Research Article

ϕ-Prime and ϕ-Primary Elements in Multiplicative Lattices

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Received 7 June 2014; Revised 4 September 2014; Accepted 9 September 2014; Published 9 October 2014

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We investigate ϕ-prime and ϕ-primary elements in a compactly generated multiplicative lattice L. By a counterexample, it is shown that a ϕ-primary element in L need not be primary. Some characterizations of ϕ-primary and ϕ-prime elements in L are obtained. Finally, some results for almost prime and almost primary elements in L with characterizations are obtained.

1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative, and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element a ∈ L is called compact if, for X ⊆ L, a ≤ ∨X implies the existence of a finite number of elements a1, a2, . . . , an in X such that a ≤ a1 ∨ a2 ∨ · · · ∨ an. The set of compact elements of L will be denoted by L∗. A multiplicative lattice is said to be compactly generated if every element of it is a join of compact elements. Throughout this paper L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact.

An element a ∈ L is said to be proper if a < 1. A proper element p ∈ L is called a prime element if ab ≤ p implies a ≤ p or b ≤ p, where a, b ∈ L, and is called a primary element if ab ≤ p implies a ≤ p or b ≤ p for some n ∈ Z+, where a, b ∈ L∗. A proper element p ∈ L is said to be weakly prime if 0 ̸= ab ≤ p implies either a ≤ p or b ≤ p, where a, b ∈ L, and is called weakly primary if 0 ̸= ab ≤ p implies a ≤ p or b ≤ p for some n ∈ Z+, where a, b ∈ L∗. For a, b ∈ L, (a : b) = ∨{x ∈ L | xb ≤ a}. The radical of a ∈ L is denoted by √a and is defined as ∨{x ∈ L∗ | xn ≤ a, for some n ∈ Z+}. An element a ∈ L is called semiprime if √a is a prime element and is called semiprime if √a = a. An element a ∈ L is called join irreducible if a = a1 ∨ a2 implies a = a1 or a = a2. A proper element m ∈ L is said to be a maximal element if m ̸= a for any other proper element a ∈ L. An element a ∈ L is said to be nilpotent if a2n = 0 for some n ∈ Z+. An element a ∈ L is called a zero divisor if ab = 0 for some 0 ̸= b ∈ L and is called an idempotent if a = a2. A multiplicative lattice is said to be a domain if it is without zero divisors and is said to be quasi-local if it contains a unique maximal element. A quasi-local multiplicative lattice L with maximal element m is denoted by (L, m). An element e ∈ L is called meet principal if a ∧ be = ((a : e) ∧ b)e for all a, b ∈ L. An element e ∈ L is called join principal if (ae ∨ b) : e = (b : e) ∨ a for all a, b ∈ L. An element e ∈ L is called weak meet principal if a ∧ e = (a : e)e for all a ∈ L. An element e ∈ L is called weak join principal if (ae : e) = (0 : e) ∨ a for all a ∈ L. An element e ∈ L is called principal if e is both meet principal and join principal.

An element a ∈ L is called semiprime if √a is a prime element and is called semiprime if √a = a. An element a ∈ L is called join irreducible if a = a1 ∨ a2 implies a = a1 or a = a2. A proper element m ∈ L is said to be a maximal element if m ̸= a for any other proper element a ∈ L. An element a ∈ L is said to be nilpotent if a2n = 0 for some n ∈ Z+. An element a ∈ L is called a zero divisor if ab = 0 for some 0 ̸= b ∈ L and is called an idempotent if a = a2. A multiplicative lattice is said to be a domain if it is without zero divisors and is said to be quasi-local if it contains a unique maximal element. A quasi-local multiplicative lattice L with maximal element m is denoted by (L, m). An element e ∈ L is called meet principal if a ∧ be = ((a : e) ∧ b)e for all a, b ∈ L. An element e ∈ L is called join principal if (ae ∨ b) : e = (b : e) ∨ a for all a, b ∈ L. An element e ∈ L is called weak meet principal if a ∧ e = (a : e)e for all a ∈ L. An element e ∈ L is called weak join principal if (ae : e) = (0 : e) ∨ a for all a ∈ L. An element e ∈ L is called principal if e is both meet principal and join principal.

A multiplicative lattice is a Noether lattice if it is modular, principally generated (every element is a join of principal elements) and satisfies the ascending chain condition. A Noether lattice L is local if it contains precisely one maximal prime. In a Noether lattice L, an element a ∈ L is said to satisfy restricted cancellation law.
if, for any \( b, c \in L \), \( ab = ac \neq 0 \) implies \( b = c \) [1]. The reader is referred to [2] for general background and terminology.

