoint of OVVI due to Domokos and Kolumbán [1] look new and interesting even though it has a limitation in application to VVI. Recently, Kazmi and Raouf [2] introduced the operator equilibrium problem which generalizes the notion of OVVI to operator vector equilibrium problems (for short, OVEP) using the operator solution. They derived some existence theorems of solution of OVEP with pseudomonotonicity, without pseudomonotonicity, and with B -pseudomonotonicity. However, they dealt with only the single-valued case of the bioperator. It is very natural and useful to extend a single-valued case to a corresponding set-valued one from both theoretical and practical points of view.

The system of vector equilibrium problems and the system of vector quasi equilibrium problems were introduced and studied by Ansari et al. [3, 4]. Inspired by above cited work, in this paper, we consider a system of operator quasi equilibrium problems (for short, SOQEP) in topological vector spaces. Using a maximal element theorem for a family of set-valued mappings according to [5] as basic tool, we derive some existence theorems for solutions to SOQEP with and without involving Φ -condensing mappings.

Further, we consider a system of generalized quasi operator equilibrium problems (for short, SGQOEP) in topological vector spaces and give some of its special cases and

derive some existence theorems for solutions to SOQEP with and without involving Φ -condensing mappings by using well-known maximal element theorem [5] for a family of set-valued mappings, and, consequently, we also get some existence theorems for solutions to a system of operator equilibrium problems.

2. Preliminaries

Let I be an index set, for each $i \in I$, and let X_i be a Hausdorff topological vector space. We denote $L(X_i, Y_i)$, the space of all continuous linear operators from X_i into Y_i , where Y_i is topological vector space for each $i \in I$. Consider a family of nonempty convex subsets $\{K_i\}_{i \in I}$ with K_i in $L(X_i, Y_i)$.

Let

$$\begin{aligned} X &= \prod_{i \in I} X_i, \\ K &= \prod_{i \in I} K_i. \end{aligned} \quad (1)$$

Let $C_i : K \rightarrow 2^{Y_i}$ be a set-valued mapping such that, for each $f \in K$, $C_i(f)$ is solid, open, and convex cone such that $0 \notin C_i(f)$ and $P_i = \bigcap_{f \in K} C_i(f)$.

For each $i \in I$, let $F_i : K \times K_i \rightarrow Y_i$ be a bifunction and let $A_i : K \rightarrow 2^{K_i}$ be a set-valued mapping with nonempty values. We consider the following *system of operator quasi*

equilibrium problems (for short, SOQEP). Find $\bar{f} \in K$ such that, for each $i \in I$,

$$\bar{f}_i \in A_i(\bar{f}), \quad F_i(\bar{f}, g_i) \notin -C_i(\bar{f}), \quad \forall g_i \in A_i(\bar{f}). \quad (2)$$

We remarked that, for the suitable choices of $I, F_i, K_i, X_i, Y_i, C_i$, and A_i , SOQEP (2) reduces to the problems considered and studied by [3–6] and the references therein.

Now, we will give the following concepts and results which are used in the sequel.

Definition 1. Let M be a nonempty and convex subset of a topological vector space, and let Z be a topological vector space with a closed and convex cone P with apex at the origin. A vector-valued function $\phi : M \rightarrow Z$ is said to be as follows:

(i) *P*-function if and only if $\forall x, y \in M$ and $\lambda \in [0, 1]$:

$$\phi(\lambda x + (1 - \lambda)y) \in \lambda\phi(x) + (1 - \lambda)\phi(y) - P; \quad (3)$$

(ii) *natural P-quasifunction* if and only if $\forall x, y \in M$ and $\lambda \in [0, 1]$:

$$\phi(\lambda x + (1 - \lambda)y) \in \text{Co}\{\phi(x), \phi(y)\} - P, \quad (4)$$

where $\text{Co}B$ denotes the convex hull of B ;

(iii) *P-quasifunction* if and only if $\forall \alpha \in Z$ and the set $\{x \in M : \phi(x) - \alpha \in -P\}$ is convex.

