**Research Article**

**Modularity in the Semilattice of $\omega$-Words**

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A partial ordering of $\omega$-words can be introduced with regard to whether an $\omega$-word can be transformed into another by a Mealy machine. It is known that the poset of $\omega$-words that is introduced by this ordering is a join-semilattice. The width of this join-semilattice has the power of continuum while the depth is at least $\aleph_0$. We have created a technique for proving that power-characteristic $\omega$-words are incomparable. We use this technique to show that this join-semilattice is not modular.

1. **Introduction**

Infinite words ($\omega$-words) provide a fertile ground of research and many classes of $\omega$-words are known. While closure properties of some classes of $\omega$-words have been studied extensively (see, e.g., [1–3]), we are interested in the general algebraic structure of $\omega$-words. Mealy machines are a simple model of a word transforming automaton with the beneficial property of always transforming an $\omega$-word into another (we write $x \rightarrow y$ if such a machine exists). When both $x \rightarrow y$ and $y \rightarrow x$ are true, we say that $x$ and $y$ are machine equivalent. A class $\mathcal{K}$ of $\omega$-words is called machine invariant if $\mathcal{K}$ contains all possible transformations of its elements.

Buls [4] has shown that machine invariant classes of $\omega$-words form a completely distributive lattice. Belovs [5] showed that the poset of machine equivalent classes of $\omega$-words is a join-semilattice and that the width of this join-semilattice has the power of continuum while the depth is at least $\aleph_0$. We show in this paper that this join-semilattice is not modular.

2. **Preliminaries**

Let $\omega$ be a word over a finite and non-empty alphabet $A$. We denote the length of $\omega$ by $|\omega|$. Similarly, the cardinality of $A$ is denoted by $|A|$. The concatenation of the words $u$, $v$ is denoted by $uv$ or simply $uv$. Define $w^0 = \lambda$ and for all $i w^{i+1} = w^i w$. We say that $u$ (resp., $v$) is a *prefix* (resp., *suffix*) of $w$. Denote by $\text{Pref}(w)$ and $\text{Suff}(w)$ the respective sets of prefixes and suffixes of $w$. Let $\mathbb{N}$ denote the set of integers $\{0, 1, 2, \ldots, n, \ldots\}$.

\[ \mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}. \]  

(1)

Call a total map $x : \mathbb{N} \rightarrow A$ an (indexed) $\omega$-word of the alphabet $A$. For any $i \geq 0$ define $x_i = x(i)$ and write

\[ x = (x_i) = x_0 x_1 \cdots x_n \cdots \]

\[ x(k + 1, n + 1) = x[k, n + 1] = x[k, n] = x_k x_{k+1} \cdots x_{n-1} x_n. \]  

(2)

The set of all $\omega$-words over $A$ is denoted by $A^{\omega}$. We say that $y$ is a *prefix* of $x$, if there exists an integer $k$ such that $y = x[0, k]$. An $\omega$-word $x$ is called *ultimately periodic* if there exist integers $p \geq 0$ and $T > 0$ such that $x_i = x_{i+T}$ for all $i \geq p$. In this case $p$ is called *preperiod* and $T$ a *period* of $x$. An ultimately periodic $\omega$-word with a preperiod of zero is called *periodic*. We say a finite word $w$ is (ultimately) periodic if it is a prefix of some (ultimately) periodic $\omega$-word. We recall the important periodicity theorem due to Fine and Wilf [6].
Theorem 1. Let \( w \) be a word having periods \( p \) and \( q \) and denote by \( \gcd(p, q) \) the greatest common divisor of \( p \) and \( q \). If \( |w| \geq p + q - \gcd(p, q) \), then \( w \) has also the period \( \gcd(p, q) \).

Corollary 2. Let \( uvw \) be words having periods, respectively, \( p \) and \( q \). If \( |v| \geq p + q - \gcd(p, q) \), then \( uvw \) has the period \( \gcd(p, q) \).

This is almost folklore in combinatorics on words. Nevertheless, for the sake of completeness, we will give the proof of this corollary.

Proof. Since \( |v| \geq p + q - \gcd(p, q) \), then (Theorem 1) \( v \) has the period \( \gcd(p, q) \).

