Research Article

The Diophantine Equation $8x^2 - y^2 + 8x(1+t) + (2t+1)^2 = 0$ and t -Balancing Numbers

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Let $t \geq 2$ be an odd integer such that $2t^2 - 1$ is a prime. In this work, we determine all integer solutions of the Diophantine equation $D^t : 8x^2 - y^2 + 8x(1+t) + (2t+1)^2 = 0$ and then we deduce the general terms of all t -balancing numbers.

1. Introduction

Balancing numbers were first considered by Behera and Panda in [1] when they considered the integer solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \quad (1)$$

for some positive integers n and r . In this case n is called a *balancing number* with *balancer* (or *cobalancing number*) r . For example, 6, 35, 204, 1189 and 6930 are balancing numbers with cobalancing numbers which are 2, 14, 84, 492 and 2870, respectively.

The n th balancing number is denoted by B_n and the n th cobalancing number is denoted by b_n . They satisfy the recurrence relations $B_{n+1} = 6B_n - B_{n-1}$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$, with initial values $B_1 = 1$, $B_2 = 6$ and $b_1 = 0$, $b_2 = 2$.

From (1), one has

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2}, \quad n = \frac{2r+1 + \sqrt{8r^2+8r+1}}{2}. \quad (2)$$

So B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Set $C_n = \sqrt{8B_n^2 + 1}$ and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$. Then C_n is called the n th *Lucas-balancing*

number and c_n is called the n th *Lucas-cobalancing number*. Binet formulas for balancing, cobalancing, Lucas-balancing, and Lucas-cobalancing numbers are $B_n = (\alpha^{2n} - \beta^{2n})/4\sqrt{2}$, $b_n = ((\alpha^{2n-1} - \beta^{2n-1})/4\sqrt{2}) - (1/2)$, $C_n = (\alpha^{2n} + \beta^{2n})/2$, and $c_n = (\alpha^{2n-1} + \beta^{2n-1})/2$, respectively, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ (for further details see also [2–7]).

Recently, there are many studies on balancing numbers. In [8], the authors generalized the theory of balancing numbers to numbers defined as follows. Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x such that $x \leq y-2$ is called a (k, l) -power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l. \quad (3)$$

Later in [9], the authors extended the concept of balancing numbers to the (a, b) -balancing numbers defined as follows. Let $a > 0$ and let $b \geq 0$ be coprime integers. If for some positive integers, n and r , one has

$$(a+b) + \cdots + (a(n-1)+b) = (a(n+1)+b) + \cdots + (a(n+r)+b), \quad (4)$$

then the number $an+b$ is called an (a, b) -balancing number and is denoted by $B_n^{(a,b)}$.

Like in (a, b) -balancing numbers, in [10], Dash et al. defined the t -balancing number and derived some results on

it. Let $t \geq 0$ be an integer. Then a positive integer n is called a t -balancing number if

$$1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t) \quad (5)$$

for some positive integer r which is called t -balancer (or t -cobalancing number). 0- and 1-balancing numbers can be given in terms of balancing numbers; indeed, $B_n^0 = b_{n+1}$, $b_n^0 = B_n$, $C_n^0 = c_{n+1}$, $c_n^0 = C_n$ and $B_n^1 = B_{n+1} - 1$, $b_n^1 = b_{n+1}$, $C_n^1 = C_{n+1}$, $c_n^1 = c_{n+1}$. So it is assumed that $t \geq 2$.

The n th t -balancing number is denoted by B_n^t and the n th t -cobalancing number is denoted by b_n^t . From (5) we see that

$$r = \frac{-(2n + 2t + 1) + \sqrt{8n^2 + 8n(1 + t) + (2t + 1)^2}}{2}, \quad (6)$$

$$n = \frac{(2r - 1) + \sqrt{8r^2 + 8tr + 1}}{2}.$$

Hence, B_n^t is a t -balancing number if and only if $8(B_n^t)^2 + 8B_n^t(1 + t) + (2t + 1)^2$ is a perfect square and b_n^t is a t -cobalancing number if and only if $8(b_n^t)^2 + 8tb_n^t + 1$ is a perfect square. Set

$$C_n^t = \sqrt{8(B_n^t)^2 + 8B_n^t(1 + t) + (2t + 1)^2}, \quad (7)$$

$$c_n^t = \sqrt{8(b_n^t)^2 + 8tb_n^t + 1}.$$

Then C_n^t is called the n th Lucas t -balancing number and c_n^t is called the n th Lucas t -cobalancing number.