Definition 2 (see [7]). Let X be a topological vector space and let L be a lattice with a minimal element, denoted by 0. A mapping $\phi : 2^X \rightarrow L$ is called a measure of noncompactness provided that the following conditions hold for any $M, N \in 2^X$:

(i) $\phi(\overline{\text{Co}}M) = \phi(M)$, where $\overline{\text{Co}}M$ denotes the closed convex hull of M ;

(ii) $\phi(M) = 0$ if and only if M is precompact;

(iii) $\phi(M \cup N) = \max\{\phi(M), \phi(N)\}$.

Definition 3 (see [7]). Let X be a topological vector space, $D \subset X$, and let ϕ be a measure of noncompactness on X . A set-valued mapping $T : D \rightarrow 2^X$ is called ϕ -condensing provided that $M \subset D$ with $\phi(T(M)) \geq \phi(M)$; then M is relative compact; that is, $\overline{M}$ is compact.

Remark 4. Note that every set-valued mapping defined on a compact set is ϕ -condensing for any measure of noncompactness ϕ . If X is locally convex, then a compact set-valued mapping (i.e., $T(D)$ is precompact) is ϕ -condensing for any measure of noncompactness ϕ . Obviously, if $T : D \rightarrow 2^X$ is ϕ -condensing and $T' : D \rightarrow 2^X$ satisfies $T'(x) \subset T(x)$, for all $x \in X$, then T' is also ϕ -condensing.

The following maximal element theorems will play key role in establishing existence results.

Theorem 5 (see [8]). For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space X_i and let $S_i, T_i : K \rightarrow 2^{K_i}$ be the two set-valued mappings. For each $i \in I$, assume that the following conditions hold:

(a) for all $x \in K$, $\text{Co}S_i(x) \subseteq T_i(x)$;

(b) for all $x \in K$, $x_i \notin T_i(x)$;

(c) for all $y_i \in K_i$, $S_i^{-1}(y_i)$ is compactly open K ;

(d) there exist a nonempty compact subset D of K and a nonempty compact convex subset $E_i \subseteq K_i$, for each $i \in I$, such that, for all $x \in K \setminus D$, there exists $i \in I$ such that $S_i(x) \cap E_i \neq \emptyset$.

Then, there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

We will use the following particular form of a maximal element theorem for a family of set-valued mappings due to Deguire et al. [5].

Theorem 6 (see [5]). Let I be any index set, for each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i , and let $S_i : K = \prod_{i \in I} K_i \rightarrow 2^{K_i}$ be a set-valued mapping. Assume that the following conditions hold:

(i) $\forall i \in I$ and $\forall x \in K$; $S_i(x)$ is convex;

(ii) $\forall i \in I$ and $\forall x \in K$; $x_i \notin S_i(x)$, where x_i is the i th component of x ;

(iii) $\forall i \in I$ and $\forall y_i \in K_i$; $S_i^{-1}(y_i)$ is open K ;

(iv) there exist a nonempty compact subset D of K and a nonempty compact convex subset $E_i \subseteq K_i$, $\forall i \in I$ such that $\forall x \in K \setminus D$ and there exists $i \in I$ such that $S_i(x) \cap E_i \neq \emptyset$.

Then, there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Remark 7. If $\forall i \in I$, K_i is nonempty, closed, and convex subset of a locally convex Hausdorff topological vector space X_i , then condition (iv) of Theorem 6 can be replaced by the following condition:

(iv)¹ the set-valued mapping $S : K \rightarrow 2^K$ is defined as $S(x) = \prod_{i \in I} S_i(x)$, $\forall x \in K$, ϕ -condensing.

3. Main Result

Throughout this paper, unless otherwise stated, for any index set I and for each $i \in I$, let Y_i be a topological vector space and let $K = \prod_{i \in I} K_i$, $C_i : K \rightarrow 2^{Y_i}$ be a set-valued mapping such that, for each $f \in K$, $C_i(f)$ is proper, solid, open, and convex cone such that $0 \notin C_i(f)$ and $P_i = \bigcap_{f \in K} C_i(f)$. We denote $L(X_i, Y_i)$, the space of all continuous linear operators from X_i into Y_i . We also assume that $\forall i \in I$, $A_i : K \rightarrow 2^{K_i}$ is a set-valued mapping such that $\forall f \in K$, $A_i(f)$ is nonempty and convex, $A_i^{-1}(g_i)$ is open in K , $f_i \in K_i$, and the set $F_i : \{f \in K : f_i \in A_i(f)\}$ is closed in K , where f_i is the i th component of f .

