Let $A$ denote the class of functions which are analytic in the unit disk $D = \{ z : |z| < 1 \}$ and given by the power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Let $C$ be the class of convex functions. In this paper, we give the upper bounds of $|a_3 - \mu a_2^2|$ for all real number $\mu$ and for any $f(z)$ in the family $\mathcal{V} = \{ f(z) : f \in A, \Re (f(z)/g(z)) > 0 \text{ for some } g \in C \}$.

1. Introduction

Let $A$ denote the class of functions which are analytic in the unit disk $D = \{ z : |z| < 1 \}$ and satisfy $f(0) = f'(0) - 1 = 0$. The set of all functions $f \in A$ that are univalent will be denoted by $S$. Let $C, S^*(\beta)$ and $K$ be the classes of convex, starlike of order $\beta$ and close-to-convex functions, respectively. Fekete and Szegö [1] proved that

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
4\mu - 3, & \mu \geq 1, \\
1 + 2 \exp \left( \frac{-2\mu}{1 - \mu} \right), & 0 \leq \mu < 1, \\
3 - 4\mu, & \mu < 0
\end{cases}
\]

holds for any $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and that this inequality is sharp. The coefficient functional $\Lambda_\mu(f) = a_3 - \mu a_2^2$ on $f$ in $A$ plays an important role in function theory. For example, $a_3 - a_2^2 = S_f(0)/6$, where $S_f$ is the Schwarzian derivative. The problem of maximizing the absolute value of the functional $\Lambda_\mu(f)$ is called the Fekete-Szegö problem. In the literature, there exist a large number of results about the Fekete-Szegö problem (see, for instance, [2–11]).

For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, let $u^\beta_\alpha$ denote the class of functions $f$ satisfying $f \in A$ and

\[
\Re \left\{ \frac{az^3 f'(z)}{g(z)} + \frac{zf'(z)}{g(z)} \right\} > 0
\]

for some $g \in S^*(\beta)$. Al-Abbadi and Darus [7] investigated the Fekete-Szegö problem on the class $u^\beta_\alpha$.

Let $C_1(\beta)$ be the class of functions $f$ in $A$ satisfying the inequality

\[
\Re \left\{ \frac{zf'(z)}{\phi(z)} e^{i\beta} \right\} > 0 \quad (|z| < 1, -\frac{\pi}{2} < \beta < \frac{\pi}{2})
\]

for some function $\phi \in C$. In [11], Srivastava et al. studied the Fekete-Szegö problem on the class

\[
C_1 = \bigcup_{\beta} C_1(\beta) \quad \left( -\frac{\pi}{2} < \beta < \frac{\pi}{2} \right)
\]

for $0 \leq \mu \leq 1$ by proving that

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{5}{3} - \frac{9}{4}\mu, & 0 \leq \mu \leq \frac{2}{9}, \\
\frac{2}{3} + \frac{1}{9}\mu, & \frac{2}{9} \leq \mu \leq \frac{2}{3}, \\
\frac{5}{6}, & \frac{2}{3} < \mu \leq 1
\end{cases}
\]

Srivastava et al. held that the inequality (5) was sharp. However, the extremal function given in [11] did not exist in the case of $2/3 < \mu \leq 1$.

In this paper, we solve the Fekete-Szegö problem for the family

\[
\mathcal{V} = \left\{ f(z) : f \in A, \Re \left( \frac{f(z)}{g(z)} \right) > 0 \text{ for some } g \in C \right\}
\]
As a corollary of the main result, we find the sharp upper bounds for absolute value of the Fekete-Szegő functional for the class $v_{\alpha}$ defined by

$$v_{\alpha} = \left\{ f(z) : f \in H, \text{Re}\left(\frac{az^2f''(z)}{g(z)} + \frac{zf'(z)}{g(z)}\right) > 0 \right\}$$

for some $g \in C$.

Clearly, $v_{\alpha}$ is a subclass of $u_{\alpha}^0$. In the case of $\alpha = 0$, we get sharp estimation of the absolute value of the Fekete-Szegő functional for the class $C(0)$ and for all real number $\mu$, which prove that the inequality (5) is not sharp actually when $2/3 < \mu \leq 1$.

