Research Article

A Class of Degenerate Nonlinear Elliptic Equations in Weighted Sobolev Space

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We prove the existence of a weak solution for the degenerate nonlinear elliptic Dirichlet boundary-value problem $Lu - \mu u g_1 + h(u, \nabla u) g_2 = f$ in Ω , $u = 0$ on $\partial\Omega$, in a suitable weighted Sobolev space, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and h is a continuous bounded nonlinearity.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$. Let L be an operator in divergence form:

$$Lu(x) = - \sum_{i,j=1}^n D_j (a_{ij}(x) D_i u(x)) \quad \text{with } D_j = \frac{\partial}{\partial x_j}, \quad (1)$$

with coefficients $a_{ij}/\omega \in L^\infty(\Omega)$ which are symmetric and satisfy the degenerate ellipticity condition:

$$\lambda |\xi|^2 \omega(x) \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \omega(x), \quad \text{a.e. } x \in \Omega, \quad (2)$$

for all $\xi \in \mathbb{R}^n$, and ω is an A_2 -weight ($\lambda > 0, \Lambda > 0$). Let $\mu \in \mathbb{R}$ and $f/\omega \in L^2(\Omega, \omega)$ and let h be a real valued continuous function defined on $\mathbb{R} \times \mathbb{R}^n$. In this paper, we study the existence of weak solution of the BVP:

$$\begin{aligned} Lu - \mu u g_1 + h(u, \nabla u) g_2 &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (3)$$

under suitable hypotheses on the functions g_1 , g_2 , and h . The present work is inspired by a semilinear problem in bounded domain given in the book by Zeidler [1]. In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic PDEs. For degenerate

problems with various types of singularities in the coefficients it is natural to look for solutions in weighted Sobolev spaces; for example, see [2–8]. Section 2 deals with preliminaries and some basic results. Section 3 contains the main result and is about the existence of a weak solution to (3) in a suitable weighted Sobolev space.

2. Preliminaries

We need the following preliminaries for the ensuing study. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open connected set). Let $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a locally integrable function with $0 < \omega < \infty$ a.e. We say that ω belongs to the Muckenhoupt class A_p , $1 < p < \infty$, or that ω is an A_p -weight, if there is a constant $c = c_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} \leq c, \quad (4)$$

for all balls B in $\mathbb{R}^n$, where $|\cdot|$ denotes the n -dimensional Lebesgue measure in $\mathbb{R}^n$. We assume that $\omega \in A_p$, $1 < p < \infty$. We will denote by $L^p(\Omega, \omega)$ ($1 \leq p < \infty$) the usual Banach space of measurable real valued functions, f , defined in Ω for which

$$\|f\|_{p,\Omega} = \left(\int_\Omega |f(x)|^p \omega(x) dx \right)^{1/p} < \infty. \quad (5)$$

For $p \geq 1$ and a positive integer k , the weighted Sobolev space $W^{k,p}(\Omega, \omega)$ is defined by

$$W^{k,p}(\Omega, \omega) := \{u \in L^p(\Omega, \omega) : D^\alpha u \in L^p(\Omega, \omega), 1 \leq |\alpha| \leq k\} \quad (6)$$

with the associated norm

$$\|u\|_{k,p,\Omega} = \|u\|_{p,\Omega} + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|_{p,\Omega}. \quad (7)$$

In order to avoid too many suffices, at each step, a generic constant is denoted by c or C_Ω . We need the following result.

Proposition 1 (the weighted Sobolev inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\omega \in A_p$ ($1 < p < \infty$). Then, there exist positive constants C_Ω and δ such that, for all $u \in C_0^\infty(\Omega)$ and all k satisfying $1 \leq k \leq n/(n-1) + \delta$,*

$$\|u\|_{k,p,\Omega} \leq C_\Omega \|\nabla u\|_{p,\Omega}. \quad (8)$$

A proof of the above statement can be found in [5, Theorem 1.3].

For $p = 2$ and $k = 1$ in the above inequality, we have

$$\|u\|_{2,\Omega} \leq C_\Omega \|u\|_{0,1,2}, \quad (9)$$

where

$$\|u\|_{0,1,2} = \|\nabla u\|_{2,\Omega} := \left(\int_\Omega |\nabla u(x)|^2 \omega dx \right)^{1/2}. \quad (10)$$

Further, we use function space $W_0^{1,2}(\Omega, \omega)$ which is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{0,1,2}$ (correctness of definition of this norm follows from inequality (9)). We also note that $W^{1,2}(\Omega, \omega)$ and $W_0^{1,2}(\Omega, \omega)$ are Hilbert spaces.