Now, we have the following existence result for SOQEP (2).

Theorem 8. For each $i \in I$, let K_i be nonempty and convex subset of a Hausdorff topological vector space X_i and let $F_i : K \times K_i \rightarrow Y_i$ be a bifunction. Suppose that the following conditions hold:

- (i) $\forall i \in I$ and $\forall f \in K$, $F_i(f, f_i) \notin -C_i(f)$, where f_i is the i th component of f ;
- (ii) $\forall i \in I$ and $\forall f \in K$; the vector-valued function $g_i \mapsto F_i(f, g_i)$ is natural P_i -quasifunction;
- (iii) $\forall i \in I$ and $\forall g_i \in K_i$; the set $\{f \in K : F_i(f, g_i) \notin -C_i(f)\}$ is closed in K ;
- (iv) there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i , for each $i \in I$ such that $\forall f \in K \setminus N$; there exists $i \in I$ and $\hat{g}_i \in B_i$ such that $\hat{g}_i \in A_i(f)$ and $F_i(f, \hat{g}_i) \in -C_i(f)$.

Then SOQEP (2) has a solution.

Proof. Let us define, for each given $i \in I$, a set-valued mapping $T_i : K \rightarrow 2^{K_i}$ by

$$T_i(f) = \{g_i \in K_i : F_i(f, g_i) \in -C_i(f)\}, \quad \forall f \in K. \quad (5)$$

First, we claim that $\forall i \in I$ and $f \in K$, $T_i(f)$ is convex. Fix an arbitrary $i \in I$ and $f \in K$. Let $g_{i,1}, g_{i,2} \in T_i(f)$ and $\lambda \in [0, 1]$; then we have

$$F_i(f, g_{i,j}) \in -C_i(f), \quad \text{for } j = 1, 2. \quad (i)$$

Since $F_i(f, \cdot)$ is natural P_i -quasifunction, there exists $\mu \in [0, 1]$ such that

$$\begin{aligned} F_i(f, \lambda g_{i,1} + (1 - \lambda) g_{i,2}) &\in \mu F_i(f, g_{i,1}) \\ &+ (1 - \mu) F_i(f, g_{i,2}) - P_i. \end{aligned} \quad (ii)$$

From the inclusion of (i) and (ii), we get

$$F_i(f, \lambda g_{i,1} + (1 - \lambda) g_{i,2}) \in -C_i(f) - C_i(f) - P_i \subseteq C_i(f). \quad (6)$$

Hence, $\lambda g_{i,1} + (1 - \lambda) g_{i,2} \in T_i(f)$ and therefore $T_i(f)$ is convex. Since $i \in I$ and $f \in K$ are arbitrary, $T_i(f)$ is convex, $\forall f \in K$ and $\forall i \in I$.

Hence, our claim is then verified.

Now $\forall i \in I$ and $\forall g_i \in K_i$; the complement of $T_i^{-1}(g_i)$ in K can be defined as

$$[T_i^{-1}(g_i)]^c = \{f \in K : F_i(f, g_i) \notin -C_i(f)\}. \quad (7)$$

From condition (iii) of the above theorem, $[T_i^{-1}(g_i)]^c$ will be closed in K .

Suppose that $\forall i \in I$ and $\forall f \in K$; we define another set-valued mapping $M_i : K \rightarrow 2^{K_i}$ by

$$M_i(f) = \begin{cases} A_i(f) \cap T_i(f) & \text{if } f \in \mathcal{F}_i \\ A_i(f); & \text{if } f \in K \setminus \mathcal{F}_i. \end{cases} \quad (8)$$

Then, it is clear that $\forall i \in I$ and $\forall f \in K$, $M_i(f)$ is convex, because $A_i(f)$ and $T_i(f)$ are both convex. Now, by condition (i), $f_i \notin M_i(f)$. Since $\forall i \in I$ and $\forall g_i \in K_i$,

$$\begin{aligned} M_i^{-1}(g_i) &= (A_i^{-1}(g_i) \cap T_i^{-1}(g_i)) \cup ((K \setminus \mathcal{F}_i) \cap A_i^{-1}(g_i)) \\ &\quad (9) \end{aligned}$$

is open in K , because $A_i^{-1}(g_i)$, $T_i^{-1}(g_i)$ and $K \setminus \mathcal{F}_i$ are open in K .

