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Research Article

On the Fekete-Szegö Problem for a Class of Analytic Functions

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Let $\mathscr{A}$ denote the class of functions which are analytic in the unit disk $D = \{z : |z| < 1\}$ and given by the power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Let C be the class of convex functions. In this paper, we give the upper bounds of $|a_3 - \mu a_2^2|$ for all real number μ and for any $f(z)$ in the family $\mathscr{V} = \{f(z) : f \in \mathscr{A}, \operatorname{Re}(f(z)/g(z)) > 0 \text{ for some } g \in C\}$.

1. Introduction

Let $\mathscr{A}$ denote the class of functions which are analytic in the unit disk $D = \{z : |z| < 1\}$ and satisfy $f(0) = f'(0) - 1 = 0$. The set of all functions $f \in \mathscr{A}$ that are univalent will be denoted by S. Let $C, S^*(\beta)$ and K be the classes of convex, starlike of order β and close-to-convex functions, respectively. Fekete and Szegö [1] proved that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 4\mu - 3, & \mu \ge 1, \\ 1 + 2\exp\left[\dfrac{-2\mu}{(1-\mu)}\right], & 0 \le \mu < 1, \\ 3 - 4\mu, & \mu < 0 \end{cases} \tag{1}$$

holds for any $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and that this inequality is sharp. The coefficient functional $\Lambda_\mu(f) = a_3 - \mu a_2^2$ on f in $\mathscr{A}$ plays an important role in function theory. For example, $a_3 - a_2^2 = S_f(0)/6$, where S_f is the Schwarzian derivative. The problem of maximizing the absolute value of the functional $\Lambda_\mu(f)$ is called the Fekete-Szegö problem. In the literature, there exist a large number of results about the Fekete-Szegö problem (see, for instance, [2–11]).

For $0 \le \alpha < 1$ and $0 \le \beta < 1$, let u_α^β denote the class of functions f satisfying $f \in \mathscr{A}$ and

$$\operatorname{Re}\left\{\frac{\alpha z^2 f''(z)}{g(z)} + \frac{z f'(z)}{g(z)}\right\} > 0 \tag{2}$$

for some $g \in S^*(\beta)$. Al-Abbadi and Darus [7] investigated the Fekete-Szegö problem on the class u_α^β.

Let $C_1(\beta)$ be the class of functions f in $\mathscr{A}$ satisfying the inequality

$$\operatorname{Re}\left\{\frac{z f'(z)}{\phi(z)} e^{i\beta}\right\} > 0 \quad \left(|z| < 1, -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right) \tag{3}$$

for some function $\phi \in C$. In [11], Srivastara et al. studied the Fekete-Szegö problem on the class

$$C_1 = \bigcup_\beta C_1(\beta) \quad \left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right) \tag{4}$$

for $0 \le \mu \le 1$ by proving that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{5}{3} - \dfrac{9}{4}\mu, & 0 \le \mu \le \dfrac{2}{9}, \\ \dfrac{2}{3} + \dfrac{1}{9\mu}, & \dfrac{2}{9} \le \mu \le \dfrac{2}{3}, \\ \dfrac{5}{6}, & \dfrac{2}{3} < \mu \le 1. \end{cases} \tag{5}$$

Srivastara et al. held that the inequality (5) was sharp. However, the extremal function given in [11] did not exist in the case of $2/3 < \mu \le 1$.

In this paper, we solve the Fekete-Szegö problem for the family

$$\mathscr{V} = \left\{f(z) : f \in \mathscr{A}, \operatorname{Re}\frac{f(z)}{g(z)} > 0 \text{ for some } g \in C\right\}. \tag{6}$$

As a corollary of the main result, we find the sharp upper bounds for absolute value of the Fekete-Szegö functional for the class v_α defined by

$$v_\alpha = \left\{ f(z) : f \in \mathscr{A}, \operatorname{Re}\left\{ \frac{\alpha z^2 f''(z)}{g(z)} + \frac{z f'(z)}{g(z)} \right\} > 0 \right. \\ \left. \text{for some } g \in C \right\}. \tag{7}$$

Clearly, v_α is a subclass of u_α^0. In the case of $\alpha = 0$, we get sharp estimation of the absolute value of the Fekete-Szegö functional for the class $C_1(0)$ and for all real number μ, which prove that the inequality (5) is not sharp actually when $2/3 < \mu \le 1$.

