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Research Article

Modularity in the Semilattice of ω-Words

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A partial ordering of ω-words can be introduced with regard to whether an ω-word can be transformed into another by a Mealy machine. It is known that the poset of ω-words that is introduced by this ordering is a join-semilattice. The width of this join-semilattice has the power of continuum while the depth is at least $\aleph_0$. We have created a technique for proving that power-characteristic ω-words are incomparable. We use this technique to show that this join-semilattice is not modular.

1. Introduction

Infinite words (ω-words) provide a fertile ground of research and many classes of ω-words are known. While closure properties of some classes of ω-words have been studied extensively (see, e.g., [1–3]), we are interested in the general algebraic structure of ω-words. Mealy machines are a simple model of a word transforming automaton with the beneficial property of always transforming an ω-word into an ω-word. A partial ordering is defined on ω-words by the existence of a Mealy machine transforming one word into another (we write $x \to y$ if such a machine exists). When both $x \to y$ and $y \to x$ are true, we say that x and y are machine equivalent. A class $\mathfrak{K}$ of ω-words is called machine invariant if $\mathfrak{K}$ contains all possible transformations of its elements.

Buls [4] has shown that machine invariant classes of ω-words form a completely distributive lattice. Belovs [5] showed that the poset of machine equivalent classes of ω-words is a join-semilattice and that the width of this join-semilattice has the power of continuum while the depth is at least $\aleph_0$. We show in this paper that this join-semilattice is not modular.

2. Preliminaries

Let A be a finite, nonempty set and A^* the free monoid generated by A. Call A an *alphabet*, its elements *letters*, and the elements of A^* *finite words*. The identity element of A^* is called the *empty word* and is denoted by λ.

Let w be a word over a finite and non-empty alphabet A. We denote the length of w by $|w|$. Similarly, the cardinality of A is denoted by $|A|$. The concatenation of the words u, v is denoted by $u\#v$ or simply uv. Define $w^0 = \lambda$ and for all i $w^{i+1} = w^i w$. We say that u (resp., v) is *a prefix* (resp., *suffix*) of uv. Denote by $\mathrm{Pref}(w)$ and $\mathrm{Suff}(w)$ the respective sets of prefixes and suffixes of w. Let $\overline{k,n}$ denote the set of integers $\{k, k+1, k+2, \ldots, n\}$:

$$\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}. \tag{1}$$

Call a total map $x : \mathbb{N} \to A$ an (indexed) *ω-word* of the alphabet A. For any $i \geq 0$ define $x_i = x(i)$ and write

$$x = (x_i) = x_0 x_1 \cdots x_n \cdots$$
$$x(k+1, n+1) = x[k, n+1) = x[k, n] = x_k x_{k+1} \ldots x_{n-1} x_n. \tag{2}$$

The set of all ω-words over A is denoted by A^ω. We say that y is a *prefix* of x, if there exists an integer k such that $y = x[0, k]$. An ω-word x is called *ultimately periodic* if there exist integers $p \geq 0$ and $T > 0$ such that $x_i = x_{i+T}$ for all $i \geq p$. In this case p is called *preperiod* and T a *period* of x. An ultimately periodic ω-word with a preperiod of zero is called *periodic*. We say a finite word w is *(ultimately) periodic* if it is a prefix of some (ultimately) periodic ω-word. We recall the important periodicity theorem due to Fine and Wilf [6].

Theorem 1. *Let w be a word having periods p and q and denote by $gcd(p,q)$ the greatest common divisor of p and q. If $|w| \geq p+q-gcd(p,q)$, then w has also the period $gcd(p,q)$.*

Corollary 2. *Let uv and vw be words having periods, respectively, p and q. If $|v| \geq p+q-gcd(p,q)$, then uvw has the period $gcd(p,q)$.*

This is almost folklore in combinatorics on words. Nevertheless, for the sake of completeness, we will give the proof of this corollary.

Proof. Since $|v| \geq p+q-\gcd(p,q)$, then (Theorem 1) v has the period $\gcd(p,q)$.