### 2. \( \phi \)-Prime and \( \phi \)-Primary Elements in \( L \)

The study of \( \phi \)-prime and \( \phi \)-primary ideals for commutative rings is carried out by Anderson and Bataineh [3], Bataineh and Kuhail [4], and Yousefian Darani [5]. We extend these concepts to compactly generated multiplicative lattices. We introduce the notion of \( \phi \)-prime, \( \phi \)-primary, and \( \omega \)-primary elements in \( L \).

**Definition 1.** Let \( \phi : L \to L \) be a function. A proper element \( p \in L \) is said to be \( \phi \)-prime if for \( a, b \in L \), \( ab \leq p \) and \( ab \notin \phi(p) \) implies either \( a \leq p \) or \( b \leq p \).

**Definition 2.** Let \( \phi : L \to L \) be a function. A proper element \( p \in L \) is said to be \( \phi \)-primary if, for \( a, b \in L \), \( ab \leq p \) and \( ab \notin \phi(p) \) implies either \( a \leq p \) or \( b \leq p \).

**Definition 3.** Let \( \phi : L \to L \) be a function. A proper element \( p \in L \) is said to be \( \phi \)-primary if, for \( a, b \in L \), \( ab \leq p \) and \( ab \notin \phi(p) \) implies either \( a \leq p \) or \( b \leq p \).

**Definition 4.** A proper element \( p \in L \) is said to be almost prime if, for \( a, b \in L \), \( ab \leq p \) and \( ab \notin \phi(p) \) implies either \( a \leq p \) or \( b \leq p \).

**Definition 5.** A proper element \( p \in L \) is said to be almost prime if, for \( a, b \in L \), \( ab \leq p \) and \( ab \notin \phi(p) \) implies either \( a \leq p \) or \( b \leq p \).

The reader can verify the following statements.

1. \( \phi(p) \leq p \) for all \( p \in L \).
2. Every \( \phi \)-prime element in \( L \) is \( \phi \)-primary.
3. Every prime element in \( L \) is \( \phi \)-prime.
4. Every prime element in \( L \) is \( \phi \)-primary.
5. Every primary element in \( L \) is \( \phi \)-primary.

The following example (take \( \phi(p) = p^2 \) for convenience) shows that

1. a \( \phi \)-primary element in \( L \) need not be \( \phi \)-prime;
2. a \( \phi \)-primary element in \( L \) need not be prime.

**Example 6.** Consider the lattice \( L \) of ideals of the ring \( R = \langle \mathbb{Z}_{24}, +, \cdot \rangle \). Then the only ideals of \( R \) are the principal ideals \( (0), (2), (3), (4), (6), (8), (12), (1) \). Clearly \( L = \langle (0), (2), (3), (4), (6), (8), (12), (1) \rangle \) is a compactly generated multiplicative lattice. Its lattice structure is as shown in Figure 1.

From the multiplication table (see Figure 1), we see that element \( c \in L \) is almost prime, while \( c \) is not prime because \((2)(6) \leq (4) = c \) but \( (2) \notin (4) = c \) and \( (6) \notin (4) = c \). Also, \( c \) is not prime.

**Theorem 7.** If a proper element \( q \in L \) is \( \phi \)-primary such that \( \sqrt{\phi(q)} = \phi(\sqrt{q}) \), then \( \sqrt{q} \) is a \( \phi \)-prime element in \( L \).

**Proof.** Let \( p = \sqrt{q} \). Assume that \( ab \leq p \) and \( ab \notin \phi(p) \) but \( a \notin p \) for some \( a, b \in L \). Then there exist \( n \in \mathbb{N} \) such that \((ab)^n \leq q \) if \((ab)^n \leq \phi(q) \), then by hypothesis \( ab \notin \phi(\sqrt{q}) \), a contradiction. So \((ab)^n \notin \phi(q) \). Since \( q \) is \( \phi \)-primary and \( a^n \notin q \) for all \( n \in \mathbb{N} \), we have \( b^n \leq \sqrt{q} \) and hence \( b \leq \sqrt{\sqrt{q}} = q \). This shows that \( q \) is a \( \phi \)-prime element in \( L \).

**Theorem 8.** Given two functions \( \gamma_1, \gamma_2 : L \to L \), we define \( \gamma_1 \leq \gamma_2 \) if \( \gamma_1(p) \leq \gamma_2(p) \) for each \( p \in L \).