For more details on A_p -weight and weighted Sobolev spaces, we refer to [5, 7, 9–11].

Proposition 2. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_n \rightarrow u$ in $L^p(\Omega, \omega)$, then there exist a subsequence $\{u_{n_k}\}$ and a function $v \in L^p(\Omega, \omega)$ such that*

- (i) $u_{n_k} \rightarrow u(x)$ as $n_k \rightarrow \infty$, ω -a.e. on Ω ;
- (ii) $|u_{n_k}(x)| \leq v(x)$, ω -a.e. on Ω .

Proof. The proof of this theorem follows in the lines of Theorem 2.8.1 in [12]. $\square$

Let $(f | x)$ denote the value of linear functional f at x .

Definition 3. Let $C, D : X \rightarrow X^*$ be the operators on the real separable reflexive Banach space X . Then,

- (i) C satisfies condition (S) if

$$u_n \rightarrow u, \quad \lim_{n \rightarrow \infty} (Cu_n - Cu | u_n - u) = 0, \quad (11)$$

implies $u_n \rightarrow u$,

- (ii) C is demicontinuous if and only if $u_n \rightarrow u$ as $n \rightarrow \infty$ implies $Cu_n \rightarrow Cu$ as $n \rightarrow \infty$,
- (iii) $C + D$ is asymptotically linear if C is linear and

$$\frac{\|Du\|}{\|u\|} \rightarrow 0, \quad \text{as } \|u\| \rightarrow \infty. \quad (12)$$

In Section 3, we use the following result.

Proposition 4. *Let $B, N : X \rightarrow X^*$ be operators on the real separable reflexive Banach space X . Then,*

- (1) the operator $B : X \rightarrow X^*$ is linear and continuous;
- (2) the operator $N : X \rightarrow X^*$ is demicontinuous and bounded;
- (3) $B + N$ is asymptotically linear;
- (4) for each $T \in X^*$ and for each $t \in [0, 1]$, the operator A_t defined by $A_t(u) := Bu + t(Nu - T)$ satisfies condition (S) in X .

If $Bu = 0$ implies $u = 0$, then, for each $T \in X^*$, the equation $Bu + Nu = T$ has a solution in X .

For a detailed proof of the above theorem, we refer to [13] or to [1, Theorem 29.C].

Definition 5. One says that $u \in W_0^{1,2}(\Omega, \omega)$ is a weak solution of (3) if

$$\begin{aligned} \int_\Omega a_{ij} D_i u(x) D_j \phi(x) dx - \int_\Omega \mu u(x) g_1(x) \phi(x) dx \\ + \int_\Omega h(u(x), \nabla u(x)) g_2(x) \phi(x) dx = \int_\Omega f(x) \phi(x) dx. \end{aligned} \quad (13)$$

for every $\phi \in W_0^{1,2}(\Omega, \omega)$.

We need the following hypotheses for further study.

(H₁) Let $(\eta, \xi) \mapsto h(\eta, \xi)$ be continuous in $\mathbb{R} \times \mathbb{R}^n$, where h is a bounded function (i.e., for a constant $A > 0$, let $|h(t)| \leq A, \forall t \in \mathbb{R}$).

(H₂) Let $\omega \in A_2$. Assume $g_1/\omega \in L^\infty(\Omega)$, $g_2/\omega \in L^2(\Omega, \omega)$, and $f/\omega \in L^2(\Omega, \omega)$.

(H₃) Assume $g_2(h(\eta, \xi) - h(\eta', \xi'))(\eta - \eta') \geq 0$, where $\eta, \eta' \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^n$.

We define the functionals $B_1, B_2 : W_0^{1,2}(\Omega, \omega) \times W_0^{1,2}(\Omega, \omega) \rightarrow \mathbb{R}$ by

$$\begin{aligned} B_1(u, \phi) &= \int_\Omega a_{ij} D_i u(x) D_j \phi(x) dx \\ &\quad - \int_\Omega \mu u(x) g_1(x) \phi(x) dx, \end{aligned} \quad (14)$$

$$B_2(u, \phi) = \int_\Omega h(u(x), \nabla u(x)) g_2(x) \phi(x) dx.$$

Also define $T : W_0^{1,2}(\Omega, \omega) \rightarrow \mathbb{R}$ by

$$T(\phi) = \int_\Omega f(x) \phi(x) dx. \quad (15)$$

A function $u \in W_0^{1,2}(\Omega, \omega)$ is a weak solution of (3) if

$$B_1(u, \phi) + B_2(u, \phi) = T(\phi), \quad \forall \phi \in W_0^{1,2}(\Omega, \omega). \quad (16)$$

