Research Article

A New Class of Meromorphic Functions Involving the Polylogarithm Function

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We introduce a new operator $D_c f(z)$ associated with polylogarithm function. By making use of the new operator, we define a certain new class of meromorphic functions and discussed some important properties of it.

1. Introduction

Historically, the classical polylogarithm function was invented in 1696, by Leibniz and Bernoulli, as mentioned in [1]. For $|z| < 1$ and c a natural number with $c \geq 2$, the polylogarithm function (which is also known as Jonquier's function) is defined by the absolutely convergent series:

$$\text{Li}_c(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^c}. \quad (1)$$

Later on, many mathematicians studied the polylogarithm function such as Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan, and many others [2], where they discovered many functional identities by using polylogarithm function. However, the work employing polylogarithm has been stopped many decades later. During the past four decades, the work using polylogarithm has again been intensified vividly due to its importance in many fields of mathematics, such as complex analysis, algebra, geometry, topology, and mathematical physics (quantum field theory) [3–5]. In 1996, Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm [6]. Recently, Al-Shaqsi and Darus generalized Ruscheweyh and Salagean operators, using polylogarithm functions on class $\mathcal{A}$ of analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. By making use of the generalized operator they introduced certain new subclasses of $\mathcal{A}$ and investigated many related

properties [7]. A year later, same authors again employed the n th order polylogarithm function to define a multiplier transformation on the class $\mathcal{A}$ in $\mathbb{U}$ [8].

To the best of our knowledge, no research work has discussed the polylogarithm function in conjunction with meromorphic functions. Thus, in this present paper, we redefine the polylogarithm function to be on meromorphic type.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \quad (2)$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}. \quad (3)$$

A function $f(z)$ in Σ is said to be meromorphically starlike of order δ if and only if

$$\Re \left\{ -\frac{zf'(z)}{f(z)} \right\} > \delta; \quad (z \in \mathbb{U}^*), \quad (4)$$

for some δ ($0 \leq \delta < 1$). We denote by $\Sigma^*(\delta)$ the class of all meromorphically starlike order δ . Furthermore, a function $f(z)$ in Σ is said to be meromorphically convex of order δ if and only if

$$\Re \left\{ -1 - \frac{zf''(z)}{f'(z)} \right\} > \delta; \quad (z \in \mathbb{U}^*), \quad (5)$$

for some δ ($0 \leq \delta < 1$). We denote by $\Sigma_K(\delta)$ the class of all meromorphically convex order δ . For functions $f \in \Sigma$ given by (2) and $g \in \Sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k, \quad (6)$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k. \quad (7)$$

Let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k; \quad a_k \geq 0, \quad (8)$$

which are analytic and univalent in $\mathbb{U}^*$.

Liu and Srivastava [9] defined a function $h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by multiplying the well-known generalized hypergeometric function ${}_qF_s$ with z^{-p} as follows:

$$\begin{aligned} h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ = z^{-p} {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z), \end{aligned} \quad (9)$$

where $\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s$ are complex parameters and $q \leq s+1$, $p \in \mathbb{N}$.

Analogous to Liu and Srivastava work [9] and corresponding to a function $\Phi_c(z)$ given by

$$\Phi_c(z) = z^{-2} \text{Li}_c(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} z^k, \quad (10)$$

we consider a linear operator $\Omega_c f(z) : \Sigma \rightarrow \Sigma$ which is defined by the following Hadamard product (or convolution):

$$\begin{aligned} \Omega_c f(z) &= \Phi_c(z) * f(z) \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} a_k z^k. \end{aligned} \quad (11)$$

Next, we define the linear operator $D_c f(z) : \Sigma \rightarrow \Sigma$ as follows:

$$\begin{aligned} D_c f(z) &= \left\{ \Omega_c f(z) - \frac{1}{2^c} a_0 \right\} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^c} a_k z^k. \end{aligned} \quad (12)$$

Now, by making use of the operator $D_c f(z)$, we define a new subclass of functions in Σ_p as follows.

Definition 1. For $\alpha > 1$ and $0 < \beta \leq 1$, let $N_c(\alpha, \beta)$ denote a subclass of Σ consisting functions of form (2) satisfying the condition that

$$\Re \left\{ z D_c f(z) - \alpha z^2 (D_c f(z))' \right\} > \beta; \quad (z \in \mathbb{U}^*), \quad (13)$$

where $D_c f(z)$ is given by (12). Furthermore, we say that a function $f \in N_{c,p}(\alpha, \beta)$, whenever $f(z)$ is of form (8).

In the following sections, we investigate coefficient inequalities, extreme points, radii of starlikeness and convexity of order δ , and integral means inequalities for the new class $N_{c,p}(\alpha, \beta)$.

2. Coefficient Inequalities

The following theorem gives a necessary and sufficient condition for a function f to be in the class $N_{c,p}(\alpha, \beta)$.