**Theorem 9.** Let \( \gamma_1, \gamma_2 : L \to L \) be functions such that \( \gamma_1 \leq \gamma_2 \).

Then every proper \( \gamma_1 \)-primary element in \( L \) is \( \gamma_2 \)-primary.

**Proof.** Let a proper element \( p \in L \) be \( \gamma_1 \)-primary. Suppose \( ab \leq p \), \( ab \notin \gamma_2(p) \) for some \( a, b \in L \). Then as \( \gamma_1 \leq \gamma_2 \) we have \( ab \notin \gamma_1(p) \). Since \( p \) is \( \gamma_1 \)-primary, it follows that either \( a \leq p \) or \( b \leq \sqrt{p} \) and hence \( p \) is \( \gamma_2 \)-primary.

**Theorem 10.** Let \( p \in L \) be a proper element. Then \( p \) is primary implies \( p \) is weakly primary implies \( p \) is \( \omega \)-primary implies \( p \) is \( n \)-almost primary (\( n \geq 2 \)) implies \( p \) is almost primary.

**Proof.** Obviously \( p \) is primary implies \( p \) is weakly primary.

Assume that \( p \) is weakly primary but not \( \omega \)-primary. Then there exist \( a, b \in L \) such that \( ab \leq p \), \( ab \notin \bigwedge_{n=1}^{\infty} p^n \), \( a \notin p \), and \( b \notin \sqrt{p} \). As \( L \) is compactly generated, there exist \( x, y \in L \) such that \( x \leq a, y \leq b, x \neq p, \) and \( y \neq \sqrt{p} \). Let \( x' \leq a \) and \( y' \leq b \) be any two compact elements of \( L \). Then \( \langle x \lor x' \rangle, \langle y \lor y' \rangle \leq L \) such that \( \langle x \lor x' \rangle \langle y \lor y' \rangle \neq \leq p \), \( \langle x \lor x' \rangle \neq p \), and \( \langle y \lor y' \rangle \neq \sqrt{p} \). Since \( p \) is weakly primary, it follows that \( \langle x \lor x' \rangle \langle y \lor y' \rangle = 0 \). So \( x' y' = 0 \), which implies \( ab = 0 \leq \bigwedge_{n=1}^{\infty} p^n \), a contradiction to \( ab \notin \bigwedge_{n=1}^{\infty} p^n \). Hence \( p \) is \( \omega \)-primary.

Next we show that \( p \) is \( \omega \)-primary implies \( p \) is \( n \)-almost primary (\( n \geq 2 \)). Assume \( p \) is \( \omega \)-primary and \( n \geq 2 \). Let \( ab \leq p \), \( ab \notin \bigwedge_{n=1}^{\infty} p^n \) for some \( a, b \in L \). Then \( ab \leq p \), \( ab \notin \bigwedge_{n=1}^{\infty} p^n \) since \( p \) is \( \omega \)-primary, it follows that either \( a \leq p \) or \( b \leq \sqrt{p} \). Hence, \( p \) is \( n \)-almost primary (\( n \geq 2 \)).
The last implication is obvious (from \( n = 2 \)).

From this implication we get the following characterization of a \( \omega \)-primary element in \( L \).

**Corollary 11.** Let \( p \in L \) be a proper element. Then \( p \) is \( \omega \)-primary if and only if \( p \) is \( n \)-almost primary for every \( n \geq 2 \).

**Proof.** Suppose \( p \in L \) is \( n \)-almost primary for every \( n \geq 2 \). Let \( ab \leq p \) and \( ab \neq \bigwedge_{n=1}^{\infty} p^n \) for some \( a, b \in L \). Then \( ab \leq p \) and \( ab \neq p^n \) for some \( n \geq 2 \). Since \( p \) is \( n \)-almost primary, we have \( a \leq p \) or \( b \leq \sqrt[p]{p} \). Hence \( p \) is \( \omega \)-primary. The converse follows from Theorem 10.

Clearly every primary element in \( L \) is \( \phi \)-primary. But the converse is not true as shown in the following example by taking \( \phi(p) = p^2 \).

**Example 12.** Consider the lattice \( L \) of ideals of the ring \( R = \langle \mathbb{Z}_{20}, + \rangle \). Then the only ideals of \( R \) are the principal ideals \((0), (2), (3), (5), (6), (10), (15), (1)\) is a compactly generated multiplicative lattice. Here the element \((6) \in L \) is almost primary but not primary.

In the following successive three theorems, we show conditions under which a \( \phi \)-primary element in \( L \) is primary.

Now we have a characterization of a \( \phi \)-primary element in \( L \).