Condition (iv) of Theorem 6 is followed from condition (iv). Hence, by fixed point Theorem 6, there exists $\bar{f} \in K$ such that $M_i(\bar{f}) = \emptyset, \forall i \in I$. Since $\forall i \in I$ and $\forall f \in K$, $A_i(f)$ is nonempty, we have $A_i(\bar{f}) \cap T_i(\bar{f}) = \emptyset, \forall i \in I$. Therefore, $\forall i \in I, \bar{f}_i \in A_i(\bar{f})$ and $F_i(\bar{f}, g_i) \in -C_i(\bar{f}), \forall g_i \in A_i(\bar{f})$.

This completes the proof. $\square$

Now, we establish an existence result for SOQEP (2) involving ϕ -condensing maps.

Theorem 9. For each $i \in I$, let K_i be a nonempty, closed, and convex subset of a locally convex Hausdorff topological vector space X_i , suppose that $F_i : K \times K_i \rightarrow Y_i$ is a bifunction, and let the set-valued mapping $A = \prod_{i \in I} A_i : K \rightarrow 2^K$ defined as $A(f) = \prod_{i \in I} A_i(f), \forall f \in K$ be ϕ -condensing. Assume that conditions (i), (ii), and (iii) of Theorem 8 hold. Then SOQEP (2) has a solution.

Proof. In view of Remark 7, it is sufficient to show that the set-valued mapping $S : K \rightarrow 2^K$ defined as $S(f) = \prod_{i \in I} S_i(f), \forall f \in K$, is ϕ -condensing, where S_i s are the same as defined in the proof of Theorem 8. By the definition of $S_i, S_i(f) \subseteq A_i(f), \forall i \in I$ and $\forall f \in K$ and therefore $S(f) \subseteq A(f), \forall f \in K$. Since A is ϕ -condensing, by Remark 7, we have S being also ϕ -condensing.

This completes the proof. $\square$

4. System of Generalized Quasi Operator Equilibrium Problem

Throughout this section, unless otherwise stated, let I be any index set. For each $i \in I$, let X_i be a Hausdorff topological vector space. We denote $L(X_i, Y_i)$, the space of all continuous linear operators from X_i into Y_i , where Y_i is topological vector space for each $i \in I$ and for each $i \in I$; let $P_i \subset Y_i$ be a closed, pointed, and convex cone with $\text{int } P_i \neq \emptyset$, where $\text{int } P_i$ denotes the interior of set $P_i, \forall i \in I$. Consider a family of nonempty convex subsets $\{K_i\}_{i \in I}$ with K_i in $L(X_i, Y_i)$. Let, for each $i \in I$, a bifunction $F_i : K \times K_i \rightarrow Y_i$ and two set-valued mappings $A_i, B_i : K \rightarrow 2^{K_i}$ be with nonempty values.

Let e_i be the unit vector in Y_i , for each $i \in I$, and also $\alpha_i e_i, \beta_i e_i \in Y_i$ such that $\alpha_i e_i \not\leq_{P_i} \beta_i e_i$, where $\alpha_i, \beta_i \in \mathbb{R}$ are two real numbers such that $\alpha_i \leq \beta_i$.