By noting $|a_{ij}(x)| \leq c\omega(x)$ and by Hölder's inequality, we get

$$\begin{aligned} |B_1(u, \phi)| &\leq \int_{\Omega} |a_{ij}(x)| |D_i u(x)| |D_j \phi(x)| dx \\ &\quad + |\mu| \int_{\Omega} |u(x)| |\phi(x)| |g_1(x)| dx \\ &\leq c \int_{\Omega} \sum_{i,j=1}^n |D_i u(x)| |D_j \phi(x)| \omega(x) dx \\ &\quad + |\mu| \int_{\Omega} |u(x)| |\phi(x)| \left| \frac{g_1(x)}{\omega(x)} \right| \omega(x) dx \\ &\leq c \left(\int_{\Omega} \sum_{i=1}^n |D_i u(x)|^2 \omega(x) dx \right)^{1/2} \\ &\quad \times \left(\int_{\Omega} \sum_{j=1}^n |D_j \phi(x)|^2 \omega(x) dx \right)^{1/2} \\ &\quad + |\mu| \left\| \frac{g_1}{\omega} \right\|_{\infty} \left(\int_{\Omega} |u(x)|^2 \omega(x) dx \right)^{1/2} \\ &\quad \times \left(\int_{\Omega} |\phi(x)|^2 \omega(x) dx \right)^{1/2} \\ &\leq \left(c + C_{\Omega} |\mu| \left\| \frac{g_1}{\omega} \right\|_{\infty} \right) \|u\|_{0,1,2} \|\phi\|_{0,1,2}, \end{aligned} \quad (17)$$

where C_{Ω} is a constant arising out of (9). Now, $B_1(\cdot, \cdot)$ is linear and bounded. Then, there exists an operator

$$B : W_0^{1,2}(\Omega, \omega) \longrightarrow [W_0^{1,2}(\Omega, \omega)]^*, \quad (18)$$

defined by $(Bu, \phi) = B_1(u, \phi)$, for all $u, \phi \in W_0^{1,2}(\Omega, \omega)$. Also, by (H_1) and (H_2) , it follows from Hölder's inequality that

$$\begin{aligned} |B_2(u, \phi)| &= \left| \int_{\Omega} g_2(x) h(u(x), \nabla u(x)) \phi(x) dx \right| \\ &\leq A \int_{\Omega} \left| \frac{g_2(x)}{\omega(x)} \right| |\phi(x)| \omega(x) dx \\ &\leq A \left\| \frac{g_2}{\omega} \right\|_{2,\Omega} \|\phi\|_{2,\Omega} \\ &\leq A \left\| \frac{g_2}{\omega} \right\|_{2,\Omega} \|\phi\|_{0,1,2}, \end{aligned} \quad (19)$$

and hence by the weighted Sobolev inequality (9)

$$|B_2(u, \phi)| \leq C_{\Omega} |u|_{0,1,2} |\phi|_{0,1,2}, \quad \forall u, \phi \in W_0^{1,2}(\Omega, \omega). \quad (20)$$

Now $B_2(u, \cdot)$ is linear and so there exists an operator $N : W_0^{1,2}(\Omega, \omega) \rightarrow [W_0^{1,2}(\Omega, \omega)]^*$ such that

$$(Nu | \phi) = B_2(u, \phi), \quad \forall u, \phi \in W_0^{1,2}(\Omega, \omega). \quad (21)$$

Further, we have

$$\begin{aligned} |T(\phi)| &\leq \int_{\Omega} |f(x)| |\phi(x)| dx \\ &\leq \left\| \frac{f}{\omega} \right\|_{2,\Omega} \|\phi\|_{2,\Omega} \leq C_{\Omega} \left\| \frac{f}{\omega} \right\|_{2,\Omega} |\phi|_{0,1,2}. \end{aligned} \quad (22)$$

Then, problem (3) is equivalent to solving the operator equation

$$Bu + Nu = T, \quad \forall u \in W_0^{1,2}(\Omega, \omega). \quad (23)$$

3. Main Results

The main result of this section is to establish the existence of a solution for the degenerate nonlinear elliptic BVP (3), when $\mu > 0$ is not an eigenvalue of

$$\begin{aligned} Lu - \mu u(x) g_1(x) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (24)$$

with certain restrictions. Also, two results are established related to the cases when g_1 does not change sign.