**Theorem 13.** Let \( L \) be a Noether lattice. Let \( 0 \neq a \in L \) be a non-nilpotent proper element satisfying the restricted cancellation law. Then \( a \) is \( \phi \)-primary for some \( \phi \leq \phi_2 \) if and only if \( a \) is primary.

**Proof.** Suppose \( a \in L \) is a primary element. Then obviously \( a \) is \( \phi \)-primary for every \( \phi \) and hence for \( \phi \leq \phi_2 \). Conversely, let \( a \) be \( \phi \)-primary for some \( \phi \leq \phi_2 \). Then, by Theorem 9, \( a \) is \( \phi_2 \)-primary (almost primary). Let \( xy \leq a \) for some \( x, y \in L_+ \). If \( xy \neq a^2 \), then as \( a \) is \( \phi_2 \)-primary we have \( x \leq a \) or \( y \leq \sqrt{a} \). If \( xy \leq a^2 \), consider \((x \vee y) = xy \vee ay \leq a \). If \((x \vee y) \neq a^2 \), then as \( a \) is \( \phi_2 \)-primary we have \((x \vee a) \leq a \) or \( y^n \leq a \) for some \( m \in \mathbb{Z}_+ \) and hence \( x \leq a \) or \( y \leq \sqrt{a} \). So assume \((x \vee a) \neq a^2 \). Then \( xy \leq a^2 \) and \( ay \leq a^2 \neq 0 \). By Lemma 1.11 of [1] we get \( y \leq a \) which shows that \( a \) is primary.

**Definition 14.** A proper element \( p \in L \) is said to be 2-potent prime if, for \( a, b \in L \), \( ab \leq p^2 \) implies either \( a \leq p \) or \( b \leq p \).

**Definition 15.** A proper element \( p \in L \) is said to be 2-potent primary if, for \( a, b \in L \), \( ab \leq p^2 \) implies either \( a \leq p \) or \( b \leq \sqrt{p} \).

**Theorem 16.** Let \( a \) be a proper element \( q \in L \) be 2-potent primary. If \( q \) is \( \phi \)-primary for some \( \phi \leq \phi_2 \), then \( q \) is primary.

**Proof.** Clearly by Theorem 9, \( q \in L \) is \( \phi_2 \)-primary (almost primary). Let \( xy \leq q \) for some \( x, y \in L_+ \). If \( xy \neq q^2 \), then as \( q \) is \( \phi_2 \)-primary we have \( x \leq q \) or \( y \leq \sqrt{q} \). If \( xy \leq q^2 \), then as \( q \) is 2-potent primary we have \( x \leq q \) or \( y \leq \sqrt{q} \). Hence \( q \) is primary.

**Theorem 17.** Let \( a \) be a proper element \( q \in L \) be \( \phi \)-primary. If \( q^2 \neq \phi(q) \), then \( q \) is primary.

**Proof.** Let \( ab \leq q \) for some \( a, b \in L_+ \). If \( ab \neq \phi(q) \), then as \( q \) is \( \phi \)-primary we have \( a \leq q \) or \( b \leq \sqrt{q} \). So assume \( ab \leq \phi(q) \).

First suppose \( a \neq \phi(q) \). Then \( ad \neq \phi(q) \) for some \( d \leq q \) in \( L \). Also \( a(d \vee d) = ad \leq q \) and \( a(b \vee d) \neq \phi(q) \). As \( q \) is \( \phi \)-primary, either \( a \leq q \) or \( (b \vee d)^k \leq q \) for some \( k \in \mathbb{Z}_+ \). Hence \( a \leq q \) or \( b^k \leq q \) for some \( k \in \mathbb{Z}_+ \). Similarly if \( bq \neq \phi(q) \), we can show that either \( a \leq q \) or \( b^k \leq q \) for some \( k \in \mathbb{Z}_+ \). So we can assume that \( ab \leq \phi(q) \) and \( bq \leq \phi(q) \). Since \( q^2 \neq \phi(q) \), there exist \( r, s \leq q \) in \( L \) such that \( rs \neq \phi(q) \).
Then \((a \lor r)(b \lor s) \leq q\), but \((a \lor r)(b \lor s) \not\leq \phi(q)\). As \(q\) is \(\phi\)-primary, we have either \((a \lor r) \leq q\) or \((b \lor s) \not\leq q\) for some \(t \in \mathbb{Z}_\ast\). Hence \(a \leq q\) or \(b \leq q\) for some \(t \in \mathbb{Z}_\ast\). Thus \(q\) is primary.