(i) At first we will prove by induction on $|u|$ that uv has the period $\gcd(p,q)$. Let $v = v_1 v_2 \cdots v_k$. If $|u| = 1$, then $u = v_0$ for some letter. Since the period of uv is p then $v_0 = v_p$.

Notice that $\gcd(p,q)$ divides p. Since the period of v is $\gcd(p,q)$, then $v_p = v_{\gcd(p,q)}$. Hence, $v_0 = v_{\gcd(p,q)}$.

(ii) If $|u| > 1$, then u can be represented as concatenation $au' = u$, where $|u'| + 1 = |u|$. By assumption the period of $u'v$ is $\gcd(p,q)$. Now from (i) follows that the period of $au'v$ is $\gcd(p,q)$ too. We have completed the inductive step.

Now we shall prove by induction on $|w|$ that vw has the period $\gcd(p,q)$.

(iii) If $|w| = 1$, then $w = v_{k+1}$ for some letter. Since the period of vw is q, then $v_{k+1-q} = v_{k+1}$. Notice

$$k+1-\gcd(p,q) \equiv k+1-q, \quad (\operatorname{mod} \gcd(p,q)). \tag{3}$$

Hence, $v_{k+1-\gcd(p,q)} = v_{k+1-q} = v_{k+1}$.

(iv) If $|w| > 1$, then w can be represented as concatenation $w'b = w$, where $|w'| + 1 = |w|$. By assumption the period of vw' is $\gcd(p,q)$. Now from (iii) follows that the period of $vw'b$ is $\gcd(p,q)$ too. We have completed the inductive step. □

A 3-sorted algebra $V = \langle Q, A, B; q_0, \circ, * \rangle$ is called an *initial Mealy machine* if Q, A, B are finite, nonempty sets, called the *set of states*, the *input alphabet,* and the *output alphabet*, respectively, $q_0 \in Q$ is called the *initial state*, $\circ : Q \times A \to Q$ is a total function called the *transition function,* and $* : Q \times A \to B$ is a total surjective function called the *output function*. We write $\langle Q, A, B; \circ, * \rangle$ or even $\langle Q, A, B; q_0 \rangle$ if there is no danger of confusion. The mappings $\circ$ and $*$ are extended to $Q \times A^*$ by defining

$$\begin{aligned} q \circ \lambda &= q, \qquad q \circ (ua) = (q \circ u) \circ a, \\ q * \lambda &= \lambda, \qquad q * (ua) = (q * u) \# ((q \circ u) * a), \end{aligned} \tag{4}$$

for all $q \in Q$, $(u,a) \in A^* \times A$. Henceforth, we shall omit parentheses if there is no danger of confusion. So, for example, we will write $q \circ u * a$ instead of $(q \circ u) * a$. Let $(x,y) \in A^\omega \times B^\omega$. If for some Mealy machine V: for all n $y[0,n] = q_0 * x[0,n]$, we say that V *transforms x to y* and write $y = q_0 * x$ or $x \overset{V}{\to} y$. We write $x \to y$ if there exists V such that $x \overset{V}{\to} y$; otherwise, we write $x \not\to y$. We write $x \rightleftarrows y$ if $x \to y$ and $y \to x$ and say that x and y are machine equivalent.

Given the integers $a_1, a_2, \ldots, a_n$, let $\operatorname{lcm}(a_1, a_2, \ldots, a_n)$ denote the least common multiple of $a_1, a_2, \ldots, a_n$. Given a number $x \in \mathbb{R}$, denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x and by $\lceil x \rceil$ the least integer greater than or equal to x.

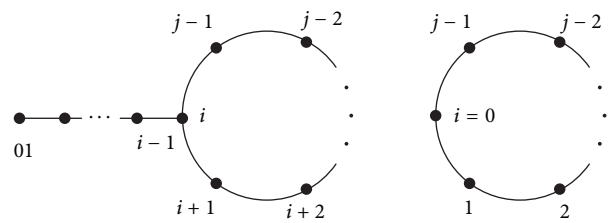

Figure 1: Indices and periods.