2. Main Result

Let $B_0$ be the class of functions $\phi(z)$ that are analytic in $D$ and satisfy $|\phi(z)| \leq |z|$ for all $|z| < 1$. The following two lemmas can be found in [2].

**Lemma 1** (see [2]). If $\phi(z) = \sum_{n=1}^{\infty} a_n z^n$ is in the class $B_0$, then, for any complex number $s$, one has $|a_2 - sa_1^2| \leq 1 + (|s| - 1)|a_1|^2 \leq \max\{1,|s|\}$. The inequality is sharp.

**Lemma 2** (see [2]). If $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is in the class $C$ and $\mu$ is a complex number, then $|c_3 - \mu c_2^2| \leq 1/3 + (|\mu| - 1/3)|c_2| \leq \max\{1/3,|\mu| - 1\}$. The inequality is sharp.

**Theorem 3.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{C}$ and $\mu$ is a real number, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 5 - 9\mu, & \text{when } \mu \leq \frac{1}{6}, \\ 2 + \frac{1}{4\mu}, & \text{when } \frac{1}{2} \leq \mu \leq \frac{2}{5}, \\ 3 - 2\mu + \frac{1}{4(1-\mu)}, & \text{when } \frac{5}{6} \leq \mu \leq \frac{6}{5}, \\ 7\mu - 3, & \text{when } \frac{1}{6} \leq \mu \leq 1, \\ 9\mu - 5, & \text{when } 1 \leq \mu. \end{cases}$$

**Proof.** By definition, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{C}$ if and only if there exists a function $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$ such that $\phi(z) = [f(z) - g(z)]/([f(z) + g(z)] = \sum_{n=1}^{\infty} a_n z^n \in B_0$. A simple computation shows $a_2 = c_2 + 2a_1, a_3 = c_3 + 2(a_1c_2 + a_2 + a_1^2)$. Thus,

$$a_3 - \mu a_2^2 = c_3 - \mu c_2^2 + 2\left[a_2 + (1-2\mu) a_1^2\right] + 2(1-2\mu) a_1 c_2.$$

So, by Lemmas 1 and 2, we have

$$|a_3 - \mu a_2^2| \leq \left|c_3 - \mu c_2^2\right| + 2\left|a_2 + (1-2\mu) a_1^2\right|$$

$$+ 2\left|1 - 2\mu\right||a_1||c_2|$$

$$\leq \left\{ \frac{1}{3} + \left[\left|\mu - 1\right| - \frac{1}{3}\right]\right\}|c_2|^2 + 2\left[1 + (1 - 2\mu)^2 - 1\right]|a_1|^2$$

$$+ 2\left|1 - 2\mu\right||a_1||c_2|.$$

Putting $|a_1| = x$ and $|c_2| = y$, we get from (10) that $|a_3 - \mu a_2^2| \leq F(x, y)$, where

$$F(x, y) = \frac{1}{3} + \left[\left|\mu - 1\right| - \frac{1}{3}\right]y^2$$

$$+ 2\left[1 + (1 - 2\mu)^2 - 1\right]x^2$$

$$+ 2\left|1 - 2\mu\right|x y.$$

Since $|a_1| \leq 1$ and $|c_2| \leq 1$, we will calculate the maximum value of $F(x, y)$ for $(x, y) \in [0, 1] \times [0, 1]$.