Theorem 2. Let $f \in \Sigma_p$ given by (8). Then $f \in N_{c,p}(\alpha, \beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{(\alpha k - 1)}{(k+2)^c} a_k \leq 1 + \alpha - \beta. \quad (14)$$

Proof. Suppose that $f \in N_{c,p}(\alpha, \beta)$. Then

$$\begin{aligned} \Re \left\{ z \left(\frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^c} a_k z^k \right) \right. \\ \left. - \alpha z^2 \left(\frac{-1}{z^2} + \sum_{k=1}^{\infty} \frac{k}{(k+2)^c} a_k z^{k-1} \right) \right\} \\ = \Re \left\{ 1 + \alpha - \sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^c} a_k z^{k+1} \right\} > \beta. \end{aligned} \quad (15)$$

If we choose z to be real and letting $z \rightarrow 1$, we get

$$1 + \alpha - \sum_{k=1}^{\infty} \frac{(\alpha k - 1)}{(k+2)^c} a_k > \beta, \quad (16)$$

which is equivalent to (14). Conversely, let us suppose that assertion (14) holds true.

Then we can write

$$\begin{aligned} & \left| z D_c f(z) - \alpha z^2 (D_c f(z))' \right| \\ &= \left| 1 + \alpha - \sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^c} a_k z^{k+1} \right| \\ &\geq 1 + \alpha - \left| \sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^c} a_k z^{k+1} \right| \\ &> \beta. \end{aligned} \quad (17)$$

Hence, $f \in N_{c,p}(\alpha, \beta)$. Finally, we note that inequality (14) is sharp; the extremal function is

$$f(z) = \frac{1}{z} + \frac{(1 + \alpha - \beta)(3)^c}{(\alpha - 1)} z; \quad \alpha > 1. \quad (18)$$

□

3. Extreme Points

In this section, we determine the extreme points for functions in the class $N_{c,p}(\alpha, \beta)$.

Theorem 3. Let $f_0(z) = 1/z$ and

$$f_k(z) = \frac{1}{z} + \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} z^k; \quad (k = 1, 2, \dots). \quad (19)$$

Then, $f \in N_{c,p}(\alpha, \beta)$ if and only if it can be represented in the form

$$f(z) = \sum_{k=0}^{\infty} \lambda_k f_k(z); \quad \left(\lambda_k \geq 0, \sum_{k=0}^{\infty} \lambda_k = 1 \right). \quad (20)$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} \lambda_k f_k(z)$, $\lambda_k \geq 0$, $k = 0, 1, 2, \dots$, $\sum_{k=0}^{\infty} \lambda_k = 1$. Then, we have

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \lambda_k f_k(z) \\ &= \lambda_0 f_0(z) + \sum_{k=1}^{\infty} \lambda_k f_k(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \left\{ \lambda_k \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} \right\} z^k. \end{aligned} \quad (21)$$

Therefore,

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{(\alpha k - 1)(1 + \alpha - \beta)(k + 2)^c}{(k + 2)^c(\alpha k - 1)} \lambda_k \\ &= (1 + \alpha - \beta) \sum_{k=1}^{\infty} \lambda_k \\ &= (1 + \alpha - \beta)(1 - \lambda_0) \\ &\leq (1 + \alpha - \beta). \end{aligned} \quad (22)$$

Hence, by Theorem 2, $f \in N_{c,p}(\alpha, \beta)$.

Conversely, we suppose that $f \in N_{c,p}(\alpha, \beta)$, since

$$a_k \leq \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)}; \quad k \geq 1. \quad (23)$$

We set

$$\lambda_k = \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c} a_k; \quad k \geq 1, \quad (24)$$

and $\lambda_0 = 1 - \sum_{k=1}^{\infty} \lambda_k$. Then we have

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \lambda_k f_k(z) \\ &= \lambda_0 f_0(z) + \sum_{k=1}^{\infty} \lambda_k f_k(z). \end{aligned} \quad (25)$$

The results follow. $\square$

4. Radii of Meromorphic Starlikeness and Meromorphic Convexity

Theorem 4. Let $f \in N_{c,p}(\alpha, \beta)$. Then f is meromorphically starlike of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_1$, where

$$r_1 = \inf_k \left[\left(\frac{1 - \delta}{k + 2 - \delta} \right) \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c} \right]; \quad (k \geq 1). \quad (26)$$

The result is sharp for the extremal function $f(z)$ given by (19).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < 1 - \delta, \quad (27)$$

which easily follows from (4), since

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} + 1 \right| &= \left| \frac{\sum_{k=1}^{\infty} (k + 1) a_k z^{k+1}}{1 + \sum_{k=1}^{\infty} a_k z^{k+1}} \right| \\ &\leq \frac{\sum_{k=1}^{\infty} (k + 1) |a_k| |z|^{k+1}}{1 - \sum_{k=1}^{\infty} |a_k| |z|^{k+1}}. \end{aligned} \quad (28)$$