3. Machine Transformations of Power-Characteristic ω-Words

Definition 3. We will call the ω-word ${}^{\zeta}x \in \{0,1\}^\omega$ the characteristic word of the power ζ if

$${}^{\zeta}x(n) = \begin{cases} 1, & \text{if } \exists k \in \mathbb{N},\ n = k^\zeta, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For example, ${}^{2}x = 110010000100\ldots$ is the characteristic word of the squares.

Convention. Henceforth, we assume that $\zeta \geq 2$ and it is a natural number.

More generally, let $f : \mathbb{N} \to \mathbb{N}$ be any total increasing function; then

$${}^{f}x(n) = \begin{cases} 1, & \text{if } \exists k \in \mathbb{N},\ n = f(k), \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Let $V = \langle Q, \{0,1\}, \{0,1\}; \circ, * \rangle$ be a Mealy machine, where

$$Q = \{q_1', q_2', \ldots, q_b'\}. \tag{7}$$

Applying the pigeonhole principle, we can state that for every $q \in Q$ there is a least integer $i \geq 0$ such that $q \circ 0^i = q \circ 0^j$ for some $i < j$. The integer i is called the *index* of q, and $j - i$ is called the *period* of q. We can visualize this as the diagrams (see Figure 1).

Claim 1. If a_s is the index and c_s is the period of q_s',

$$a_s \leq m_1 < m_2, \quad m_1 \equiv m_2 \quad (\operatorname{mod} c_s), \tag{8}$$

then

$$q_s' \circ 0^{m_1} = q_s' \circ 0^{m_2}. \tag{9}$$

Claim 2. If $\max(a_1, a_2, \ldots, a_b) \leq m_1 < m_2$ and $m_1 \equiv m_2$ $(\operatorname{mod} m)$, where

$$m = \operatorname{lcm}(c_1, c_2, \ldots, c_b), \tag{10}$$

then

$$\forall q \in Q, \quad q \circ 10^{m_1} = q \circ 10^{m_2}. \tag{11}$$

Let $\alpha(X)$ be an integer polynomial; that is, $\alpha(X) \in \mathbb{Z}[X]$. The following theorem is known from elementary number theory.

Theorem 4. *If $i \equiv j \pmod m$, then $\alpha(i) \equiv \alpha(j) \pmod m$.*

If we take $\alpha_k = (k+1)^\zeta - k^\zeta - 1$, then we can express

$$^\zeta x = 110^{\alpha_1} 10^{\alpha_2} \cdots 10^{\alpha_k} \cdots = u_0 u_1 \cdots u_k \cdots, \tag{12}$$

where $u_k = 10^{\alpha_k}$. Hence, $^\zeta x[0, k^\zeta) = u_0 u_1 \cdots u_{k-1}$.

Corollary 5. *If $i \equiv j \pmod m$, then $\alpha_i \equiv \alpha_j \pmod m$.*

Let

$$\begin{aligned} w_0 &= u_0, \\ w_{k+1} &= w_k u_{k+1}. \end{aligned} \tag{13}$$

Then

$$^\zeta x\left[0, k^\zeta\right) = u_0 u_1 \cdots u_{k-1} = w_{k-1}. \tag{14}$$

Let $q \in Q$. We define a sequence

$$q_0, q_1, \ldots, q_k, \ldots, \tag{15}$$

where $q_k = q \circ w_k$.

Corollary 6. *The sequence $q_0, q_1, \ldots, q_k, \ldots$ is ultimately periodic.*

Proof. Let $m = \operatorname{lcm}(c_1, c_2, \ldots, c_b)$. There exists n such that

$$\left|u_{mn}\right| > \max\left(a_1, a_2, \ldots, a_b\right). \tag{16}$$

Now consider the sequence $q_{mn}, q_{m(n+1)}, \ldots, q_{m(n+b)}$. Since $|\overline{0, b}| = b+1 > |Q|$, then—by the pigeonhole principle—there must exist two equal states

$$q_{m(n+i)} = q_{m(n+j)}, \quad \text{with } 0 \le i < j \le b. \tag{17}$$

Hence

$$\begin{aligned} q_{m(n+i)+1} &= q_{m(n+i)} \circ u_{m(n+i)+1} = q_{m(n+j)} \circ u_{m(n+i)+1} \\ &\underset{\text{Claim 2}}{=} q_{m(n+j)} \circ u_{m(n+j)+1} = q_{m(n+j)+1}. \end{aligned} \tag{18}$$

The rest follows by induction. □

The following lemma is very easy, but it turns out to be useful.