(i) At first we will prove by induction on \( |u| \) that \( u \) has the period \( \gcd(p, q) \). Let \( v = v_1v_2 \cdots v_r \). If \( |u| = 1 \), then \( u = v_0 \) for some letter. Since the period of \( uv \) is \( p \) then \( v_0 = v_p \).

Notice that \( \gcd(p, q) \) divides \( p \). Since the period of \( v \) is \( \gcd(p, q) \), then \( v_p = \gcd(p, q) \). Hence, \( v_0 = \gcd(p, q) \).

(ii) If \( |u| > 1 \), then \( u \) can be represented as concatenation \( au' = u \), where \( |u'| + 1 = |u| \). By assumption the period of \( u' \) is \( \gcd(p, q) \). Now from (i) follows that the period of \( au' \) is \( \gcd(p, q) \) too. We have completed the inductive step.

Now we shall prove by induction on \( |w| \) that \( w \) has the period \( \gcd(p, q) \).

(iii) If \( |w| = 1 \), then \( w = v_{k+1} \) for some letter. Since the period of \( vw \) is \( q \), then \( v_{k+1} = v_{k+1} \). Notice

\[
k + 1 - \gcd(p, q) \equiv k + 1 - q \quad (\text{mod} \ \gcd(p, q)).
\]

Hence, \( v_{k+1-\gcd(p, q)} = v_{k+1-\gcd(p, q)} = v_{k+1} \).

(iv) If \( |w| > 1 \), then \( w \) can be represented as concatenation \( u'v = w \), where \( |u'| + 1 = |w| \). By assumption the period of \( u' \) is \( \gcd(p, q) \). Now from (iii) follows that the period of \( u'v \) is \( \gcd(p, q) \) too. We have completed the inductive step. \( \square \)

A 3-sorted algebra \( V = \langle Q, A, B; q_0, *, \rangle \) is called an initial Mealy machine if \( Q, A, B \) are finite, nonempty sets, called the set of states, the input alphabet, and the output alphabet, respectively, \( q_0 \in Q \) is called the initial state, \( * : Q \times A \rightarrow Q \) is a total function called the transition function, and \( * : Q \times A \rightarrow B \) is a total surjective function called the output function. We write \( (Q, A, B; *, *) \) or even \( (Q, A, B; q_0) \) if there is no danger of confusion. The mappings \( \circ \) and \( \ast \) are extended to \( Q \times A^* \) by defining

\[
q \circ \lambda = q, \quad q \ast (ua) = (q \ast u) \ast a, \quad q \ast \lambda = \lambda, \quad q \ast (ua) = (q \ast u) \ast ((q \ast u) \ast a),
\]

for all \( q \in Q, (u, a) \in A^* \times A \). Henceforth, we shall omit parentheses if there is no danger of confusion. So, for example, we will write \( q \circ u \ast a \) instead of \( (q \circ u) \ast a \). Let \( (x, y) \in A^w \times B^j \). If for some Mealy machine \( V \) for all \( n \ \forall \ 0 \leq n \leq |x| \) we say that \( V \) transforms \( x \) to \( y \). We write \( x \rightarrow y \). If there exists \( V \) such that \( x \rightarrow y \); otherwise, we write \( x \not\rightarrow y \). We write \( x \equiv y \) if \( x \rightarrow y \) and \( y \rightarrow x \). This means that \( x \) and \( y \) are characteristic equivalents.

Given the integers \( a_1, a_2, \ldots, a_n \), let \( \text{lcm}(a_1, a_2, \ldots, a_n) \) denote the least common multiple of \( a_1, a_2, \ldots, a_n \). Given a number \( x \in \mathbb{R} \), denote by \( \lfloor x \rfloor \) the greatest integer less than or equal to \( x \) and by \( \lceil x \rceil \) the least integer greater than or equal to \( x \).