Now, we consider the system of generalized quasi operator equilibrium problems (for short, SGQOEP). Find $\bar{f} \in K$ such that, for each $i \in I$,

$$\bar{f}_i \in B_i(\bar{f}), \quad \alpha_i e_i \not\leq_{P_i} F_i(\bar{f}, g_i) \not\leq_{P_i} \beta_i e_i; \quad \forall g_i \in A_i(\bar{f}). \quad (10)$$

4.1. Special Cases

- (I) If $A_i = B_i, \forall i$, then SGQOEP (10) reduces to finding of $\bar{f} \in K$ such that, for each $i \in I$,

$$\alpha_i e_i \not\leq_{P_i} F_i(\bar{f}, g_i) \not\leq_{P_i} \beta_i e_i; \quad \forall g_i \in A_i(\bar{f}). \quad (11)$$

(II) If, in Case (I), we take $Y_i = \mathbb{R}$, then $P_i = \mathbb{R}_+$ and $e_i = 1$; then problem (10) reduces to the system of generalized quasi operator equilibrium problems with lower and upper bounds (for short, SGQOEPLUB). Find $\bar{f} \in K$ such that, for each $i \in I$,

$$\bar{f}_i \in B_i(\bar{f}), \quad \alpha_i \leq F_i(\bar{f}, g_i) \leq \beta_i; \quad \forall g_i \in A_i(\bar{f}). \quad (12)$$

Now, we establish the existence result for SGQOEP (10).

Theorem 10. For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space X_i and $F_i, P_i, Q_i : K \times K_i \rightarrow K_i$ are the bifunctions, $B_i : K \rightarrow 2^{K_i}$ is a set-valued mapping such that the set $F_i = \{f \in K : f_i \in B_i(f)\}$ is compactly closed, $A_i : K \rightarrow 2^{K_i}$ is a set-valued mapping with nonempty values such that, for each $g_i \in K_i$, $A_i^{-1}(g_i)$ is compactly open in K , and $\forall i \in I, e_i \in Y_i$ are the unit vector such that $\alpha_i e_i \not\leq_{P_i} \beta_i e_i$, where $\alpha_i, \beta_i \in \mathbb{R}$ are two real numbers such that $\alpha_i \leq \beta_i$. For each $i \in I$, assume that the following conditions hold:

- (i) for all $f \in K$, $\text{Co}A_i(f) \subseteq B_i(f)$;
- (ii) for all $f \in K$, $\alpha_i e_i >_{P_i} P_i(f, f_i)$ or $Q_i(f, f_i) >_{P_i} \beta_i e_i$;
- (iii) for all $f \in K$ and for every nonempty finite subset $N_i \subseteq \{g_i \in K_i : F_i(f, g_i) <_{P_i} \alpha_i e_i \text{ or } F_i(f, g_i) >_{P_i} \beta_i e_i\}$, we have

$$\text{Co}N_i \subseteq \{g_i \in K_i : P_i(f, g_i) \not\leq_{P_i} \alpha_i e_i \text{ and } Q_i(f, g_i) \not\leq_{P_i} \beta_i e_i\}; \quad (13)$$

- (iv) for all $g_i \in K_i$, the set $\{f \in K : \beta_i e_i \not\leq_{P_i} F_i(f, g_i) \not\leq_{P_i} \alpha_i e_i\}$ is compactly closed in K ;
- (v) there exist a nonempty compact subset D of K and a nonempty compact convex subset $E_i \subset K_i$, for each $i \in I$, such that, for all $f \in K \setminus D$, there exists $i \in I$ such that $\hat{g}_i \in E_i$ satisfying $\hat{g}_i \in A_i(f)$ and either $F_i(f, \hat{g}_i)_{P_i} < \alpha_i e_i$ or $F_i(f, \hat{g}_i)_{P_i} > \beta_i e_i$.

Then the problem SGQOEP (10) has a solution.

Proof. For each $i \in I$ and for all $f \in K$, define two set-valued mappings $G_i, H_i : K \rightarrow 2^{K_i}$ by

$$\begin{aligned} G_i(f) &= \{g_i \in K_i : F_i(f, g_i) <_{P_i} \alpha_i e_i \text{ or } F_i(f, g_i) >_{P_i} \beta_i e_i\}, \\ H_i(f) &= \{g_i \in K_i : P_i(f, g_i)_{P_i} \not\leq \alpha_i e_i \text{ and } Q_i(f, g_i) \not\leq_{P_i} \beta_i e_i\}. \end{aligned} \quad (14)$$

Condition (iii) implies that, for each $i \in I$ and for all $f \in K$, $\text{Co}G_i(f) \subseteq H_i(f)$.

