Research Article

Residuation Properties and Weakly Primary Elements in Lattice Modules

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We obtain some elementary residuation properties in lattice modules and obtain a relation between a weakly primary element in a lattice module M and weakly prime element of a multiplicative lattice L .

1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative, and join distributive multiplication in which the largest element I acts as a multiplicative identity. An element $a \in L$ is called proper if $a < I$. A proper element p of L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. If $a \in L$, $b \in L$, $(a : b)$ is the join of all elements c in L such that $cb \leq a$. A proper element p of L is said to be primary if $ab \leq p$ implies $a \leq p$ or $b^n \leq p$ for some positive integer n . If $a \in L$, the radical of a is denoted by $\sqrt{a} = \vee \{x \in L \mid x^n \leq a, n \in \mathbb{Z}_+\}$. An element $a \in L$ is called compact if $a \leq \vee_{\alpha} b_{\alpha}$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \dots \vee b_{\alpha_n}$ for some finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact and every finite product of compact elements is compact. We will denote by L_* the set compact elements of L . A nonempty subset F of L_* is called a filter of L_* if the following conditions are satisfied:

- (1) $x, y \in F$ implies $xy \in F$;
- (2) $x \in F, x \leq y$ implies $y \in F$.

Let $F(L_*)$ denote the set of all filters of L . For a nonempty subset $\{F_{\alpha}\} \subseteq F(L_*)$, define $\mathbb{W}F_{\alpha} = \{x \in L_* \mid x \geq f_1 f_2 \dots f_n \in F_{\alpha_i}, \text{ for some } i = 1, 2, \dots, n\}$. Then it is observed that $F(L_*) = \langle F(L_*), \mathbb{W}, \cap \rangle$ is a complete distributive lattice with $\mathbb{W}$ as the supremum and the set theoretic $\cap$ as the infimum. For $a \in L_*$

the smallest filter containing a is denoted by $[a]$ and it is given by $[a] = \{x \in L_* \mid x \geq a^n \text{ for some nonnegative integer } n\}$. For a filter $F \in F(L_*)$ we denote $0_F = \vee \{x \in L_* \mid xs = 0, \text{ for } s \in F\}$.

Let M be a complete lattice and L a multiplicative lattice. Then M is called L -module or module over L if there is a multiplication between elements of L and M written as aB where $a \in L$ and $B \in M$ which satisfies the following properties:

- (1) $(\vee_{\alpha} a_{\alpha})A = \vee_{\alpha} a_{\alpha}A, \forall a_{\alpha} \in L, A \in M$;
- (2) $a(\vee_{\alpha} A_{\alpha}) = \vee_{\alpha} aA_{\alpha}, \forall a \in L, A_{\alpha} \in M$;
- (3) $(ab)A = a(bA), \forall a, b \in L, A \in M$;
- (4) $IB = B$;
- (5) $OB = O_M$, for all $a, a_{\alpha}, b \in L$ and $A, A_{\alpha} \in M$, where I is the supremum of L and O is the infimum of L . We denote by 0_M and I_M the least element and the greatest element of M . Elements of L will generally be denoted by $a, b, c, \dots$ and elements of M will generally be denoted by $A, B, C, \dots$

Let M be an L -module. If $N \in M$ and $a \in L$ then $(N : a) = \vee \{X \in M \mid aX \leq N\}$. If $A, B \in M$, then $(A : B) = \vee \{x \in L \mid xB \leq A\}$. An L -module M is called a multiplication L -module if for every element $N \in M$ there exists an element $a \in L$ such that $N = aI_M$. In this paper a lattice module M will be a multiplication lattice module. A proper element N of

M is said to be prime if $aX \leq N$ implies $X \leq N$ or $aI_M \leq N$; that is, $a \leq (N : I_M)$ for every $a \in L$, $X \in M$. If N is a prime element of M then $(N : I_M)$ is prime element of L [1]. An element $N < I_M$ in M is said to be primary if $aX \leq N$ implies $X \leq N$ or $a^n I_M \leq N$; that is, $a^n \leq (N : I_M)$ for some integer n . An element N of M is called a radical element if $(N : I_M) = \sqrt{(N : I_M)}$.

2. Residuation Properties

We prove some elementary properties of residuation. Such results are obtained by Anderson and others [2] for compactly generated multiplicative lattices.