**Case 1.** Suppose $\mu \leq 1/2$. Then it follows from (11) that

$$F(x, y) = \frac{7}{3} + \left(\frac{2}{3} - \mu\right)y^2 - 4\mu x^2 + 2(1 - 2\mu) xy.$$

Since

$$F_x(x, y) = -8\mu x + 2(1 - 2\mu) y,$$

$$F_y(x, y) = 2(1 - 2\mu) x + \left(\frac{4}{3} - 2\mu\right) y,$$

$$\left|\frac{\mu}{2} - \frac{2}{3}\right| = \frac{16\mu}{3} - 4 < 0,$$

$F(x, y)$ does not have a local maximum at any point of the open rectangle $(0, 1) \times (0, 1)$. Hence, $F(x, y)$ must attain its maximum at a boundary point. Since $F(0, y) \leq 3 - \mu, F(1, y) \leq F(1, 1) = 5 - 9\mu$ for $\mu \leq 1/2$ and

$$F(x, 0) \leq \begin{cases} \frac{7}{3} - 4\mu, & \text{if } \mu < 0, \\ \frac{7}{3}, & \text{if } 0 \leq \mu < \frac{1}{2}, \end{cases}$$

$$F(x, 1) \leq \begin{cases} F(1, 1) = 5 - 9\mu, & \text{if } \mu < \frac{1}{6}, \\ F\left(\frac{1 - 2\mu}{4\mu}, 1\right) = \frac{1}{4\mu} + 2, & \text{if } \frac{1}{6} \leq \mu \leq \frac{1}{2}, \end{cases}$$

we have

$$|a_3 - \mu a_2^2| \leq F(x, y) \leq \begin{cases} 5 - 9\mu, & \text{when } \mu \leq \frac{1}{6}, \\ 2 + \frac{1}{4\mu}, & \text{when } \frac{1}{6} < \mu \leq \frac{1}{2}. \end{cases}$$
Case 2. Suppose $1/2 < \mu \leq 1$. Then, we get from (11) that
\[
F(x, y) = \frac{7}{3} + \left(\frac{2}{3} - \mu\right)y^2 + 4(\mu - 1)x^2 + 2(2\mu - 1)xy.
\] (16)

Since
\[
F_x(x, y) = 8(\mu - 1)x + 2(2\mu - 1)y,
\]
\[
F_y(x, y) = 2(2\mu - 1)x + \left(\frac{4}{3} - 2\mu\right)y,
\] (17)
\[
8(\mu - 1) \quad 2(2\mu - 1) \\
2(2\mu - 1) \quad 2\mu - \frac{8}{3}
\]
\[
= 4\left(-8\mu^2 + \frac{32}{3}\mu - \frac{11}{3}\right) < 0,
\]
\[
F(0, y) \leq \frac{7}{3} + \left(\frac{2}{3} - \mu\right)y^2 + 4(\mu - 1)x^2 + 2(2\mu - 1)xy.
\]
\[
F(1, y) \leq \frac{7}{3} + \left(\frac{2}{3} - \mu\right)y^2 + 4(\mu - 1)x^2 + 2(2\mu - 1)xy.
\]
\[
F(1, 1) = \frac{7}{3} + \left(\frac{2}{3} - \mu\right)y^2 + 4(\mu - 1)x^2 + 2(2\mu - 1)xy.
\]
\[
F(1, y) \leq F(1, 1) = 7\mu - 3, \quad \text{if} \quad \frac{1}{2} < \mu \leq 1,
\]
we get
\[
\left|a_3 - \mu a_2^2\right| \leq F(x, y)
\]
\[
\leq \begin{cases} 
3 - 2\mu + \frac{1}{4}\left(1 - \mu\right), & \text{when } \frac{1}{2} < \mu \leq \frac{5}{6}, \\
7\mu - 3, & \text{when } \frac{5}{6} < \mu \leq 1.
\end{cases}
\] (19)

Case 3. Suppose $\mu > 1$. Then, (11) gives
\[
F(x, y) = \frac{7}{3} + \left(\mu - \frac{4}{3}\right)y^2 + 4(\mu - 1)x^2 + 2(2\mu - 1)xy.
\] (20)

Since
\[
F_x(x, y) = 8(\mu - 1)x + 2(2\mu - 1)y,
\]
\[
F_y(x, y) = 2(2\mu - 1)x + \left(2\mu - \frac{8}{3}\right)y,
\] (21)
\[
8(\mu - 1) \quad 2(2\mu - 1) \\
2(2\mu - 1) \quad 2\mu - \frac{8}{3}
\]
\[
= 4\left(13 - 16\mu\right) < 0,
\]
\[
F(x, y) \text{ must attain its maximum at a boundary point of the rectangle } [0, 1] \times [0, 1].
\]