From the above theorem it follows that

1. if a proper element \(q \in L\) is \(\phi\)-primary but not primary, then \(q^2 \leq \phi(q)\);
2. a \(\phi\)-primary element \(q < 1\) in \(L\) with \(q^2 \not\leq \phi(q)\) is primary.

**Corollary 18.** If a proper element \(q \in L\) is \(\phi\)-primary but not primary, then \(\sqrt{q} = \sqrt{\phi(q)}\).

**Proof.** From Theorem 17 we have \(q^2 \leq \phi(q)\). So \(q \leq \sqrt{\phi(q)}\) which gives \(\sqrt{q} \leq \sqrt{\phi(q)} = \sqrt{\phi(q)}\). Since \(\phi(q) \leq q\), we have \(\sqrt{\phi(q)} \leq \sqrt{q}\). Hence \(\sqrt{q} = \sqrt{\phi(q)}\).

**Theorem 21.** Let \(q \in L\) be a proper element and let \(\phi : L \to L\) be a function. Then the following statements are equivalent:

1. \(q\) is \(\phi\)-primary;
2. for every \(a \in L\) such that \(a \not\leq \sqrt{q}\), either \((q : a) = q\) or \((q : a) = (\phi(q) : a)\);
3. for any two elements \(r, s \in L\), \(rs \leq q\) and \(rs \not\leq \phi(q)\) implies either \(s \leq q\) or \(r \not\leq \sqrt{q}\).

**Proof.** (i) \(\Rightarrow\) (ii). Suppose (i) holds. Let \(h \in L\) be compact such that \(h \leq (q : a)\) and \(h \not\leq \sqrt{q}\). Then \(ah \leq q\). If \(ah \leq \phi(q)\), then \(h \leq (\phi(q) : a)\). If \(ah \not\leq \phi(q)\), then since \(q\) is \(\phi\)-primary and \(a \not\leq \sqrt{q}\) it follows that \(h \leq q\). Hence by Lemma 1 of [6] either \((q : a) \leq (\phi(q) : a)\) or \((q : a) \leq q\). Consequently either \((q : a) = (\phi(q) : a)\) or \((q : a) = q\).

(i) \(\Rightarrow\) (ii). Suppose (ii) holds. Let \(rs \leq q\), \(rs \not\leq \phi(q)\), and \(r \not\leq \sqrt{q}\) for some \(r, s \in L\). Then by (ii) we have either \((q : r) = (\phi(q) : r)\) or \((q : r) = q\). If \((q : r) = (\phi(q) : r)\), then as \(s \leq (q : r)\) it follows that \(s \leq (\phi(q) : r)\) which contradicts \(rs \not\leq \phi(q)\). If \((q : r) = q\), then \(s \leq (q : r)\) gives \(s \leq q\).

(ii) \(\Rightarrow\) (i). Suppose (i) holds. Let \(ab \leq q\), \(ab \not\leq \phi(q)\), and \(a \not\leq \sqrt{q}\) for some \(a, b \in L\). Then as \(L\) is compactly generated, there exist \(x, x', y' \in L\) such that \(x \leq a, x' \leq a\), and \(y' \leq b\). But \(x \not\leq \sqrt{q}, x'y' \not\leq \phi(q)\). Let \(y \leq b\) be any compact element of \(L\). Then \((x \lor x'), (y \lor y') \in L\), such that \((x \lor x')(y \lor y') \leq ab \not\leq \phi(q), (x \lor x')(y \lor y') \not\leq \sqrt{q}\). So by (i) \((y \lor y') \leq q\) which implies \(b \leq q\). Therefore, \(q\) is \(\phi\)-primary.

**Definition 22.** Let \(\phi : L \to L\) be a function. A proper element \(p \in L\) is said to be \(\phi\)-prime (or briefly \(\omega\)-prime) if, for \(a, b \in L\), \(ab \leq p\) and \(ab \not\leq \phi(q)\) implies either \(a \leq p\) or \(b \leq p\).

Similarly \(\phi\)-prime elements in \(L\) are defined by following settings in the definition for \(\phi\)-prime element. For any proper element \(p \in L\),

(i) \(\phi_0(p) = 0\) and then \(p\) is called weakly prime element;
(ii) \(\phi_2(p) = p^2\) and then \(p\) is called almost prime element;
(iii) \(\phi_n(p) = p^n (n \geq 2)\) and then \(p\) is called \(n\)-almost prime element.

The analogous theorems and corollaries (obtained for \(\phi\)-primary elements in \(L\)) for \(\phi\)-prime elements in \(L\) are as follows whose proofs being on similar arguments are omitted.