3. Machine Transformations of Power-Characteristic \( \omega \)-Words

Definition 3. We will call the \( \omega \)-word \( \xi x \in \{0,1\}^\omega \) the characteristic word of the power \( \xi \) if

\[
\xi x(n) = \begin{cases} 1, & \text{if } \exists k \in \mathbb{N}, n = k^\xi, \\ 0, & \text{otherwise}. \end{cases}
\]

For example, \( \xi x = 110010000100 \ldots \) is the characteristic word of the squares.

Convention. Henceforth, we assume that \( \xi \geq 2 \) and it is a natural number.

More generally, let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be any total increasing function; then

\[
f(x) = \begin{cases} 1, & \text{if } \exists k \in \mathbb{N}, n = f (k), \\ 0, & \text{otherwise}. \end{cases}
\]

Let \( V = \langle Q, \{0,1\}, \{0,1\}; *, * \rangle \) be a Mealy machine, where

\[
Q = \{q_1, q_2, \ldots, q_{|Q|}\}.
\]

Applying the pigeonhole principle, we can state that for every \( q \in Q \) there is a least integer \( i \geq 0 \) such that \( q \circ 0^i = q \circ 0^j \) for some \( i < j \). The integer \( i \) is called the index of \( q \), and \( j - i \) is called the period of \( q \). We can visualize this as the diagrams (see Figure 1).

Claim 1. If \( a_i \) is the index and \( c_i \) is the period of \( d_i \),

\[
a_i \leq m_1 < m_2, \quad m_1 \equiv m_2 \quad (\text{mod} \ c_i),
\]

then

\[
d_i \circ 0^{m_1} = d_i \circ 0^{m_2}.
\]

Claim 2. If \( \max(a_1, a_2, \ldots, a_b) \leq m_1 < m_2 \) and \( m_1 \equiv m_2 \pmod{m} \), then

\[
m = \text{lcm}(c_i, c_2, \ldots, c_b),
\]
then
\[ \forall q \in Q, \quad q \circ 1^m = q \circ 1^{m_1}. \] (11)

Let \( \alpha(X) \) be an integer polynomial; that is, \( \alpha(X) \in \mathbb{Z}[X] \). The following theorem is known from elementary number theory.

**Theorem 4.** If \( i \equiv j \pmod{m} \), then \( \alpha(i) \equiv \alpha(j) \pmod{m} \).

If we take \( \alpha_k = (k+1)^k - k^k - 1 \), then we can express
\[ \zeta x = 110^a \cdot 10^b \cdots 10^{a_k} \cdots = u_0 u_1 \cdots u_{k-1}, \] (12)
where \( u_k = 10^{a_k} \). Hence, \( \zeta x[0,k] = u_0 u_1 \cdots u_{k-1} = w_{k-1} \).

**Corollary 5.** If \( i \equiv j \pmod{m} \), then \( \alpha_i \equiv \alpha_j \pmod{m} \).

Let
\[ w_0 = u_0, \quad w_{k+1} = w_k u_{k+1}. \] (13)

Then
\[ \zeta x[0,k] = u_0 u_1 \cdots u_{k-1} = w_{k-1}. \] (14)

Let \( q \in Q \). We define a sequence
\[ q_0, q_1, \ldots, q_k, \ldots, \] (15)
where \( q_k = q \circ w_k \).

**Corollary 6.** The sequence \( q_0, q_1, \ldots, q_k, \ldots \) is ultimately periodic.

**Proof.** Let \( m = \text{lcm}(c_1, c_2, \ldots, c_q) \). There exists \( n \) such that
\[ |u_{mn}| > \max(a_1, a_2, \ldots, a_q). \] (16)

Now consider the sequence \( q_{mn}, q_{m(n+1)}, \ldots, q_{m(n+b)} \). Since \( |0,b| = b + 1 > |Q| \), then—by the pigeonhole principle—there must exist two equal states
\[ q_{m(n+i)} = q_{m(n+j)}, \quad 0 \leq i < j \leq b. \] (17)

Hence
\[ q_{m(n+i)+1} = q_{m(n+i)} \circ u_{m(n+i)+1} = q_{m(n+i)} \circ u_{m(n+j)+1} \] (18)

Claim 2.

The rest follows by induction. \( \square \)

The following lemma is very easy, but it turns out to be useful.