From condition (ii), we have $f_i \notin H_i(f)$ for all $f \in K$ and for each $i \in I$.

Thus, for each $i \in I$ and for all $g_i \in K_i$,

$$G_i^{-1}(g_i) = \{f \in K : F_i(f, g_i) <_{P_i} \alpha_i e_i \text{ or } F_i(f, g_i) >_{P_i} \beta_i e_i\}. \quad (15)$$

We have complement of $G_i^{-1}(g_i)$ in K :

$$[G_i^{-1}(g_i)]^c = \{\beta_i e_i \not\leq_{P_i} F_i(\bar{f}, g_i) \not\leq_{P_i} \alpha_i e_i\}, \quad (16)$$

which is compactly closed by virtue of condition (iv). Therefore, for each $i \in I$ and for all $g_i \in K_i$, $G_i^{-1}(g_i)$ is compactly open in K .

For each $i \in I$, define two set-valued mappings $S_i, T_i : K \rightarrow 2^{K_i}$ by

$$\begin{aligned} S_i(f) &= \begin{cases} G_i(f) \cap A_i(f); & \text{if } f \in \mathcal{F}_i \\ A_i(f); & \text{if } f \in K \setminus \mathcal{F}_i, \end{cases} \\ T_i(f) &= \begin{cases} H_i(f) \cap B_i(f); & \text{if } f \in \mathcal{F}_i \\ B_i(f); & \text{if } f \in K \setminus \mathcal{F}_i. \end{cases} \end{aligned} \quad (17)$$

Thus, for each $i \in I$ and for all $f \in K$, $\text{Co}G_i(f) \subseteq H_i(f)$ and in view of condition (i), we obtain $\text{Co}S_i(f) \subseteq T_i(f)$. It is easy to see that

$$S_i^{-1}(g_i) = (A_i^{-1}(g_i) \cap G_i^{-1}(g_i)) \cup ((K \setminus \mathcal{F}_i) \cap A_i^{-1}(g_i)) \quad (18)$$

for each $i \in I$ and for all $f \in K$. Thus, for each $i \in I$ and for all $G_i^{-1}(g_i), A_i^{-1}(g_i)$ and $K \setminus \mathcal{F}_i$ are compactly open in K . We have $S_i^{-1}(g_i)$ being compactly open in K . Also $f_i \notin T_i(f)$ for all $f \in K$ and for each $i \in I$.

Then, by Theorem 5, there exists $\bar{f} \in K$ such that $S_i(\bar{f}) = \emptyset$ for each $i \in I$. If $\bar{f} \in K \setminus \mathcal{F}_i$, then $A_i(\bar{f}) = S_i(\bar{f}) = \emptyset$, which contradicts the fact that $A_i(f)$ is nonempty for each $i \in I$ and for all $f \in X$. Hence, $\bar{f} \in \mathcal{F}_i$, for each $i \in I$. Therefore, $\bar{f}_i \in B_i(\bar{f})$ and $G_i(\bar{f}) \cap A_i(\bar{f}) = \emptyset$, for all $i \in I$. Thus, for each $i \in I$, $\bar{f}_i \in B_i(\bar{f})$ and $\beta_i e_i \not\leq_{P_i} F_i(\bar{f}, g_i) \not\leq_{P_i} \alpha_i e_i$ for all $g_i \in A_i(\bar{f})$. This completes the proof. $\square$

Now, we establish an existence result for SGQOEP (10) involving ϕ -condensing maps.

Theorem 11. For each $i \in I$, assume that conditions (i)–(iv) of Theorem 10. hold. Let ϕ be a measure of noncompactness on $\prod_{i \in I} X_i$. Further, assume that the set-valued mapping $B : K \rightarrow 2^K$ defined as K_i is a nonempty, closed, and convex subset of a locally convex Hausdorff topological vector space X_i and $F_i : K \times K_i \rightarrow Y_i$ is a bifunction and let the set-valued mapping $A = \prod_{i \in I} A_i : K \rightarrow 2^K$ defined as $B(f) = \prod_{i \in I} B_i(f), \forall f \in K$ be ϕ -condensing. Then, there exists a solution $\bar{f} \in K$ of SGQOEP (10).