Theorem 1. *Let L be a multiplicative lattice and M a multiplication lattice module over L . For $x, y \in L$ and $Z, A, B \in M$, where $(0_M : I_M)$ is a radical element, one has the following identities:*

- (i) $x \leq y$ implies $(0_M : y) \leq (0_M : x)$ and $0_M : (0_M : x) \leq 0_M : (0_M : y)$;
- (ii) $x \leq 0_M : (0_M : x)$;
- (iii) $0_M : [0_M : (0_M : x)] = (0_M : x)$;
- (iv) $(0_M : x) = (0_M : x^n)$ for every $n \in \mathbb{Z}_+$;
- (v) $0_M : (0_M : x) \wedge 0_M : (0_M : y) = 0_M : (0_M : xy) = 0_M : [0_M : (x \wedge y)]$;
- (vi) $(0_M : a) = 0_M$ implies $(0_M : a^n) = 0_M$ for every $n \in \mathbb{Z}_+$;
- (vii) $x \vee y = 1$ implies $(0_M : x) \vee (0_M : y) = 0_M : (x \wedge y) = 0_M : xy$;
- (viii) For Z in M , $Z \leq 0_M : (0_M : Z)$;
- (ix) $A \leq B$ implies $(0_M : B) \leq (0_M : A)$;
- (x) $0_M : [0_M : (0_M : A)] = 0_M : A$;
- (xi) $0_M : xI_M = 0_M : x^n I_M$ for some positive integer n .

Proof. (i) We have $(0_M : y) = \vee\{A \in M \mid yA = 0_M\}$. As $x \leq y$, $xA \leq yA$, so $yA = 0_M$ implies $xA = 0_M$. This shows that $(0_M : y) \leq (0_M : x)$. Next part $0_M : (0_M : x) \leq 0_M : (0_M : y)$ is obvious.

(ii) We have $x(0_M : x) = x\{\vee\{A \in M \mid xA = 0_M\}\} = \vee\{xA \mid xA = 0_M\} = 0_M$. This implies that $x \leq 0_M : (0_M : x)$.

(iii) We have, by (ii), $x \leq 0_M : (0_M : x)$. So by (i), $0_M : [0_M : (0_M : x)] \leq (0_M : x)$. If $A \in M$, $A(0_M : A) = \vee\{xA \mid xA = 0_M\} = 0_M$. So, $A \leq 0_M : (0_M : A)$. Hence, $(0_M : x) \in M$ implies $(0_M : x) \leq 0_M : [0_M : (0_M : x)]$. Therefore, $0_M : [0_M : (0_M : x)] = (0_M : x)$.

(iv) We have $(0_M : x) = \vee\{A \in M \mid xA \leq 0_M\}$. As $x^n \leq x$, we have $(0_M : x) \leq (0_M : x^n)$ for every $n \in \mathbb{Z}_+$. Let $Z = (0_M : x^n)$ where $Z \in M$. As M is a multiplication lattice module $Z = aI_M$ for some $a \in L$. So $x^n a^n I_M \leq x^n a I_M = 0_M$. Hence $xa \leq \sqrt{(0_M : I_M)} = (0_M : I_M)$. So $xaI_M \leq 0_M$; that is, $Z \leq (0_M : x)$, and hence $(0_M : x) = (0_M : x^n)$ for each positive integer n .

(v) Note that $xy \leq x, y$. Therefore by (i), $0_M : (0_M : xy) \leq 0_M : (0_M : x) \wedge 0_M : (0_M : y)$. Let $z = 0_M : (0_M : x) \wedge 0_M : (0_M : y)$. Now we have $z \leq 0_M : (0_M : x)$ and $z \leq 0_M : (0_M : y)$. Then, $z(0_M : x) = 0_M$ and $z(0_M : y) = 0_M$. This implies

that $(0_M : x) \leq (0_M : z)$, $(0_M : y) \leq (0_M : z)$. Therefore, we have $(0_M : xy) = (0_M : x) : y \leq (0_M : z) : y = (0_M : yz)$ [3]. Hence, $(0_M : xy) \leq (0_M : y) : z \leq (0_M : z) : z = (0_M : z^2) = (0_M : z)$, by (iv). That is, $z(0_M : xy) = 0_M$ or $z \leq 0_M : (0_M : xy)$. Consequently, $0_M : (0_M : xy) = 0_M : (0_M : x) \wedge 0_M : (0_M : y)$. Note that $x \wedge y \leq x, y$. Then by (i), we have $0_M : (0_M : (x \wedge y)) \leq 0_M : (0_M : x) \wedge 0_M : (0_M : y)$. Let $z = 0_M : (0_M : x) \wedge 0_M : (0_M : y)$. We have $0_M : (x \wedge y) \leq 0_M : xy \leq 0_M : z^2 = 0_M : z$, by (iv). This shows that $z[0_M : (x \wedge y)] \leq 0_M$. Thus, $z \leq 0_M : [0_M : (x \wedge y)]$. Consequently, $0_M : [0_M : (x \wedge y)] = 0_M : (0_M : x) \wedge 0_M : (0_M : y)$.