**Theorem 23.** Let \(\gamma_1, \gamma_2 : L \to L\) be functions such that \(\gamma_1 \leq \gamma_2\). Then every proper \(\gamma_1\)-prime element in \(L\) is \(\gamma_2\)-prime.

**Theorem 24.** Let \(p \in L\) be a proper element. Then \(p\) is prime implies \(p\) is weakly prime implies \(p\) is \(\omega\)-prime implies \(p\) is \(n\)-almost prime \((n \geq 2)\) implies \(p\) is almost prime.

Now we have the characterization of an \(\omega\)-prime element in \(L\).

**Corollary 25.** Let \(p \in L\) be a proper element. Then \(p\) is \(\omega\)-prime if and only if \(p\) is \(n\)-almost prime for every \(n \geq 2\).

Clearly every prime element in \(L\) is \(\phi\)-prime. But the converse is not true as shown in the following example by taking \(\phi(p) = p^2\).

**Example 26.** Consider \(L\) as in Example 6. Here the element \((0) \in L\) is weakly prime and hence almost prime, while \((0)\) is not prime since \((2)(12) \leq (0)\), but neither \((2) \leq (0)\) nor \((12) \leq (0)\).

In the following successive three theorems we show conditions under which a \(\phi\)-prime element in \(L\) is prime.

The next theorem gives a characterization of a \(\phi\)-prime element in \(L\).

**Theorem 27.** Let \(L\) be a Noether lattice. Let \(0 \neq a \in L\) be a non-nilpotent proper element satisfying the restricted cancellation law. Then \(a\) is \(\phi\)-prime for some \(\phi \leq \phi_2\) if and only if \(a\) is prime.

**Theorem 28.** Let a proper element \(q \in L\) be 2-potent prime. If \(q\) is \(\phi\)-prime for some \(\phi \leq \phi_2\), then \(q\) is prime.
Theorem 29. Let a proper element \( q \in L \) be \( \phi \)-prime. If \( q^2 \not\in \phi(q) \), then \( q \) is prime.

From the above theorem it follows that
1. if a proper element \( q \in L \) is \( \phi \)-prime but not prime, then \( q^2 \not\in \phi(q) \);
2. a \( \phi \)-prime element \( q < 1 \) in \( L \) with \( q^2 \not\in \phi(q) \) is prime.

Corollary 30. If a proper element \( q \in L \) is \( \phi \)-prime but not prime, then \( \sqrt{q} = \sqrt{\phi(q)} \).

Corollary 31. If a proper element \( q \in L \) is \( \phi \)-prime where \( \phi \preceq \phi_3 \), then \( q \) is \( \omega \)-prime.

Corollary 32. If a proper element \( q \in L \) is \( \phi_0 \)-prime but not prime, then \( q^2 = 0 \).

Theorem 33. Let \( q \in L \) be a proper element and let \( \phi : L \to L \) be a function. Then the following statements are equivalent:
1. \( q \) is \( \phi \)-prime;
2. for every \( a \in L \) such that \( a \not\in q \), either \( (q : a) = q \) or \( (q : a) = (\phi(q) : a) \);
3. for any two elements \( r, s \in L \) such that \( rs \leq q \) and \( q \not\in \phi(q) \) implies either \( s \leq q \) or \( r \leq q \).

3. Almost Prime and Almost Primary Elements in \( L \)

The study of almost prime and almost primary ideals for commutative rings is carried out by Bhatwadekar and Sharma [7] and Bataine and Kuhail [4], respectively. From the previous section we know that almost prime and almost primary elements in \( L \) are particular cases of \( \phi_0 \)-prime and \( \phi_\omega \)-primary elements in \( L \), respectively. In this section, we obtain some results on almost prime elements and on almost primary elements in \( L \) with characterizations.

The reader can verify the following statements.

1. Every almost prime element in \( L \) is almost primary.
2. Every prime element in \( L \) is almost prime.
3. Every idempotent element in \( L \) is almost primary.
4. Every prime element in \( L \) is almost primary.
5. Every idempotent element in \( L \) is almost primary.

But the converse is not true as shown in the following example.

Example 34. Consider the lattice \( L \) of ideals of the ring \( R = \langle Z_8, +, \cdot \rangle \). Then the only ideals of \( R \) are the principal ideals \((0), (2), (4), (1)\). Clearly \( L = \{(0), (2), (4), (1)\} \) is a compactly generated multiplicative lattice. Here the element \( (4) \in L \) is almost primary but not idempotent.

6. Every primary element in \( L \) is almost primary.

But the converse is not true, which is clear from Example 12.