**Lemma 7.** Let \( V = \langle Q, A, B; q_0 \rangle \) be a Mealy machine. If \( |Q| = m \) and \( q_0 \circ q' = w \), then there exists \( \nu \) such that
\[ w = u \nu' \nu, \quad \text{where } |u| + |\nu| \leq m \text{ and } \nu' \in \text{Pref}(\nu). \] (19)

**Proof.** (i) If \( s \leq m \), then \( |w| = |0'| = s \leq m \), and we can choose \( u = w, v = \nu = \lambda \).

(ii) Let \( s > m \) and \( q_0, q_1, \ldots, q_m \) states, where
\[ \forall i \in \overline{0,m}, \quad q_i = q_0 \circ \nu. \] (20)

Since \( |0, m| = m + 1 > |Q| \), then—by the pigeonhole principle—there must exist two equal states; namely, there exist \( i \) and \( j \), \( 0 \leq i < j \leq m \), such that
\[ q_i = q_j. \] (21)

Putting
\[ u = q_0 \circ q^i, \quad v = q_i \circ q^{j-i}, \quad \nu = \begin{bmatrix} s - i \\ j - i \end{bmatrix}, \quad \nu' = q_i \circ q^{j-i}, \] (22)

then \( |u| + |\nu| = |0'| + |0| = s \leq m \) and \( w = u \nu' \nu \).

**Proposition 8.** If \( f : x \rightarrow y \), \( \zeta x \rightarrow y \) and
\[ \forall \nu \exists \alpha f(\nu) \leq (a + \alpha) \nu \leq f(k + 1), \] (23)
then \( y \) is ultimately periodic.

**Proof.** Since \( \zeta x \rightarrow y \) and \( \zeta x \rightarrow y \), there exist Mealy machines
\[ V = \langle Q, [0,1], B; q_0, \circ, * \rangle, \quad V' = \langle Q', [0,1], B; q_1, \circ, * \rangle \] (24)

such that \( \zeta x \rightarrow y \) and \( f : x \rightarrow y' \).

(i) First, we express \( \zeta x \rightarrow y \) and \( f : x \rightarrow y' \).

(ii) By assumption (see (23)) we can choose integers \( k \) and \( a \) such that
\[ f(k) \leq a^k < (a + T^1) \nu \leq f(k + 1) \] (27)

and, moreover, \( a > p + 7 \) and \( (a + 1)^k - a^k > 3 \max(|Q|, |Q'|) + 7. \) Now, \( f(x(f, k + 1)) \) is a word of the form \( 0^i \), and thus
(by Lemma 7)
\[ y(f(k), f(k + 1)) \] (28)

must be ultimately periodic with both its period and preperiod not greater than \( |Q'| \). We denote this preperiod by \( p' \) and the least period by \( T' \). Since
\[ p' \leq |Q'| < (a + 1)^k - a^k - 7, \] (29)
then \( y[(a + 1) \xi, (a + 1 + T) \xi] \) is periodic with the period \( T' \). Notice that
\[
y[(a + i) \xi, (a + i + 1) \xi] = q_{a+1} * 10^{a+i+1} \xi
\]
and that the sequence of states \( q_{a}, q_{a+1}, q_{a+2}, \ldots, q_{a+n}, \ldots \) is also periodic. Therefore
\[
y[(a + i + T) \xi, (a + i + 1 + T) \xi] = q_{a+i+1} * 10^{a+i+1+T} \xi
\]
where \( |\dot{u}| = |\dot{v}| = \max(|Q|, |Q'|) \). So we have two periodic words \( \dot{u} \dot{v} \) and \( \dot{w} \dot{w} \). Hence by Corollary 2
\[
y[(a + i + T) \xi, (a + i + 1 + T) \xi] = \mu \mu^{-1} X^{-j} \xi
\]
where
\[
\mu = \sum_{j=1}^{\xi} \mu^{-1} X^{-j} \xi
\]
We have shown in (ii) that \( y[X \xi, (X + 1) \xi] \) is periodic. Therefore there is a \( \nu \) such that
\[
y[X \xi, (X + 1) \xi] = \nu^{1} \nu',
\]
where \( |\nu| = T' \) and \( \nu' \in \text{Pref}(\nu) \). Since \( T \) divides \( \mu \), then \( q_{X-1} = q_{X-1+\nu} \mu \). But then
\[
y[(X + \mu) \xi, (X + \mu + 1) \xi] = q_{X+\mu-1} * \xi X[(X + \mu) \xi, (X + \mu + 1) \xi] = q_{X+\mu-1} * 10^{a+\mu+1} \xi
\]
and therefore \( \nu' = \nu'' \).