(vi) Assume that $(0_M : a) = 0_M$. We have, by (iv), $(0_M : a) = (0_M : a^n)$ for every $n \in \mathbb{Z}_+$. But $(0_M : a) = 0_M$. Thus, $(0_M : a^n) = 0_M$ for every $n \in \mathbb{Z}_+$.

(vii) Assume that $x \vee y = 1$. As $x \wedge y \leq x, y$, we have $(0_M : x) \leq (0_M : (x \wedge y))$, $(0_M : y) \leq (0_M : (x \wedge y))$. Therefore, $(0_M : x) \vee (0_M : y) \leq 0_M : (x \wedge y)$. Let Z in M be such that $Z \leq 0_M : (x \wedge y)$. Then, $Zxy \leq Z(x \wedge y) \leq 0_M$. This gives $Zx \leq (0_M : y)$ and $Zy \leq (0_M : x)$. Then we have, for Z in M , $Z = Z.1 = Z(x \vee y) = Zx \vee Zy \leq (0_M : x) \vee (0_M : y)$. Consequently, $0_M : (x \wedge y) = (0_M : x) \vee (0_M : y)$. For the next part, $xy \leq x, y$ where $x, y \in L$. We have $(0_M : x) \leq (0_M : xy)$, $(0_M : y) \leq (0_M : xy)$. Therefore, we get $(0_M : x) \vee (0_M : y) \leq (0_M : xy)$. Let Z be an element of M such that $Z \leq 0_M : xy$. Then, $xyZ \leq 0_M$. This gives $xZ \leq (0_M : y)$ and $yZ \leq (0_M : x)$. Then we have $Z = 1.Z = (x \vee y)Z = xZ \vee yZ \leq (0_M : x) \vee (0_M : y)$. Consequently, $0_M : (xy) = (0_M : x) \vee (0_M : y)$.

(viii) We have, for Z in M , $Z(0_M : Z) = Z[\vee\{a \in L \mid aZ = 0_M\}] = \vee\{aZ \mid a \in L \text{ and } aZ = 0_M\}$. This implies that $Z(0_M : Z) = 0_M$. This shows that, for Z in M , $Z \leq 0_M : (0_M : Z)$.

(ix) Let $A \leq B$. We have $x \leq (0_M : B)$ implies $xB = 0_M$. $A \leq B$ implies $xA \leq xB = 0_M$. So, $x \leq (0_M : A)$. Therefore, $(0_M : B) \leq (0_M : A)$.

(x) As $(0_M : A) \in L$, by (ii), we get $(0_M : A) \leq 0_M : [0_M : (0_M : A)]$. But $A \leq 0_M : (0_M : A)$. Therefore, $(0_M : A) \geq 0_M : [0_M : (0_M : A)]$. Thus, $0_M : [0_M : (0_M : A)] = (0_M : A)$.

(xi) As $x^n \leq x$, we have $x^n I_M \leq xI_M$. So $(0_M : xI_M) \leq (0_M : x^n I_M)$. Let $z = (0_M : x^n I_M)$ where $z \in L$. Then $zx^n I_M \leq 0_M$; that is, $x^n z I_M = 0_M$. Therefore, $x^n z^n I_M = (xz)^n I_M \leq x^n z I_M = 0_M$. Hence, $xz \leq \sqrt{(0_M : I_M)} = (0_M : I_M)$, since 0_M is a radical element. This implies that $z \leq (0_M : xI_M)$. Consequently, $(0_M : x^n I_M) \leq (0_M : xI_M)$. Hence, $(0_M : xI_M) = (0_M : x^n I_M)$. $\square$

Let M_* denote the set of compact elements of M . We define $0_{FM} = \vee\{X \in M_* \mid sX = 0_M \text{ for some } s \in F\}$. Such concept is defined for multiplicative lattices by Anderson and others [2].