Lemma 7. *Let $V = \langle Q, A, B; q_0\rangle$ be a Mealy machine. If $|Q| = m$ and $q_0 * 0^s = w$, then there exists $\varkappa$ such that*

$$w = uv^\varkappa \dot{v}, \quad \textit{where } |u| + |v| \le m \textit{ and } \dot{v} \in \textit{Pref}(v). \tag{19}$$

Proof. (i) If $s \le m$, then $|w| = |0^s| = s \le m$, and we can choose $u = w$, $v = \dot{v} = \lambda$.

(ii) Let $s > m$ and $q_0, q_1, \ldots, q_m$ states, where

$$\forall i \in \overline{0, m} \quad q_i = q_0 \circ 0^i. \tag{20}$$

Since $|\overline{0, m}| = m+1 > |Q|$, then—by the pigeonhole principle—there must exist two equal states; namely, there exist i and j, $0 \le i < j \le m$, such that

$$q_i = q_j. \tag{21}$$

Putting

$$\begin{aligned} u &= q_0 * 0^i, \qquad v = q_i * 0^{j-i}, \\ \varkappa &= \left\lfloor \frac{s-i}{j-i} \right\rfloor, \qquad \dot{v} = q_i * 0^{s-i-\varkappa(j-i)}, \end{aligned} \tag{22}$$

then $|u| + |v| = |0^i| + |0^{j-i}| = i + (j-i) = j \le m$ and $w = uv^\varkappa \dot{v}$. □

Proposition 8. *If $^f x \to y$, $^\zeta x \to y$ and*

$$\forall \varkappa\, \exists ak \quad f(k) \le a^\zeta < (a+\varkappa)^\zeta \le f(k+1), \tag{23}$$

then y is ultimately periodic.

Proof. Since $^\zeta x \to y$ and $^f x \to y$, there exist Mealy machines

$$V = \langle Q, \{0,1\}, B; q, \circ, *\rangle, \qquad V' = \left\langle Q', \{0,1\}, B; q', \circ', *'\right\rangle \tag{24}$$

such that $^\zeta x \xrightarrow{V} y$ and $^f x \xrightarrow{V'} y$.

(i) First, we express $^\zeta x = u_0 u_1 \cdots u_n \cdots$ with

$$u_n = 10^{\alpha_n}, \qquad \alpha_n = (n+1)^\zeta - n^\zeta - 1, \tag{25}$$

and look at the sequence $q_0, q_1, \ldots, q_n, \ldots$, where

$$q_n = q \circ \left(u_0 u_1 \cdots u_n\right). \tag{26}$$

We have shown (Corollary 6) that the sequence $q_0, q_1, \ldots, q_n, \ldots$ is ultimately periodic. Assume its period is T and the preperiod p.

(ii) By assumption (see (23)) we can choose integers k and a such that

$$f(k) \le a^\zeta < (a+T+1)^\zeta \le f(k+1) \tag{27}$$

and, moreover, $a > p+7$ and $(a+1)^\zeta - a^\zeta > 3 \cdot \max(|Q|, |Q'|) + 7$. Now, $^f x(f(k), f(k+1))$ is a word of the form 0^d, and thus (by Lemma 7)

$$y\left(f(k), f(k+1)\right) \tag{28}$$

must be ultimately periodic with both its period and preperiod not greater than $|Q'|$. We denote this preperiod by p' and the least period by T'. Since

$$p' \le \left|Q'\right| < (a+1)^\zeta - a^\zeta - 7, \tag{29}$$

then $y[(a+1)^\zeta,(a+1+T)^\zeta)$ is periodic with the period T'. Notice that

$$y\left[(a+i)^\zeta,(a+i+1)^\zeta\right) = q_{a+i-1} * {}^\zeta x\left[(a+i)^\zeta,(a+i+1)^\zeta\right) = q_{a+i-1} * 10^{\alpha_{a+i}} \tag{30}$$