(iv) Finally, we can select integers \( \bar{k}, \bar{a} \) such that \( k < \bar{k} \) and
\[
f(\bar{k}) \leq \bar{a} < (\bar{a} + \mu + 1) \xi \leq f(\bar{k} + 1)
\]
Now we repeat the proof from (ii). So we can conclude that there is the least period \( T'' \leq |Q'| \) of the word \( y[(\bar{a} + 1) \xi, (\bar{a} + 1 + \mu) \xi] \). A period of
\[
y[(\bar{a} + 1) \xi, (\bar{a} + 2) \xi]
\]
is \( T'' \) too. Hence (Theorem 1) \( T'' = T' \). Denote \( y[(\bar{a} + 1) \xi, (\bar{a} + 1) \xi + T'] = u \). As it was shown in (iii) we can choose \( \Sigma_{1}, \Sigma_{2} \) such that
\[
y[(\bar{a} + 1) \xi, (\bar{a} + 2) \xi] = u^{1} \nu' u',
\]
But then \( y[(\bar{a} + 1) \xi, (\bar{a} + 2 + \mu) \xi] = u^{1} \nu' u' \), which means that
\[
y[(\bar{a} + 1) \xi, (\bar{a} + 2 + \mu) \xi]
\]
is periodic with the period \( T'' \). Now suppose that \( y[(\bar{a} + 1) \xi, (\bar{a} + \mu) \xi] = u \nu' \), where \( n > \mu + 1 \) and \( \nu \in \text{Pref}(\nu) \). Then there exists such \( \nu' \) that \( \nu' u' = u \). From (see formula (33)) \( (\bar{a} + n) \xi - (\bar{a} + n - \mu) \xi \equiv 0 \pmod{T'} \), we can conclude that
\[
\nu' \in \text{Pref}(y[(\bar{a} + n - \mu) \xi, (\bar{a} + n - \mu + 1) \xi])
\]
It follows from what we have shown in (iii) that there are \( \Sigma_{1}, \Sigma_{2} \) such that
\[
y[(\bar{a} + n - \mu) \xi, (\bar{a} + n - \mu + 1) \xi] = \nu'' \nu',
\]
with \( |\nu| = T' \) and \( \nu' \in \text{Pref}(\nu) \). But then \( \nu' = \nu'' u' \) and
\[
y[(\bar{a} + 1) \xi, (\bar{a} + n + 1) \xi] = u^{\bar{a}} \nu (\bar{a}) u^{\bar{a}} \nu' = u^{\bar{a}} \nu (\bar{a}) u^{\bar{a}} \nu'
\]
This means that \( y[(\bar{a} + 1) \xi, (\bar{a} + n + 1) \xi] \) is periodic with period \( T'' \). Now, by induction, we have \( y[(\bar{a} + 1) \xi, (\bar{a} + i) \xi] \) is periodic with the period \( T'' \) for any \( i > 1 \). Hence, \( y \) is ultimately periodic.

\[
4. \text{Modularity in the Semilattice of } \omega\text{-Words}
\]
Our main object of investigation is the machine poset of infinite words. In order to avoid some set-theoretical problems, we make some assumptions. Let us take the set
\[ \mathcal{R} = \bigcup_{k=0}^{\infty} (0, k)^\infty. \]

We shall assume that the states of the involved Mealy machines as well as their input and output alphabets all are from the set \( \mathbb{N} \). If another input or output alphabet \( A \) is used, we assume that there exists a bijection \( \beta : A \rightarrow \mathbb{N} \), and that this bijection is applied to the input or output word, respectively.