In the following successive three theorems we show conditions under which an almost primary element in \( L \) is primary.

Theorem 35. Let a proper element \( p \in L \) be 2-potent primary. Then \( p \) being almost primary implies that \( p \) is primary.

Proof. The proof is obvious.

Theorem 36. Let a proper element \( p \in L \) be 2-potent prime. Then
1. \( p \) being almost prime implies that \( p \) is prime;
2. \( p \) being almost primary implies that \( p \) is primary.

Proof. The proof is obvious.

From the following examples it is clear that
1. an almost primary element in \( L \) need not be 2-potent prime;
2. a 2-potent prime element in \( L \) which is almost primary need not be prime.

Example 37. Consider \( L \) as in Example 12. Here the element \( (6) \in L \) is almost primary but not 2-potent prime.

Example 38. Consider \( L \) as in Example 34. Here the element \( (4) \in L \) is 2-potent prime, almost primary but not prime.

Theorem 39. Let \( L \) be a local Noetherian domain. If a proper element \( p \in L \) is \( n \)-almost primary \((n \geq 2)\), then \( p \) is primary.

Proof. Let \( ab \leq p \) for some \( a, b \in L \). If \( ab \not\in p^n \) for all \( n \geq 2 \), then as \( p \in L \) is \( n \)-almost primary we have \( a \leq p \) or \( b \leq \sqrt{p} \). If \( ab \not\in p^n \) for all \( n \geq 1 \), then as \( L \) is a local Noetherian domain, from Corollary 3.3 of [8], it follows that \( ab \not\in \bigwedge_{n=1}^{\infty} p^n = \{0\} \). Thus \( ab = 0 \). Since \( L \) is domain, we have either \( a \leq 0 \) or \( b \leq 0 \) which implies \( a \leq p \) or \( b \leq p \) and hence \( p \) is primary.

Theorem 40. Let \( (L, m) \) be a quasi-local Noetherian lattice. If a proper element \( p \in L \) is such that \( p^2 = m^2 \leq p \leq m \), then \( p \) is almost primary.

Proof. Let \( xy \leq p \) and \( xy \not\in p^2 \) for some \( x, y \in L \). If \( x \not\in m \), then \( x = 1 \). So \( xy \leq p \) gives \( y \leq p \). Similarly \( y \not\in m \) gives \( x \leq p \). Now if \( x \leq m \), then \( x^2 \leq m^2 \leq p^2 \leq p \) and hence \( x \leq \sqrt{p} \). Similarly \( y \leq m \) gives \( y \leq \sqrt{p} \). Hence in any case \( p \) is almost primary.

The next theorem gives the characterization of an almost primary element in \( L \).

Theorem 41. Let a proper element \( q \in L \) be join irreducible. Then the following statements are equivalent:
1. \( q \) is almost primary;
2. for every \( a \in L \) such that \( a \not\in \sqrt{q} \), \( (q : a) = q \vee (q^2 : a) \);
\( \circ \) for every \( a \in L \) such that \( a \notin \sqrt{q} \), either \((q : a) = q \) or \((q : a) = (q^2 : a) \).

Proof. \( \circ \Rightarrow \circ \). Suppose \( \circ \) holds. Let \( h \in L \) be compact such that \( h \leq (q : a) \) and \( a \notin \sqrt{q} \). Then \( ah \leq q \). If \( ah \notin q^2 \), then \( h \leq (q^2 : a) \). If \( ah \notin q^2 \), then since \( q \) is almost primary and \( a \notin \sqrt{q} \) it follows that \( h \leq q \). Hence by Lemma 1 of [6] either \((q : a) \leq (q^2 : a) \leq q \lor (q^2 : a) \) or \((q : a) \leq q \lor (q^2 : a) \).

But as \( q \lor (q^2 : a) \leq (q : a) \), we have \((q : a) = q \lor (q^2 : a) \).

\( \circ \Rightarrow \circ \). Suppose \( \circ \) holds. Let \( a \notin \sqrt{q} \) for some \( a \in L \). Then by \( \circ \) \((q : a) = q \lor (q^2 : a) \). Now as \( q \) is join irreducible, we have either \((q : a) = q \lor (q : a) = (q^2 : a) \).

\( \circ \Rightarrow \circ \). Suppose \( \circ \) holds. Let \( rs \leq q, rs \notin q^2 \), and \( r \notin \sqrt{q} \) for some \( r, s \in L \). Then by \( \circ \) we have either \((q : r) = (q^2 : r) \) or \((q : r) = q \). If \((q : r) = (q^2 : r) \), then as \( s \leq (q : r) \) it follows that \( s \leq (q^2 : r) \) which contradicts \( rs \notin q^2 \). If \((q : r) = q \), then \( s \leq (q : r) \) gives \( s \leq q \). Therefore \( q \) is almost primary.