Theorem 6. Assume the hypotheses (H_1) – (H_3) and the inequality

$$\lambda > \mu C_{\Omega} \left\| \frac{g_1}{\omega} \right\|_{\infty}, \quad (25)$$

where C_{Ω} is a constant arising out of (9). Let $\mu > 0$ not be an eigenvalue of (24). Then, the BVP (3) has a solution $u \in W_0^{1,2}(\Omega, \omega)$.

Proof. Idea of proof is such. First we write a weak solution of the BVP (3) as solution of operator equation

$$u \in W_0^{1,2}(\Omega, \omega) : Bu + Nu = T \quad \text{in } [W_0^{1,2}(\Omega, \omega)]^*, \quad (26)$$

where $T \in [W_0^{1,2}(\Omega, \omega)]^*$, $B : W_0^{1,2}(\Omega, \omega) \rightarrow [W_0^{1,2}(\Omega, \omega)]^*$ is linear and continuous, and $N : W_0^{1,2}(\Omega, \omega) \rightarrow [W_0^{1,2}(\Omega, \omega)]^*$ is demicontinuous and bounded and satisfies few more conditions. Further, we put Proposition 4 to this operator equation. The realization of this idea is split into 5 steps for convenience.

Step 1. We note that the operator B is linear. It follows from (17) and (20) that the operators B, N are bounded.

Step 2. Let $u_k \rightarrow u$ in $W_0^{1,2}(\Omega, \omega)$. We now claim

$$Nu_k \rightharpoonup Nu \quad \text{in } W_0^{1,2}(\Omega, \omega) \quad \text{as } k \rightarrow \infty \quad (27)$$

or

$$\begin{aligned} |(Nu_k - Nu | \phi)| &\rightarrow 0, \\ \forall \phi \in W_0^{1,2}(\Omega, \omega) &\quad \text{as } k \rightarrow \infty. \end{aligned} \quad (28)$$

If $u_k \rightarrow u$ in $W_0^{1,2}(\Omega, \omega)$, then $|\nabla u_k| \rightarrow |\nabla u|$ in $L^2(\Omega, \omega)$ and $u_k \rightarrow u$ in $L^2(\Omega, \omega)$. Using Proposition 2, there exist a subsequence $\{u_{n_k}\}$ and functions v_1 and v_2 in $L^2(\Omega, \omega)$ such that

$$\begin{aligned} u_k(x) &\rightarrow u(x) \quad \omega\text{-a.e. in } \Omega, \\ |u_{n_k}(x)| &\leq v_1(x); \quad \omega\text{-a.e. in } \Omega, \\ \nabla u_k(x) &\rightarrow \nabla u(x) \quad \omega\text{-a.e. in } \Omega, \\ |\nabla u_{n_k}(x)| &\leq v_2(x); \quad \omega\text{-a.e. in } \Omega. \end{aligned} \quad (29)$$

Now

$$\begin{aligned} |(Nu_k - Nu | \phi)| &= \left| \int_{\Omega} h(u_k(x), \nabla u_k(x)) \right. \\ &\quad \left. - h(u(x), \nabla u(x)) g_2(x) \phi_k(x) \right|. \end{aligned} \quad (30)$$

Since h is bounded, we get

$$\begin{aligned} |h(u_k(x), \nabla u_k(x)) - h(u(x), \nabla u(x)) g_2(x) \phi(x)| \\ \leq 2A |g_2 \phi|. \end{aligned} \quad (31)$$

Also,

$$\int_{\Omega} |g_2 \phi| \leq \left\| \frac{g_2}{\omega} \right\|_{2,\Omega} \|\phi\|_{2,\Omega} < \infty. \quad (32)$$

By (H_1) and (29), we infer that

$$\begin{aligned} [h(u_k(x), \nabla u_k(x)) - h(u(x), \nabla u(x))] g_2(x) \phi(x) &\rightarrow 0 \\ \text{a.e. in } \Omega, \text{ as } k \rightarrow \infty. \end{aligned} \quad (33)$$

Letting $k \rightarrow \infty$, by dominated convergence theorem, we obtain

$$\begin{aligned} |(Nu_k - Nu | \phi)| \\ = \left| \int_{\Omega} [h(u_k(x), \nabla u_k(x)) - h(u(x), \nabla u(x))] \right. \\ \left. \times g_2(x) \phi(x) \right| \rightarrow 0. \end{aligned} \quad (34)$$

Hence, we have claim (28) or equivalently N is demicontinuous.