and that the sequence of states $q_a, q_{a+1}, q_{a+2}, \ldots, q_{a+T}, \ldots$ is also periodic. Therefore

$$\begin{aligned} y\left[(a+i+T)^\zeta,(a+i+1+T)^\zeta\right) &= q_{a+i-1+T} * {}^\zeta x\left[(a+i+T)^\zeta,(a+i+1+T)^\zeta\right) \\ &= q_{a+i-1+T} * 10^{\alpha_{a+i+T}} = q_{a+i-1} * 10^{\alpha_{a+i+T}} = \dot{u}\dot{v}\dot{w}, \end{aligned} \tag{31}$$

where $|\dot{u}| = |\dot{v}| = \max(|Q|,|Q'|)$. So we have two periodic words $\dot{u}\dot{v}$ and $\dot{v}\dot{w}$. Hence by Corollary 2

$$y\left[(a+i+T)^\zeta,(a+i+1+T)^\zeta\right) \tag{32}$$

is periodic with period T'. Therefore we can conclude that $y[(a+i)^\zeta,(a+i+1)^\zeta)$ is periodic with period T' for all $i>0$; besides, T' is the least period for all $i>0$.

(iii) Let $X>a$ and $\mu = \operatorname{lcm}(T,T')$. Then

$$\begin{aligned} (X+\mu)^\zeta - X^\zeta &= \sum_{j=0}^{\zeta}\binom{\zeta}{j}\mu^j X^{\zeta-j} - X^\zeta \\ &= \mu\sum_{j=1}^{\zeta}\binom{\zeta}{j}\mu^{j-1}X^{\zeta-j} = \mu P(X), \end{aligned} \tag{33}$$

where

$$P(X) = \sum_{j=1}^{\zeta}\binom{\zeta}{j}\mu^{j-1}X^{\zeta-j}. \tag{34}$$

We have shown in (ii) that $y[X^\zeta,(X+1)^\zeta)$ is periodic. Therefore there is a v such that

$$y\left[X^\zeta,(X+1)^\zeta\right) = v^r v', \tag{35}$$

where $|v| = T'$ and $v' \in \operatorname{Pref}(v)$. Since T divides μ, then $q_{X-1} = q_{X-1+\mu}$. But then

$$\begin{aligned} y\left[(X+\mu)^\zeta,(X+\mu+1)^\zeta\right) &= q_{X+\mu-1} * {}^\zeta x\left[(X+\mu)^\zeta,(X+\mu+1)^\zeta\right) \\ &= q_{X-1} * {}^\zeta x\left[(X+\mu)^\zeta,(X+\mu+1)^\zeta\right) \\ &= q_{X-1} * 10^{\alpha_{X+\mu}} = v^{r'}v'', \end{aligned} \tag{36}$$

for some number r' and $v'' \in \operatorname{Pref}(v)$. It follows from (33) that

$$(X+\mu+1)^\zeta - (X+\mu)^\zeta \equiv (X+1)^\zeta - X^\zeta \pmod{T'} \tag{37}$$

and therefore $v' = v''$.

(iv) Finally, we can select integers $\check{k}, \check{a}$ such that $k < \check{k}$ and

$$f(\check{k}) \le \check{a}^\zeta < (\check{a}+\mu+1)^\zeta \le f(\check{k}+1). \tag{38}$$

Now we repeat the proof from (ii). So we can conclude that there is the least period $T'' \le |Q'|$ of the word $y[(\check{a}+1)^\zeta, (\check{a}+1+\mu)^\zeta)$. A period of

$$y\left[(\check{a}+1)^\zeta,(\check{a}+2)^\zeta\right) \tag{39}$$

is T' too. Hence (Theorem 1) $T'' = T'$. Denote $y[(\check{a}+1)^\zeta, (\check{a}+1)^\zeta + T') = u$. Since (from formula (33))

$$(\check{a}+1+\mu)^\zeta - (\check{a}+1)^\zeta \equiv 0 \pmod{T'}, \tag{40}$$

there is an integer s such that $y[(\check{a}+1)^\zeta,(\check{a}+1+\mu)^\zeta) = u^s$. As it was shown in (iii) we can choose s_1', s_2' such that

$$\begin{aligned} y\left[(\check{a}+1)^\zeta,(\check{a}+2)^\zeta\right) &= u^{s_1'}u', \\ y\left[(\check{a}+1+\mu)^\zeta,(\check{a}+2+\mu)^\zeta\right) &= u^{s_2'}u'. \end{aligned} \tag{41}$$