2. Main Result

Let B_0 be the class of functions $\phi(z)$ that are analytic in D and satisfy $|\phi(z)| \le |z|$ for all $|z| < 1$. The following two lemmas can be found in [2].

Lemma 1 (see [2]). *If $\phi(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ is in the class B_0, then, for any complex number s, one has $|\alpha_2 - s\alpha_1^2| \le 1 + (|s| - 1)|\alpha_1^2| \le \max\{1, |s|\}$. The inequality is sharp.*

Lemma 2 (see [2]). *If $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is in the class C and μ is a complex number, then $|c_3 - \mu c_2^2| \le 1/3 + (|\mu - 1| - 1/3)|c_2^2| \le \max\{1/3, |\mu - 1|\}$. The inequality is sharp.*

Theorem 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathscr{V}$ and μ is a real number, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 5 - 9\mu, & \text{when } \mu \le \dfrac{1}{6}, \\ 2 + \dfrac{1}{4\mu}, & \text{when } \dfrac{1}{6} \le \mu \le \dfrac{1}{2}, \\ 3 - 2\mu + \dfrac{1}{4(1-\mu)}, & \text{when } \dfrac{1}{2} < \mu \le \dfrac{5}{6}, \\ 7\mu - 3, & \text{when } \dfrac{5}{6} < \mu \le 1, \\ 9\mu - 5, & \text{when } 1 \le \mu. \end{cases} \tag{8}$$

Proof. By definition, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathscr{V}$ if and only if there exists a function $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$ such that $\phi(z) = [f(z) - g(z)]/[f(z) + g(z)] = \sum_{n=1}^{\infty} \alpha_n z^n \in B_0$. A simple computation shows $a_2 = c_2 + 2\alpha_1, a_3 = c_3 + 2(\alpha_1 c_2 + \alpha_2 + \alpha_1^2)$. Thus,

$$a_3 - \mu a_2^2 = c_3 - \mu c_2^2 + 2\left[\alpha_2 + (1 - 2\mu)\alpha_1^2\right] + 2(1 - 2\mu)\alpha_1 c_2. \tag{9}$$

So, by Lemmas 1 and 2, we have

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \left|c_3 - \mu c_2^2\right| + 2\left|\alpha_2 + (1 - 2\mu)\alpha_1^2\right| \\ &\qquad + 2\left|1 - 2\mu\right\|\alpha_1\|c_2\| \\ &\quad \le \frac{1}{3} + \left[|\mu - 1| - \frac{1}{3}\right]|c_2|^2 + 2\left[1 + (|1 - 2\mu| - 1)|\alpha_1|^2\right] \\ &\qquad + 2\left|1 - 2\mu\right\|\alpha_1\|c_2\|. \end{aligned} \tag{10}$$

Putting $|\alpha_1| = x$ and $|c_2| = y$, we get from (10) that $|a_3 - \mu a_2^2| \le F(x, y)$, where

$$\begin{aligned} F(x, y) = \frac{1}{3} &+ \left[|\mu - 1| - \frac{1}{3}\right] y^2 \\ &+ 2\left[1 + (|1 - 2\mu| - 1)x^2\right] \\ &+ 2|1 - 2\mu| xy. \end{aligned} \tag{11}$$

Since $|\alpha_1| \le 1$ and $|c_2| \le 1$, we will calculate the maximum value of $F(x, y)$ for $(x, y) \in [0, 1] \times [0, 1]$.

Case 1. Suppose $\mu \le 1/2$. Then it follows from (11) that

$$F(x, y) = \frac{7}{3} + \left(\frac{2}{3} - \mu\right) y^2 - 4\mu x^2 + 2(1 - 2\mu) xy. \tag{12}$$

Since

$$\begin{gathered} F_x(x, y) = -8\mu x + 2(1 - 2\mu) y, \\ F_y(x, y) = 2(1 - 2\mu) x + \left(\frac{4}{3} - 2\mu\right) y, \\ \begin{vmatrix} -8\mu & 2(1 - 2\mu) \\ 2(1 - 2\mu) & \dfrac{4}{3} - 2\mu \end{vmatrix} = \frac{16}{3}\mu - 4 < 0, \end{gathered} \tag{13}$$