Step 3. Next, we claim that $B+N$ is asymptotically linear. Since h is bounded, we observe that, for all $u \in W_0^{1,2}(\Omega, \omega)$,

$$|(Nu | \phi)| = |B_2(u, \phi)| \leq AC_{\Omega} \left\| \frac{g_2}{\omega} \right\|_{2,\Omega} \|\phi\|_{0,1,2} \quad (35)$$

which implies $\|Nu\| \leq C'$, where $C' = AC_{\Omega} \|g_2/\omega\|_{2,\Omega}$. Consequently,

$$\frac{\|Nu\|}{\|u\|_{0,1,2}} \rightarrow 0 \quad \text{as } \|u\|_{0,1,2} \rightarrow \infty, \quad (36)$$

which shows that $B+N$ is asymptotically linear.

Step 4. We denote $A_t(u) = Bu + t(Nu - T)$, $t \in [0, 1]$. Let $u_k \rightarrow u$ in $W_0^{1,2}(\Omega, \omega)$ and

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (Bu_k + t(Nu_k - T) - (Bu + t(Nu - T) | u_k - u)) \\ &= \lim_{k \rightarrow \infty} (Bu_k - Bu + t(Nu_k - Nu) | u_k - u) \\ &= \lim_{k \rightarrow \infty} (Bu_k - Bu | u_k - u) + t \lim_{k \rightarrow \infty} (Nu_k - Nu | u_k - u). \end{aligned} \quad (37)$$

We claim that $u_k \rightarrow u$ strongly in $W_0^{1,2}(\Omega, \omega)$ or $A_t(u)$ satisfies condition (S). By (2) and inequality (9) of Proposition 1, it follows that

$$\begin{aligned} B_1(u_k - u, u_k - u) \\ = \int_{\Omega} a_{ij} D_i(u_k - u) D_j(u_k - u) dx - \int_{\Omega} \mu(u_k - u)^2 g_1 dx \\ \geq \lambda \int_{\Omega} |\nabla(u_k - u)|^2 \omega dx - \int_{\Omega} \mu(u_k - u)^2 g_1 dx \\ \geq \left(\lambda - \mu C_{\Omega} \left\| \frac{g_1}{\omega} \right\|_{\infty} \right) \int_{\Omega} |\nabla(u_k - u)|^2 \omega dx \\ = \beta_{\Omega} \|u_k - u\|_{0,1,2}^2, \end{aligned} \quad (38)$$

where $\beta_{\Omega} = (\lambda - \mu C_{\Omega} \|g_1/\omega\|_{\infty})$, a positive constant depending on Ω . Since B is linear, from (38) we have

$$\begin{aligned} (Bu_k - Bu | u_k - u) &= (B(u_k - u) | u_k - u) \\ &= B_1(u_k - u, u_k - u) \\ &\geq \beta_{\Omega} \|u_k - u\|_{0,1,2}^2. \end{aligned} \quad (39)$$

By hypothesis (H_3) , we have now

$$\begin{aligned} (Nu_n - Nu | u_n - u) \\ = \int_{\Omega} [g_2 h(u_k, \nabla u_k) - h(u, \nabla u)] (u_n - u) \geq 0. \end{aligned} \quad (40)$$

From (37), (39), and (40), we note

$$0 \leq \beta_{\Omega} \lim_{k \rightarrow \infty} \|u_k - u\|_{0,1,2}^2 = 0. \quad (41)$$

Since, by condition (25), $\beta_{\Omega} > 0$, we have

$$\|u_k - u\|_{0,1,2}^2 \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (42)$$

Consequently, $\|u_k - u\|_{0,1,2} \rightarrow 0$, as $k \rightarrow \infty$, which implies that, for each $t \in [0, 1]$, $A_t(u)$ satisfies condition (S).

Step 5. Since, by given hypothesis, $\mu > 0$ is not an eigenvalue of (24), $Bu = 0$ implies $u = 0$. By Proposition 4, $Bu + Nu = T$ has a solution $u \in W_0^{1,2}(\Omega, \omega)$ which equivalently shows that the BVP (3) has a solution $u \in W_0^{1,2}(\Omega, \omega)$. $\square$

Remark 7. In the following results, we dispense with condition (25), when g_1 does not change sign. The two results are related to the cases when $g_1 > 0$ with $\mu < 0$ and $g_1 < 0$ with $\mu > 0$.