We suppose that the reader is familiar with the basic notions of ordered sets [7]. If \( \rightarrow \) is used as an algebraic relation on \( \mathcal{R} \), then the algebraic structure \( \langle \mathcal{R}, \rightarrow \rangle \) defines a preorder [5], while the quotient set \( \overline{\mathcal{R}} = \mathcal{R}/\sim \) becomes the ordered set \( \langle \overline{\mathcal{R}}, \rightarrow \rangle \). It has been shown that this poset \( \overline{\mathcal{R}} \) is a join-semilattice [5], where the join \( \langle [x_i] \rangle \vee \langle [y_j] \rangle = \langle [x_i, y_j] \rangle \).

**Definition 9.** A join-semilattice \( \langle D, \leq \rangle \) is distributive when

\[ \forall xab \left( x \leq a \lor b \Rightarrow \exists a'b' \left( a' \leq a \& b' \leq b \& x = a' \lor b' \right) \right). \]  
(46)

A join-semilattice \( \langle D, \leq \rangle \) is modular when

\[ \forall xab \left( a \leq x \leq a \lor b \Rightarrow \exists b' \left( x = a \lor b' \right) \right). \]  
(47)

**Theorem 10.** The join-semilattice \( \langle \overline{\mathcal{R}}, \rightarrow \rangle \) is not modular.

**Proof.** We start by showing that \( ^2 x \lor ^4 x \rightarrow x' \), where

\[ x'(n) = \begin{cases} 
1, & \text{if } \exists k \in \mathbb{N}, n = k^2, \\
1, & \text{if } \exists k \in \mathbb{N}, n = (k^2 + 1)^2, \\
0, & \text{otherwise}.
\end{cases} \]  
(48)

By definition, \( ^2 x \lor ^4 x(n) = ^2 x(n), ^4 x(n) \). Define the Mealy machine

\[ V = \langle \{ q_0, q_1, q_2 \}, \{ (0, 0), (0, 1), (1, 1) \}, \{ 0, 1 \}, q_0, \circ, \ast \rangle \]  
(49)

by

\[ q_1 = q_0 \circ (0) = q_0 \circ (1) = q_0 \circ (1) = q_0 \circ (1), \]
\[ q_2 = q_1 \circ (1) = q_2 \circ (0) = q_2 \circ (1) = q_2 \circ (1), \]
\[ 0 = q_1 \ast (0) = q_1 \ast (1) = q_1 \ast (0), \]
\[ = q_2 \ast (0) = q_2 \ast (1) = q_2 \ast (1), \]
\[ 1 = q_0 \ast (0) = q_0 \ast (1) = q_0 \ast (0), \]
\[ = q_0 \ast (1) = q_1 \ast (1) = q_2 \ast (1). \]  
(50)

We illustrate this by the diagram in Figure 2. It follows straightforwardly from the construction that \( ^2 x \lor ^4 x \rightarrow x' \). Now suppose that there exists \( y \) such that \( ^2 x \rightarrow y \) and \( x' \equiv ^4 x \). Then \( x' \rightarrow y \) too. Notice that \( x' = g^n x \) for

\[ g(k) = \begin{cases} 
0, & \text{if } k = 0, \\
(k + 1)^4, & \text{if } k \text{ is odd}, \\
(k^2 + 1)^2, & \text{if } k \text{ is even}.
\end{cases} \]  
(51)

Hence, by Proposition 8 by is ultimately periodic. But if so, then \( ^4 x \lor y \equiv ^4 x \). However this is a contradiction because then \( x' = ^4 x \).

**Corollary 11** (see [8]). The join-semilattice \( \langle \overline{\mathcal{R}}, \rightarrow \rangle \) is not distributive.

We recall that every distributive join-semilattice is modular.

**5. Conclusion**

We agree that it is not sufficient to state that \( \overline{\mathcal{R}} \) is very complicated as it was done in Wikipedia writing review about Turing degrees [9]. Nevertheless it is clear that \( \overline{\mathcal{R}} \) is not modular and that there is no classical algebraic notion that describes such semilattice axiomatically at this moment. So we have challenge to extract axioms for semilattice \( \overline{\mathcal{R}} \).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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