Proof. In view of Remark 7, it is sufficient to show that the set-valued mapping $T : K \rightarrow 2^K$ defined as $T(f) = \prod_{i \in I} T_i(f), \forall f \in K$, is ϕ -condensing, where T_i s are the same as defined in the proof of Theorem 10. By the definition of $T_i, T_i(f) \subseteq B_i(f), \forall i \in I$ and $\forall f \in K$ and therefore $S(f) \subseteq A(f), \forall f \in K$. Since B is ϕ -condensing, by Remark 7, we have T being also ϕ -condensing.

This completes the proof. $\square$

Next, we derive the existence result for the solution of SGQOEPLUB (12).

Corollary 12. For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space X_i and $F_i, L_i, Q_i : K \times K_i \rightarrow \mathbb{R}$ are the bifunctions, $B_i : K \rightarrow 2^{K_i}$ is a set-valued mapping such that the set $\mathcal{F}_i = \{f \in K : f_i \in B_i(f)\}$ is compactly closed, $A_i : K \rightarrow 2^{K_i}$ is a set-valued mapping with nonempty values such that, for each $g_i \in K_i$, $A_i^{-1}(g_i)$ is compactly open in K , and $\alpha_i, \beta_i \in \mathbb{R}$ are two real numbers such that $\alpha_i \leq \beta_i$. For each $i \in I$, assume that the following conditions hold:

- (i) for all $f \in K$, $\text{Co}A_i(f) \subseteq B_i(f)$;
- (ii) for all $f \in K$, $L_i(f, f_i) < \alpha_i$ or $Q_i(f, f_i) > \beta_i$;
- (iii) for all $f \in K$ and for every nonempty finite subset $N_i \subseteq \{g_i \in K_i : F_i(f, g_i) < \alpha_i \text{ or } F_i(f, g_i) > \beta_i\}$, we have

$$\text{Co}N_i \subseteq \{g_i \in K_i : L_i(f, g_i) \geq \alpha_i \text{ and } Q_i(f, g_i) \leq \beta_i\}; \quad (19)$$

- (iv) for all $g_i \in K_i$, the set $\{f \in K : \alpha_i \leq F_i(f, g_i) \leq \beta_i\}$ is compactly closed in K ;
- (v) there exist a nonempty compact subset D of K and a nonempty compact convex subset $E_i \subset K_i$, for each $i \in I$, such that, for all $f \in K \setminus D$, there exists $i \in I$ such that $\hat{g}_i \in E_i$ satisfying $\hat{g}_i \in A_i(f)$ and either $F_i(f, \hat{g}_i) < \alpha_i$ or $F_i(f, \hat{g}_i) > \beta_i$.

Then the problem SGQOEPLUB (12) has a solution.

Proof. For each $i \in I$ and for all $f \in K$, define two set-valued mappings $G_i, H_i : K \rightarrow 2^{K_i}$ by

$$\begin{aligned} G_i(f) &= \{g_i \in K_i : F_i(f, g_i) < \alpha_i \text{ or } F_i(f, g_i) > \beta_i\}, \\ H_i(f) &= \{g_i \in K_i : L_i(f, g_i) \geq \alpha_i \text{ and } Q_i(f, g_i) \leq \beta_i\}. \end{aligned} \quad (20)$$

Condition (iii) implies that, for each $i \in I$ and for all $f \in K$, $\text{Co}G_i(f) \subseteq H_i(f)$.

From condition (ii), we have $f_i \notin H_i(f)$ for all $f \in K$ and for each $i \in I$.

Thus, for each $i \in I$ and for all $g_i \in K_i$,

$$G_i^{-1}(g_i) = \{f \in K : F_i(f, g_i) < \alpha_i \text{ or } F_i(f, g_i) > \beta_i\}. \quad (21)$$

We have complement of $G_i^{-1}(g_i)$ in K :

$$\left[G_i^{-1}(g_i)\right]^c = \{\alpha_i \leq F_i(f, g_i) \leq \beta_i\}, \quad (22)$$

which is compactly closed by virtue of condition (iv). Therefore, for each $i \in I$ and for all $g_i \in K_i$, $G_i^{-1}(g_i)$ is compactly open in K .