Since
\[
F(x, 0) \leq 4\mu - \frac{5}{3}, \quad \text{if } \mu \geq 1,
\]
\[
F(x, 1) \leq F(1, 1) = 9\mu - 5, \quad \text{if } \mu \geq 1,
\]
\[
F(0, y) \leq \begin{cases} 
\frac{7}{3}, & \text{if } \mu \leq \frac{4}{3}, \\
1 + \mu, & \text{if } \mu > \frac{4}{3},
\end{cases}
\] (22)
\[
F(1, y) \leq F(1, 1) = 9\mu - 5, \quad \text{if } \mu \geq 1,
\]
\[
\text{it follows that}
\]
\[
\left|a_3 - \mu a_2^2\right| \leq F(x, y) \leq 9\mu - 5, \quad \text{when } \mu < 1.
\] (23)

Combining (15), (19) with (23), we get (8). Since inequalities in Lemmas 1 and 2 are sharp, it follows that inequality (8) is also sharp. The proof is completed.

Since $f \in \nu_a$ if and only if $F(z) = ax^2 f''(z) + zf'(z) \in \mathcal{V}$, by a simple calculation, we have the following.

**Corollary 4.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \nu_a$, then
\[
\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 
\frac{5}{3} \left(1 + 2\alpha\right) - \frac{9\mu}{4(1 + \alpha)^2}, & \mu \leq 2(1 + \alpha)^2 \left(1 + \alpha\right), \\
\frac{2}{3} \left(1 + 2\alpha\right) + \frac{9\mu}{4(1 + \alpha)^2} \mu, & \mu \geq \frac{2}{3} \left(1 + 2\alpha\right) + \frac{9\mu}{4(1 + \alpha)^2} \mu, \\
\frac{1}{3} \mu^2 - \frac{2(1 + \alpha)^2}{(1 + \alpha)^2}, & \mu \geq \frac{1}{3} \mu^2 - \frac{2(1 + \alpha)^2}{(1 + \alpha)^2},
\end{cases}
\] (24)

Letting $\alpha = 0$ in (24), we get the following.
Corollary 5. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_1(0) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{5}{3} - \frac{9}{4\mu}, & \text{when } \mu \leq \frac{2}{9}, \\
\frac{2}{3} + \frac{1}{9\mu}, & \text{when } \frac{2}{9} \leq \mu \leq \frac{2}{3}, \\
1 - \frac{1}{2\mu} + \frac{1}{12 - 9\mu}, & \text{when } \frac{2}{3} < \mu \leq \frac{10}{9}, \\
\frac{7}{4\mu} - 1, & \text{when } \frac{10}{9} < \mu \leq \frac{4}{3}, \\
\frac{9}{4\mu} - \frac{5}{3}, & \text{when } \frac{4}{3} < \mu \leq \mu. 
\end{cases}
\] (25)

Remark 6. In [11], Srivastava et al. gave a function \( f(z) = z + \sum_{n=3}^{\infty} a_n z^n \in A \) satisfying \( \phi(z) = \frac{f(z) - g(z)}{f(z) + g(z)} = \sum_{n=1}^{\infty} \alpha_n z^n \in B_0 \), where \( g(z) = z + z^2 + z^3 + \cdots \in C \), \( \alpha_2 = 1 - \alpha_1^2 \) and

\[
\alpha_1 = \frac{(2 - 3\lambda) \pm i \sqrt{6\mu - 4}}{6\mu}. \] (26)

Srivastava held that \( f \in C_1(0) \) and \( |a_3 - \mu a_2^2| = 5/6 \) when \( 2/3 \leq \mu \leq 1 \). But \( \phi \in B_0 \) implies that \( |\alpha_2| \leq 1 - |\alpha_1^2| \). So \( \phi(z) \) satisfying the above conditions does not exist.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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