The proof of the following results is similar to Theorem 6 and, hence, we give a sketch of the proof.

Theorem 8. Suppose that (H_1) – (H_3) hold. Let $g_1 > 0$ and $\mu < 0$; then the BVP (3) has a solution $u \in W_0^{1,2}(\Omega, \omega)$.

Proof. As in Theorem 6, the basic idea is to reduce the problem (3) to an operator equation $Bu + Nu = T$. We note that the operator B is linear and continuous and N is bounded. Let $u_k \rightarrow u$ in $W_0^{1,2}(\Omega, \omega)$ and as in (37)

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (A_t(u_k) - A_t(u) | u_k - u) \\ &= \lim_{k \rightarrow \infty} (Bu_k - Bu | u_k - u) \\ &\quad + t \lim_{k \rightarrow \infty} ((Nu_k - Nu) | u_k - u). \end{aligned} \quad (43)$$

Since $\mu < 0$ and $g_1 > 0$, then, by (2) and the weighted Sobolev inequality (9), it follows that

$$\begin{aligned} B_1(u_k - u, u_k - u) &= \int_{\Omega} a_{ij} D_i(u_k - u) D_j(u_k - u) dx \\ &\quad - \int_{\Omega} \mu(u_k - u)^2 g_1 dx \\ &\geq \lambda \int_{\Omega} |\nabla(u_k - u)|^2 \omega dx - \int_{\Omega} \mu(u_k - u)^2 g_1 dx \\ &\geq \lambda \int_{\Omega} |\nabla(u_k - u)|^2 \omega dx = \lambda \|u_k - u\|_{0,1,2}^2. \end{aligned} \quad (44)$$

Since B is linear, from (44) we have

$$\begin{aligned} (Bu_k - Bu | u_k - u) &= (B(u_k - u) | u_k - u) \\ &= B_1(u_k - u, u_k - u) \geq \lambda \|u_k - u\|_{0,1,2}^2. \end{aligned} \quad (45)$$

By hypothesis (H_3) , we have now

$$\begin{aligned} (Nu_n - Nu | u_n - u) &= \int_{\Omega} [g_2(x) h(u_k(x), \nabla u_k(x)) \\ &\quad - h(u(x), \nabla u(x))] (u_n - u) \geq 0. \end{aligned} \quad (46)$$

From (43), (45), and (46), we note that

$$0 \leq \lambda \lim_{k \rightarrow \infty} \|u_k - u\|_{0,1,2}^2 = 0. \quad (47)$$

Since $\lambda > 0$, we have $\|u_k - u\|_{0,1,2}^2 \rightarrow 0$ and hence $\|u_k - u\|_{0,1,2} \rightarrow 0$, as $k \rightarrow \infty$, which implies that, for each $t \in [0, 1]$, $A_t(u)$ satisfies condition (S). Also, we note that $B + N$ is asymptotically linear. Now $Bu = 0$ implies

$$\int_{\Omega} a_{ij} |D_i u(x)|^2 dx - \int_{\Omega} \mu u^2(x) g_1(x) dx = 0 \quad (48)$$

or

$$\begin{aligned} \lambda \int_{\Omega} |Du(x)|^2 dx &\leq \int_{\Omega} a_{ij} |D_i u(x)|^2 dx \\ &= \mu \int_{\Omega} u^2(x) g_1(x) dx \end{aligned} \quad (49)$$

which shows that $u = 0$ (since $\mu < 0$ and $g_1 > 0$). By Proposition 4, $Bu + Nu = T$ has a solution $u \in W_0^{1,2}(\Omega, \omega)$. $\square$

With suitable changes in the proof of Theorem 8, we arrive at the following result.

Theorem 9. Let the hypotheses of Theorem 8 hold, except that $g_1 < 0$ and $\mu > 0$. Then, the BVP (3) has a solution $u \in W_0^{1,2}(\Omega, \omega)$.

Proof. The proof of this result is in the same lines to that of Theorem 8 and hence omitted. $\square$

Remark 10. In Theorems 8 and 9, we have studied the BVP (3) both positive and negative values of μ . In Theorem 9, with positive value of μ we do not need the extra condition (25) at the cost of g_1 negative.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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