For each $i \in I$, define two set-valued mappings $S_i, T_i : K \rightarrow 2_i^K$ by

$$\begin{aligned} S_i(f) &= \begin{cases} G_i(f) \cap A_i(f) & \text{if } f \in \mathcal{F}_i \\ A_i(f); & \text{if } f \in K \setminus \mathcal{F}_i, \end{cases} \\ T_i(f) &= \begin{cases} H_i(f) \cap B_i(f) & \text{if } f \in \mathcal{F}_i \\ B_i(f); & \text{if } f \in K \setminus \mathcal{F}_i. \end{cases} \end{aligned} \quad (23)$$

Thus, for each $i \in I$ and for all $f \in K$, $\text{Co}G_i(f) \subseteq H_i(f)$ and in view of condition (i), we obtain $\text{Co}S_i(f) \subseteq T_i(f)$. It is easy to see that

$$S_i^{-1}(g_i) = (A_i^{-1}(g_i) \cap G_i^{-1}(g_i)) \bigcup ((K \setminus \mathcal{F}_i) \cap A_i^{-1}(g_i)) \quad (24)$$

for each $i \in I$ and for all $f \in K$. Thus, for each $i \in I$ and for all $G_i^{-1}(g_i), A_i^{-1}(g_i)$ and $K \setminus \mathcal{F}_i$ are compactly open in K . We have $S_i^{-1}(g_i)$ being compactly open in K . Also $f_i \notin T_i(f)$ for all $f \in K$ and for each $i \in I$.

Then, by Theorem 5, there exists $\bar{f} \in K$ such that $S_i(\bar{f}) = \emptyset$ for each $i \in I$. If $\bar{f} \in K \setminus \mathcal{F}_i$, then $A_i(\bar{f}) = S_i(\bar{f}) = \emptyset$, which contradicts the fact that $A_i(f)$ is nonempty for each $i \in I$ and for all $f \in X$. Hence, $\bar{f} \in \mathcal{F}_i$, for each $i \in I$. Therefore, $\bar{f}_i \in B_i(\bar{f})$ and $G_i(\bar{f}) \cap A_i(\bar{f}) = \emptyset$, for all $i \in I$. Thus, for each $i \in I$, $\bar{f}_i \in B_i(\bar{f})$ and $\alpha_i \leq F_i(\bar{f}, g_i) \leq \beta_i$ for all $g_i \in A_i(\bar{f})$. This completes the proof. $\square$

Now, we establish an existence result for SGQOEPLUB (12) involving ϕ -condensing maps.

Theorem 13. For each $i \in I$, assume that conditions (i)–(iv) of Corollary 12 hold. Let ϕ be a measure of noncompactness on $\prod_{i \in I} X_i$. Further, assume that the set-valued mapping $B : K \rightarrow 2^K$ defined as K_i is a nonempty, closed, and convex subset of a locally convex Hausdorff topological vector space X_i and $F_i : K \times K_i \rightarrow \mathbb{R}$ is a bifunction and let the set-valued mapping $A = \prod_{i \in I} A_i : K \rightarrow 2^K$ defined as $B(f) = \prod_{i \in I} B_i(f), \forall f \in K$, be ϕ -condensing. Then, there exists a solution $\bar{f} \in K$ of SGQOEPLUB (12).

Proof. In view of Remark 7, it is sufficient to show that the set-valued mapping $T : K \rightarrow 2^K$ defined as $T(f) = \prod_{i \in I} T_i(f), \forall f \in K$, is ϕ -condensing, where T^i 's are the same as defined in the proof of Theorem 10. By the definition of $T_i, T_i(f) \subseteq B_i(f), \forall i \in I$ and $\forall f \in K$ and therefore $S(f) \subseteq A(f), \forall f \in K$. Since B is ϕ -condensing, by Remark 7, we have T being also ϕ -condensing.

This completes the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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