Considering that $\sum_{k=1}^{\infty} |a_k| < 1$, the above expression is less than $1 - \delta$ if and only if

$$\begin{aligned} &\sum_{k=1}^{\infty} (k + 1) |a_k| |z|^{k+1} \\ &\leq (1 - \delta) \left(1 - \sum_{k=1}^{\infty} |a_k| |z|^{k+1} \right), \end{aligned} \quad (29)$$

or

$$\sum_{k=1}^{\infty} \frac{k + 2 - \delta}{1 - \delta} |a_k| |z|^{k+1} \leq 1. \quad (30)$$

By Theorem 2, we have

$$\sum_{k=1}^{\infty} \frac{(\alpha k - 1)}{(k + 2)^c (1 + \alpha - \beta)} a_k \leq 1; \quad (31)$$

then, (27) holds true if

$$\frac{k + 2 - \delta}{1 - \delta} |z|^{k+1} < \frac{(\alpha k - 1)}{(k + 2)^c (1 + \alpha - \beta)}; \quad k \geq 1, \quad (32)$$

which is equivalent to

$$|z|^{k+1} < \frac{1 - \delta}{k + 2 - \delta} \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c}, \quad (33)$$

which yields the starlikeness of the family and completes the proof. $\square$

Theorem 5. Let $f \in N_{c,p}(\alpha, \beta)$. Then f is meromorphically convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_2$, where

$$r_2 = \inf_k \left[\left(\frac{1 - \delta}{k + 2 - \delta} \right) \frac{(\alpha k - 1)}{k(1 + \alpha - \beta)(k + 2)^c} \right]; \quad (k \geq 1). \quad (34)$$

The result is sharp for the extremal function $f(z)$ given by

$$f_k(z) = \frac{1}{z} + \frac{k(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} z^k; \quad (k = 1, 2, \dots). \quad (35)$$

Proof. By using the technique employed in the proof of Theorem 4, we can show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| < 1 - \delta, \quad (36)$$

for $|z| < r_2$, and prove that the assertion of the theorem is true. $\square$

5. Integral Means Inequalities

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}^*$. Then the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}^*$, written by

$$f(z) < g(z); \quad (z \in \mathbb{U}^*), \quad (37)$$

if there exists a function $w(z)$ which is analytic in $\mathbb{U}^*$ with $w(0) = 0$ and $|w(z)| < 1$ with $z \in \mathbb{U}^*$ and such that $f(z) = g(w(z))$ for $z \in \mathbb{U}^*$. From the definition of the subordinations, it is easy to show that subordination (37) implies that

$$f(0) = g(0), \quad f(\mathbb{U}^*) \subset g(\mathbb{U}^*). \quad (38)$$

Theorem 6 (see [10]). If f and g are any two functions, analytic in $\mathbb{U}$, with $f < g$, then, for $\eta > 0$ and $z = re^{i\theta}$; ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |g(z)|^\eta d\theta. \quad (39)$$

Theorem 7. Suppose $f \in N_{c,p}(\alpha, \beta)$ and f_k is defined by

$$f_k(z) = \frac{1}{z} + \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} z^k; \quad (k = 1, 2, \dots). \quad (40)$$

If there exists an analytic function $w(z)$ such that

$$[w(z)]^{k+1} = \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c} \sum_{k=1}^{\infty} a_k z^{k+1}, \quad (41)$$

then, for $z = re^{i\theta}$ and ($0 < r < 1$),

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_k(re^{i\theta})|^\eta d\theta; \quad (\eta > 0). \quad (42)$$

Proof. We need to show that

$$\begin{aligned} & \int_0^{2\pi} \left| 1 + \sum_{k=1}^{\infty} a_k z^{k+1} \right|^\eta d\theta \\ & \leq \int_0^{2\pi} \left| 1 + \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} z^{k+1} \right|^\eta d\theta. \end{aligned} \quad (43)$$

From Theorem 6, it suffices to prove that

$$1 + \sum_{k=1}^{\infty} a_k z^{k+1} < 1 + \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} z^{k+1}. \quad (44)$$

If we set w such that

$$[w(z)]^{k+1} = \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c} \sum_{k=1}^{\infty} a_k z^{k+1}, \quad (45)$$

we get

$$1 + \sum_{k=1}^{\infty} a_k z^{k+1} = 1 + \frac{(1 + \alpha - \beta)(k + 2)^c}{(\alpha k - 1)} [w(z)]^{k+1}. \quad (46)$$

Clearly, $w(0) = 0$; then from Theorem 2 we can write

$$\begin{aligned} & |w(z)|^{k+1} \\ & = \left| \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c} \sum_{k=1}^{\infty} a_k z^{k+1} \right| \\ & \leq \frac{(\alpha k - 1)}{(1 + \alpha - \beta)(k + 2)^c} \sum_{k=1}^{\infty} |a_k| |z|^{k+1} \\ & < |z| < 1. \end{aligned} \quad (47)$$

That completes the proof. $\square$

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

Both authors read and approved the final paper.

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