We obtain the necessary and sufficient condition for an element of a lattice module M to be contained in 0_{FM} .

Theorem 2. *Let $F \subseteq L$ be a filter of $F(L_*)$ and let X be a compact element of M . Then $X \leq 0_{FM}$ if and only if $sX = 0_M$ for some $s \in F$.*

Proof. As X is compact element and $X \leq 0_{FM}$, there exist finite number of elements $X_1, X_2, \dots, X_n$ such that $X \leq X_1 \vee X_2 \vee \dots \vee X_n$ where $s_i X_i = 0_M$ for some $s_i \in F$. Let

$s = s_1 s_2 \cdots s_n$. Then by hypothesis, finite product of compact elements is compact, and hence, s is a compact element. As F is a filter, $s \in F$. Also, $s \leq s_i$ for each i . Hence, we have $sX \leq s(X_1 \vee X_2 \vee \cdots \vee X_n) = sX_1 \vee \cdots \vee sX_n \leq s_1 X_1 \vee s_2 X_2 \vee \cdots \vee s_n X_n = 0_M$. Conversely, suppose $sX = 0_M$ for some $s \in F$. Hence, $X \leq 0_{FM}$ obviously. $\square$

Now we obtain an equivalent formulation for 0_{FM} .

Theorem 3. For $F \in F(L_*)$, $0_{FM} = \vee\{(0_M : x) \mid x \in F\}$.

Proof. Let $A = \vee\{(0_M : x) \mid x \in F\}$. Let B be a compact element of M such that $B \leq 0_{FM}$. Then by Theorem 2 there exists $x \in F$ such that $xB = 0_M$. So, $B \leq (0_M : x)$. Hence, $B \leq A = \vee\{(0_M : x) \mid x \in F\}$. Therefore, $0_{FM} \leq \vee\{(0_M : x) \mid x \in F\} = A$. Let B be a compact element such that $B \leq A$. Then, there exist $x_1, x_2, \dots, x_n \in F$ such that $B \leq (0_M : x_1) \vee (0_M : x_2) \vee \cdots \vee (0_M : x_n)$. By compactness of B , there exist compact elements $Y_1, Y_2, \dots, Y_n \in M$ such that $x_i Y_i = 0_M$ and $B \leq Y_1 \vee Y_2 \vee \cdots \vee Y_n$. As F is a filter, we have $x = x_1 x_2 \cdots x_n \in F$. Hence, $xB \leq x(Y_1 \vee Y_2 \vee \cdots \vee Y_n) \leq xY_1 \vee xY_2 \vee \cdots \vee xY_n = 0_M$. Thus, $B \leq 0_{FM}$. This shows that $0_{FM} = \vee\{(0_M : x) \mid x \in F\}$. $\square$

Theorem 4. For $F_1, F_2 \in F(L_*)$

- (1) $F_1 \subseteq F_2$ implies $0_{F_1 M} \leq 0_{F_2 M}$;
- (2) $0_{F_1 M} \wedge 0_{F_2 M} = 0_{(F_1 \cap F_2) M}$.

Proof. (1) Let $F_1 \subseteq F_2$ and let X be a compact element of M such that $X \leq 0_{F_1 M}$. Then by Theorem 2 $sX = 0_M$, for some $s \in F_1$. By assumption, $s \in F_2$, $sX = 0_M$. Hence, $X \leq 0_{F_2 M}$ and $0_{F_1 M} \leq 0_{F_2 M}$.

(2) We have $F_1 \cap F_2 \subseteq F_1, F_2$. Hence by 1, $0_{(F_1 \cap F_2) M} \leq 0_{F_1 M}, 0_{F_2 M}$. Thus, $0_{(F_1 \cap F_2) M} \leq 0_{F_1 M} \wedge 0_{F_2 M}$. Let X be a compact element in M such that $X \leq 0_{F_1 M} \wedge 0_{F_2 M}$. Then $X \leq 0_{F_1 M}$ and $X \leq 0_{F_2 M}$. By Theorem 2, there exist elements $s \in F_1$ and $r \in F_2$ such that $sX = 0_M$ and $rX = 0_M$. So $X \leq (0_M : s)$ and $X \leq (0_M : r)$. Consequently, $X \leq (0_M : s) \wedge (0_M : r) = 0_M : (s \vee r)$ [3]. As r and s are compact elements, $s \vee r$ is also compact. As F_1 and F_2 are filters, $s \vee r \in F_1$ and $s \vee r \in F_2$. This implies that $s \vee r \in F_1 \cap F_2$. Thus, $X \leq 0_M : (s \vee r)$ where $s \vee r \in F_1 \cap F_2$. That is, $(s \vee r)X = 0_M$. Hence, by Theorem 2, $X \leq 0_{(F_1 \cap F_2) M}$. That is, $0_{F_1 M} \wedge 0_{F_2 M} \leq 0_{(F_1 \cap F_2) M}$. Thus, $0_{F_1 M} \wedge 0_{F_2 M} = 0_{(F_1 \cap F_2) M}$. $\square$