$F(x, y)$ does not have a local maximum at any point of the open rectangle $(0, 1) \times (0, 1)$. Hence, $F(x, y)$ must attain its maximum at a boundary point. Since $F(0, y) \le 3 - \mu, F(1, y) \le F(1, 1) = 5 - 9\mu$ for $\mu \le 1/2$ and

$$\begin{gathered} F(x, 0) \le \begin{cases} \dfrac{7}{3} - 4\mu, & \text{if } \mu < 0, \\ \dfrac{7}{3}, & \text{if } 0 \le \mu \le \dfrac{1}{2}, \end{cases} \\ F(x, 1) \le \begin{cases} F(1, 1) = 5 - 9\mu, & \text{if } \mu < \dfrac{1}{6}, \\ F\left(\dfrac{1 - 2\mu}{4\mu}, 1\right) = \dfrac{1}{4\mu} + 2, & \text{if } \dfrac{1}{6} \le \mu \le \dfrac{1}{2}, \end{cases} \end{gathered} \tag{14}$$

we have

$$\left|a_3 - \mu a_2^2\right| \le F(x, y) \le \begin{cases} 5 - 9\mu, & \text{when } \mu \le \dfrac{1}{6}, \\ 2 + \dfrac{1}{4\mu}, & \text{when } \dfrac{1}{6} < \mu \le \dfrac{1}{2}. \end{cases} \tag{15}$$

Case 2. Suppose $1/2 < \mu \le 1$. Then, we get from (11) that

$$F(x,y) = \frac{7}{3} + \left(\frac{2}{3} - \mu\right) y^2 + 4(\mu - 1) x^2 + 2(2\mu - 1) xy. \tag{16}$$

Since

$$\begin{gathered} F_x(x,y) = 8(\mu - 1) x + 2(2\mu - 1) y, \\ F_y(x,y) = 2(2\mu - 1) x + \left(\frac{4}{3} - 2\mu\right) y, \\ \begin{vmatrix} 8(\mu - 1) & 2(2\mu - 1) \\ 2(2\mu - 1) & \frac{4}{3} - 2\mu \end{vmatrix} = 4\left(-8\mu^2 + \frac{32}{3}\mu - \frac{11}{3}\right) < 0, \end{gathered} \tag{17}$$

$F(x,y)$ must attain its maximum at a boundary point of the rectangle $[0,1] \times [0,1]$.

Since

$$\begin{gathered} F(x,0) \le \frac{7}{3}, \quad \text{if } \frac{1}{2} < \mu \le 1, \\ F(x,1) \le \begin{cases} F\left(\frac{1-2\mu}{4(\mu-1)}, 1\right) \\ \quad = 3 - 2\mu + \frac{1}{4(1-\mu)}, & \text{if } \frac{1}{2} < \mu \le \frac{5}{6}, \\ F(1,1) = 7\mu - 3, & \text{if } \frac{5}{6} < \mu \le 1, \end{cases} \\ F(0,y) \le \begin{cases} 3 - \mu, & \text{if } \frac{1}{2} < \mu \le \frac{2}{3}, \\ \frac{7}{3}, & \text{if } \frac{2}{3} < \mu \le 1, \end{cases} \\ F(1,y) \le F(1,1) = 7\mu - 3, \quad \text{if } \frac{1}{2} < \mu \le 1, \end{gathered} \tag{18}$$

we get

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\le F(x,y) \\ &\le \begin{cases} 3 - 2\mu + \frac{1}{4(1-\mu)}, & \text{when } \frac{1}{2} < \mu \le \frac{5}{6}, \\ 7\mu - 3, & \text{when } \frac{5}{6} < \mu \le 1. \end{cases} \end{aligned} \tag{19}$$

Case 3. Suppose $\mu > 1$. Then, (11) gives

$$F(x,y) = \frac{7}{3} + \left(\mu - \frac{4}{3}\right) y^2 + 4(\mu - 1) x^2 + 2(2\mu - 1) xy. \tag{20}$$

Since

$$\begin{gathered} F_x(x,y) = 8(\mu - 1) x + 2(2\mu - 1) y, \\ F_y(x,y) = 2(2\mu - 1) x + \left(2\mu - \frac{8}{3}\right) y, \\ \begin{vmatrix} 8(\mu - 1) & 2(2\mu - 1) \\ 2(2\mu - 1) & 2\mu - \frac{8}{3} \end{vmatrix} = \frac{4}{3}(13 - 16\mu) < 0, \end{gathered} \tag{21}$$