But then $y[(\check{a}+1)^\zeta,(\check{a}+2+\mu)^\zeta) = u^s u^{s'_2}u'$, which means that

$$y\left[(\check{a}+1)^\zeta,(\check{a}+2+\mu)^\zeta\right) \tag{42}$$

is periodic with the period T'. Now suppose that $y[(\check{a}+1)^\zeta, (\check{a}+n)^\zeta) = u^\sigma\check{u}$, where $n>\mu+1$ and $\check{u} \in \operatorname{Pref}(u)$. Then there exists such $\check{v} \in \operatorname{Suff}(u)$ that $\check{u}\check{v} = u$. From (see formula (33)) $(\check{a}+n)^\zeta - (\check{a}+n-\mu)^\zeta \equiv 0 \pmod{T}'$, we can conclude that

$$\check{v} \in \operatorname{Pref}\left(y\left[(\check{a}+n-\mu)^\zeta,(\check{a}+n-\mu+1)^\zeta\right)\right). \tag{43}$$

It follows from what we have shown in (iii) that there are σ_1, σ_2 such that

$$\begin{aligned} y\left[(\check{a}+n-\mu)^\zeta,(\check{a}+n-\mu+1)^\zeta\right) &= v^{\sigma_1}v', \\ y\left[(\check{a}+n)^\zeta,(\check{a}+n+1)^\zeta\right) &= v^{\sigma_2}v' \end{aligned} \tag{44}$$

with $|v| = T'$ and $v' \in \operatorname{Pref}(v)$. But then $v = \check{v}\check{u}$ and

$$\begin{aligned} y\left[(\check{a}+1)^\zeta,(\check{a}+n+1)^\zeta\right) &= u^\sigma\check{u}v^{\sigma_2}v' = u^\sigma\check{u}(\check{v}\check{u})^{\sigma_2}v' \\ &= u^\sigma(\check{u}\check{v})^{\sigma_2}\check{u}v' = u^{\sigma+\sigma_2}\check{u}v'. \end{aligned} \tag{45}$$

This means that $y[(\check{a}+1)^\zeta,(\check{a}+n+1)^\zeta)$ is periodic with period T'. Now, by induction, we have $y[(\check{a}+1)^\zeta,(\check{a}+i)^\zeta]$ is periodic with the period T' for any $i>1$. Hence, y is ultimately periodic. □

4. Modularity in the Semilattice of ω-Words

Our main object of investigation is the machine poset of infinite words. In order to avoid some set-theoretical problems, we make some assumptions. Let us take the set

$\mathfrak{N} = \bigcup_{k=0}^{\infty} (\overline{0,k})^{\omega}$. We shall assume that the states of the involved Mealy machines as well as their input and output alphabets all are from the set $\mathbb{N}$. If another input or output alphabet A is used, we assume that there exists a bijection $\beta : A \to \overline{0, |A| - 1}$ and that this bijection is applied to the input or output word, respectively.

We suppose that the reader is familiar with the basic notions of ordered sets [7]. If $\rightharpoonup$ is used as an algebraic relation on $\mathfrak{N}$, then the algebraic structure $\langle \mathfrak{N}, \rightharpoonup \rangle$ defines a preorder [5], while the quotient set $\widetilde{\mathfrak{N}} = \mathfrak{N}/ \rightleftharpoons$ becomes the ordered set $\langle \mathfrak{N}, \rightharpoonup \rangle$. It has been shown that this poset $\widetilde{\mathfrak{N}}$ is a join-semilattice [5], where the join $[(x_i)] \vee [(y_i)] = [(x_i, y_i)]$.