3. Weakly Prime and Weakly Primary Elements

We introduce the concepts of weakly prime and weakly primary elements in lattice modules. These concepts are generalisations of weakly prime elements in multiplicative lattices [4] and weakly primary ideals [5].

Definition 5. A proper element N of M is said to be weakly prime element if $0 \neq aX \leq N$ implies $X \leq N$ or $aI_M \leq N$ for every $a \in L, X \in M$.

Definition 6. A proper element N of M is said to be weakly primary element if $0 \neq aX \leq N$ implies $X \leq N$ or $a^n I_M \leq N$ for every $a \in L, X \in M$.

Example 7. Let M denote a cyclic Z -module $Z/6Z$ and let $L = L(Z)$ denote the set of all ideals of Z and let $L(Z/6Z)$ denote the set of all submodules of $Z/6Z$. Then $L(Z/6Z)$ is a lattice module over L . Let $N = \{0\}$. Then N is a weakly prime element of M .

Example 8. Let $R = Z$ denote the set of integers and let $L(R)$ denote the set of all ideals of R . Then $L(R)$ is a multiplicative lattice. Let $M = Z \times Z$. Then M is a module over Z . Let $L(M)$ denote the set of all submodules of M . Then $L(M)$ is a lattice module over $L(R)$. Let $L = (4, 0)Z + (0, 1)Z$; then L is a weakly primary element of $L(M)$; however it is not weakly prime element of $L(M)$, because $0 \neq (2)(2, 1) \subseteq L$. But neither $(2)M \subseteq L$ nor $(2, 1) \in L$; see [6].

The next result gives the relation between weakly primary element of M and weakly primary element of L .

The study of such concepts for submodules is carried out by Eid [6].

Theorem 9. Let L be a multiplicative lattice, M a lattice module over L , and N a weakly primary element of M . Then $(N : I_M)$ is a weakly primary element of L .

Proof. Suppose N is a weakly primary element of M . Let $0 \neq ab \leq (N : I_M)$ and $a \not\leq (N : I_M)$ where $a, b \in L$. This implies that $aI_M \not\leq N$. But N is a weakly primary element and $0 \neq abI_M \leq N$ and $aI_M \not\leq N$. So $b^n I_M \leq N$ for some $n \in \mathbb{Z}_+$; that is, $b^n \leq (N : I_M)$. Hence, $(N : I_M)$ is a weakly primary element of L . $\square$

Now we obtain the relation between weakly primary element of M and weakly prime element of L where L is assumed to be without divisors of zero.

Theorem 10. Let L be a multiplicative lattice and M a lattice module over L and suppose N is a weakly primary element of M . Then $\sqrt{(N : I_M)}$ is a weakly prime element of L .

Proof. Suppose N is a weakly primary element of M . Then by Theorem 9 $(N : I_M)$ is a weakly primary element of L . Let $0 \neq ab \leq \sqrt{(N : I_M)}$ with $a \not\leq \sqrt{(N : I_M)}$. Hence $a^n \not\leq (N : I_M)$ for any $n \in \mathbb{Z}_+$. That is, $a^n I_M \not\leq N$ for any $n \in \mathbb{Z}_+$. We have $0 \neq (ab)^k \leq (N : I_M)$ for some $k \in \mathbb{Z}_+$. That is, $0 \neq a^k b^k I_M \leq N$ where $a^k I_M \not\leq N$ and N is a weakly primary element M . Hence, $(b^k)^l I_M \leq N$ for some $l \in \mathbb{Z}_+$. That is, $b^{kl} I_M \leq N$ where $kl = m \in \mathbb{Z}_+$. Thus $\sqrt{(N : I_M)}$ is a weakly prime element of L . $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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