$F(x,y)$ must attain its maximum at a boundary point of the rectangle $[0,1] \times [0,1]$. Since

$$\begin{gathered} F(x,0) \le 4\mu - \frac{5}{3}, \quad \text{if } \mu \ge 1, \\ F(x,1) \le F(1,1) = 9\mu - 5, \quad \text{if } \mu \ge 1, \\ F(0,y) \le \begin{cases} \frac{7}{3}, & \text{if } 1 < \mu \le \frac{4}{3}, \\ 1 + \mu, & \text{if } \mu > \frac{4}{3}, \end{cases} \\ F(1,y) \le F(1,1) = 9\mu - 5, \quad \text{if } \mu \ge 1, \end{gathered} \tag{22}$$

it follows that

$$\left|a_3 - \mu a_2^2\right| \le F(x,y) \le 9\mu - 5, \quad \text{when } 1 < \mu. \tag{23}$$

Combining (15), (19) with (23), we get (8). Since inequalities in Lemmas 1 and 2 are sharp, it follows that inequality (8) is also sharp. The proof is completed. □

Since $f \in v_\alpha$ if and only if $F(z) = \alpha z^2 f''(z) + z f'(z) \in \mathscr{V}$, by a simple calculation, we have the following.

Corollary 4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in v_\alpha$, then*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\le \begin{cases} \frac{5}{3(1+2\alpha)} - \frac{9\mu}{4(1+\alpha)^2}, \\ \qquad \mu \le \frac{2(1+\alpha)^2}{9(1+2\alpha)}, \\ \frac{2}{3(1+2\alpha)} + \frac{(1+\alpha)^2}{9(1+2\alpha)^2\mu}, \\ \qquad \frac{2(1+\alpha)^2}{9(1+2\alpha)} \le \mu \le \frac{2(1+\alpha)^2}{3(1+2\alpha)}, \\ \frac{1}{1+2\alpha} - \frac{\mu}{2(1+\alpha)^2} \\ \quad + \frac{(1+\alpha)^2}{12(1+2\alpha)(1+\alpha)^2 - 9(1+2\alpha)^2\mu}, \\ \qquad \frac{2(1+\alpha)^2}{3(1+2\alpha)} \le \mu \le \frac{10(1+\alpha)^2}{9(1+2\alpha)}, \\ \frac{7}{4(1+\alpha)^2}\mu - \frac{1}{1+2\alpha}, \\ \qquad \frac{10(1+\alpha)^2}{9(1+2\alpha)} \le \mu \le \frac{4(1+\alpha)^2}{3(1+2\alpha)}, \\ \frac{9}{4(1+\alpha)^2}\mu - \frac{5}{3(1+2\alpha)}, \\ \qquad \frac{4(1+\alpha)^2}{3(1+2\alpha)} \le \mu. \end{cases} \end{aligned} \tag{24}$$

Letting $\alpha = 0$ in (24), we get the following.

Corollary 5. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_1(0)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{5}{3} - \frac{9}{4}\mu, & \text{when } \mu \leq \frac{2}{9}, \\ \frac{2}{3} + \frac{1}{9\mu}, & \text{when } \frac{2}{9} \leq \mu \leq \frac{2}{3}, \\ 1 - \frac{1}{2}\mu + \frac{1}{12 - 9\mu}, & \text{when } \frac{2}{3} < \mu \leq \frac{10}{9}, \\ \frac{7}{4}\mu - 1, & \text{when } \frac{10}{9} < \mu \leq \frac{4}{3}, \\ \frac{9}{4}\mu - \frac{5}{3}, & \text{when } \frac{4}{3} \leq \mu. \end{cases} \tag{25}$$

Remark 6. In [11], Srivastava et al. gave a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}$ satisfying $\phi(z) = [f(z) - g(z)]/[f(z) + g(z)] = \sum_{n=1}^{\infty} \alpha_n z^n \in B_0$, where $g(z) = z + z^2 + z^3 + \cdots \in C$, $\alpha_2 = 1 - \alpha_1^2$ and

$$\alpha_1 = \frac{(2 - 3\lambda) \pm i\sqrt{6\mu - 4}}{6\mu}. \tag{26}$$

Srivastava held that $f \in C_1(0)$ and $|a_3 - \mu a_2^2| = 5/6$ when $2/3 \leq \mu \leq 1$. But $\phi \in B_0$ implies that $|\alpha_2| \leq 1 - |\alpha_1^2|$. So $\phi(z)$ satisfying the above conditions does not exist.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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