In view of Proposition 2 of [9] and the above Theorems 21 and 41 we have the following result.

Corollary 42. Let \((L, m)\) be a quasi-local Noether lattice. Let a proper element \( q \in L \) be weak meet principal and let \( f \leq q \) be a nonzero weak join principal element in \( L \). Then the following statements are equivalent:

1. \( q \) is almost primary;
2. for every \( a \in L \) such that \( a \notin \sqrt{q} \), \((q : a) = q \lor (q^2 : a) \);
3. for every \( a \in L \) such that \( a \notin \sqrt{q} \), \((q : a) = q \lor (q^2 : a) \);
4. for any two elements \( r, s \in L^+, rs \leq q \) and \( rs \notin q^2 \) implies either \( s \leq q \) or \( r \leq \sqrt{q} \).

Proof. The proof is obvious.

The following theorem gives the similar characterization of an almost primary element in \( L \).

Theorem 43. A proper element \( q \in L \) is almost primary if and only if for every \( a \in L \) such that \( a \notin q \) either \((q : a) \leq \sqrt{q} \) or \((q : a) = (q^2 : a) \).

Proof. Assume that the proper element \( q \in L \) is almost primary. Let \( h \in L \) be compact such that \( h \leq (q : a) \) and \( a \notin q \). Then \( ah \leq q \). If \( ah \notin q^2 \), then \( h \leq (q^2 : a) \). If \( ah \notin q^2 \), then since \( q \) is almost primary and \( a \notin q \) it follows that \( h \leq \sqrt{q} \). Hence by Lemma 1 of [6] either \((q : a) \leq (q^2 : a) \) or \((q : a) \notin \sqrt{q} \). But as \((q^2 : a) \leq (q : a) \), we have \((q : a) \leq \sqrt{q} \) or \((q : a) = (q^2 : a) \). Conversely assume that for every \( a \in L \) such that \( a \notin q \) either \((q : a) \leq \sqrt{q} \) or \((q : a) = (q^2 : a) \). Let \( rs \leq q, rs \notin q^2 \), and \( r \notin \sqrt{q} \) for some \( r, s \in L \). Then either \((q : r) = (q^2 : r) \) or \((q : r) \leq \sqrt{q} \). If \((q : r) = (q^2 : r) \), then as \( s \leq (q : r) \) it follows that \( s \leq (q^2 : r) \) which contradicts \( rs \notin q^2 \). If \((q : r) \leq \sqrt{q} \), then \( s \leq (q : r) \) gives \( s \leq \sqrt{q} \). Therefore \( q \) is almost primary.

The analogous theorems and corollaries (obtained for almost primary elements in \( L \)) for almost prime elements in \( L \) are as follows whose proofs being on similar arguments are omitted.

Theorem 44. Let \( L \) be a local Noetherian domain. If a proper element \( p \in L \) is \( n \)-almost prime \((n \geq 2) \), then \( p \) is prime.

Theorem 45. Let \((L, m)\) be a quasi-local Noether lattice. If a proper element \( p \in L \) is such that \( p^2 = m^2 \leq p \leq m \), then \( p \) is almost prime.

The next theorem gives the characterization of an almost prime element in \( L \).

Theorem 46. Let a proper element \( q \in L \) be join irreducible. Then the following statements are equivalent:

1. \( q \) is almost prime;
2. for every \( a \in L \) such that \( a \notin q \), \((q : a) = q \lor (q^2 : a) \);
3. for every \( a \in L \) such that \( a \notin q \), either \((q : a) = q \) or \((q : a) = (q^2 : a) \).

Corollary 47. Let \((L, m)\) be a quasi-local Noether lattice. Let a proper element \( q \in L \) be weak meet principal and let \( f \leq q \) be a nonzero weak join principal element in \( L \). Then the following statements are equivalent:

1. \( q \) is almost prime;
2. for every \( a \in L \) such that \( a \notin q \), \((q : a) = q \lor (q^2 : a) \);
3. for every \( a \in L \) such that \( a \notin q \), either \((q : a) = q \) or \((q : a) = (q^2 : a) \);
4. for any two elements \( r, s \in L^+, rs \leq q \) and \( rs \notin q^2 \) implies either \( s \leq q \) or \( r \leq q \).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors wish to thank the referee for useful comments.

References

Algebra