Definition 9. A join-semilattice $\langle D, \leq \rangle$ is distributive when

$$\forall xab \left(x \leq a \vee b \Longrightarrow \exists a'b' \left(a' \leq a \,\&\, b' \leq b \,\&\, x = a' \vee b' \right) \right). \tag{46}$$

A join-semilattice $\langle D, \leq \rangle$ is modular when

$$\forall xab \left(a \leq x \leq a \vee b \Longrightarrow \exists b' \leq b \left(x = a \vee b' \right) \right). \tag{47}$$

Theorem 10. *The join-semilattice $\langle \widetilde{\mathfrak{N}}, \rightharpoonup \rangle$ is not modular.*

Proof. We start by showing that ${}^2x \vee {}^4x \rightharpoonup x'$, where

$$x'(n) = \begin{cases} 1, & \text{if } \exists k \in \mathbb{N},\ n = k^4, \\ 1, & \text{if } \exists k \in \mathbb{N},\ n = \left(k^2 + 1\right)^2, \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

By definition, $({}^2x \vee {}^4x)(n) = ({}^2x(n), {}^4x(n))$. Define the Mealy machine

$$V = \left\langle \{q_0, q_1, q_2\}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \{0, 1\}; q_0, \circ, * \right\rangle \tag{49}$$

by

$$\begin{aligned} q_1 &= q_0 \circ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = q_0 \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = q_0 \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = q_0 \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= q_1 \circ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = q_1 \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = q_1 \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = q_2 \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ q_2 &= q_1 \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = q_2 \circ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = q_2 \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = q_2 \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ 0 &= q_1 * \begin{pmatrix} 0 \\ 0 \end{pmatrix} = q_1 * \begin{pmatrix} 0 \\ 1 \end{pmatrix} = q_1 * \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= q_2 * \begin{pmatrix} 0 \\ 0 \end{pmatrix} = q_2 * \begin{pmatrix} 0 \\ 1 \end{pmatrix} = q_2 * \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ 1 &= q_0 * \begin{pmatrix} 0 \\ 0 \end{pmatrix} = q_0 * \begin{pmatrix} 0 \\ 1 \end{pmatrix} = q_0 * \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= q_0 * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = q_1 * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = q_2 * \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{50}$$

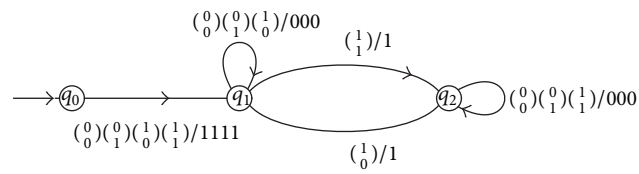

Figure 2: ${}^2x \vee {}^4x \overset{V}{\rightharpoonup} x'$.

We illustrate this by the diagram in Figure 2. It follows straightforwardly from the construction that ${}^2x \vee {}^4x \overset{V}{\rightharpoonup} x' \rightharpoonup {}^4x$. Now suppose that there exists y such that ${}^2x \rightharpoonup y$ and $x' \rightleftharpoons {}^4x \vee y$. But then $x' \rightharpoonup y$ too. Notice that $x' = {}^g x$ for

$$g(k) = \begin{cases} 0, & \text{if } k = 0, \\ \dfrac{(k+1)^4}{16}, & \text{if } k \text{ is odd}, \\ \left(\dfrac{k^2}{4} + 1 \right)^2, & \text{if } k \text{ is even}. \end{cases} \tag{51}$$

Hence, by Proposition 8 y is ultimately periodic. But if so, then ${}^4x \vee y \rightleftharpoons {}^4x$. However this is a contradiction because then $x' \rightleftharpoons {}^4x$. □

Corollary 11 (see [8]). *The join-semilattice $\langle \widetilde{\mathfrak{N}}, \rightharpoonup \rangle$ is not distributive.*

We recall that every distributive join-semilattice is modular.

5. Conclusion

We agree that it is not sufficient to state that $\widetilde{\mathfrak{N}}$ is very complicated as it was done in Wikipedia writing review about Turing degrees [9]. Nevertheless it is clear that $\widetilde{\mathfrak{N}}$ is not modular and that there is no classical algebraic notion that describes such semilattice axiomatically at this moment. So we have challenge to extract axioms for semilattice $\widetilde{\mathfrak{N}}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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