2. Main Results

To determine the general terms of all t -balancing numbers, we have to determine all integer (in fact only positive) solutions of some specific Diophantine equations. Let us explain this as follows. We see as above that $8x^2 + 8x(1 + t) + (2t + 1)^2$ is a perfect square for a t -balancing number x . So we set $8x^2 + 8x(1 + t) + (2t + 1)^2 = y^2$ for some integer $y \neq 0$. Hence, we get the Diophantine equation:

$$D^t : 8x^2 - y^2 + 8x(1 + t) + (2t + 1)^2 = 0. \quad (8)$$

If we make the change of variables $x = v + h$ and $y = u + k$ for some h and k , then we get

$$\widetilde{D}^t : 8(v + h)^2 - (u + k)^2 + 8(v + h)(1 + t) + (2t + 1)^2 = 0. \quad (9)$$

Here we obtain $h = -(t + 1)/2$ and $k = 0$. Thus (9) becomes

$$\widetilde{D}^t : u^2 - 8v^2 = 2t^2 - 1 \quad (10)$$

which is a Pell equation (see [11–14]). Before considering its integer solutions, we need some notations. Now let Δ be a nonsquare discriminant. Then the Δ -order O_Δ is defined for nonsquare discriminants Δ to be the ring $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\Delta}/4$ if $\Delta \equiv 0 \pmod{4}$, or $(1 + \sqrt{\Delta})/2$

if $\Delta \equiv 1 \pmod{4}$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. The unit group O_Δ^* is defined to be the group of units of the ring O_Δ .

For the quadratic form $F(x, y) = ax^2 + bxy + cy^2$, we can write $F = (xa + y((b + \sqrt{\Delta})/2))(xa + y((b - \sqrt{\Delta})/2))/a$. So the module M_F of F is the O_Δ -module $M_F = \{xa + y((b + \sqrt{\Delta})/2) : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta})$. Therefore, we get $(u + v\rho_\Delta)(xa + y((b + \sqrt{\Delta})/2)) = x'a + y'((b + \sqrt{\Delta})/2)$, where

$$(x', y') = \begin{cases} (x, y) \begin{pmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{pmatrix} & \text{if } \Delta \equiv 0 \pmod{4} \\ (x, y) \begin{pmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{pmatrix} & \text{if } \Delta \equiv 1 \pmod{4}. \end{cases} \quad (11)$$

So there is a bijection $\Psi : \{(x, y) : F_\Delta(x, y) = m\} \rightarrow \{\gamma \in M_F : N(\gamma) = am\}$ for solving the Pell equation $F(x, y) = m$; that is, $ax^2 + bxy + cy^2 = m$. The action of $O_{\Delta,1}^* = \{\alpha \in O_\Delta^* : N(\alpha) = 1\}$ on the set $\Omega = \{(x, y) : F_\Delta(x, y) = m\}$ of integral solutions of the equation $F(x, y) = m$ is most interesting when Δ is a positive nonsquare since $O_{\Delta,1}^*$ is infinite. Therefore, the orbit of each solution will be infinite and so the set Ω is either empty or infinite. Since $O_{\Delta,1}^*$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_Δ be the smallest unit of O_Δ , that is, greater than 1, and let $\tau_\Delta = \varepsilon_\Delta$ if $N(\varepsilon_\Delta) = 1$, or ε_Δ^2 if $N(\varepsilon_\Delta) = -1$. Then every $O_{\Delta,1}^*$ orbit of integral solutions of $F(x, y) = m$ contains a solution $(x, y) \in \mathbb{Z}^2$ such that $0 \leq y \leq U$, where $U = |am\tau_\Delta/\Delta|^{1/2}(1 - (1/\tau_\Delta))$ if $am > 0$ or $U = |am\tau_\Delta/\Delta|^{1/2}(1 + (1/\tau_\Delta))$ if $am < 0$. So for finding a set of representatives of the $O_{\Delta,1}^*$ orbits of integral solutions of $F(x, y) = m$, we must find, for each integer y such that $0 \leq y \leq U$, all integers x that satisfy $F(x, y) = m$. If $F(x, y) = m$, then $\Delta y^2 + 4am = (2ax + by)^2$ and so $x = \pm(\sqrt{\Delta y^2 + 4am} - by)/2a$.

For the Pell equation $\widetilde{D}^t$, there are one, two, three, or more sets of representatives (or sets of solutions) depending on t . For example, for $t = 3$, the set of representatives is $\{(\pm 5, 1)\}$; for $t = 9$, the set of representatives is $\{(\pm 13, 1), (\pm 17, 4)\}$; for $t = 89$, the set of representatives is $\{(\pm 127, 6), (\pm 129, 10), (\pm 143, 24), (\pm 177, 44)\}$; and for $t = 93$, the set of representatives is $\{(\pm 133, 7), (\pm 155, 29), (\pm 186, 46)\}$. So we cannot determine for which values of t there are one, two, three, or more sets of representatives. Consequently, we cannot determine all integer solutions of $\widetilde{D}^t$ in one way. To determine all integer solutions of $\widetilde{D}^t$, we have to put some restrictions on t . From now on, t is assumed to be odd such that $2t^2 - 1$ is a prime.

Thus for the Pell equation $\widetilde{D}^t$, we get $\tau_{32} = 3 + \sqrt{8}$ and $32y^2 + 8t^2 - 4$ is a square only for $y = (t - 1)/2$ in the range

$0 \leq y \leq U$. Hence, $x = \pm(2t - 1)$. Therefore, there is exactly one $O_{32,1}^*$ set of representatives of the orbits and that $\{(\pm(2t - 1), (t - 1)/2)\}$ is a set of representatives; that is, there are two classes of solutions. Here we see that

- (1) $(2t - 1, (t - 1)/2)M^n$ generates the solutions (u_{2n+1}, v_{2n+1}) for $n \geq 0$,
- (2) $(1 - 2t, (1 - t)/2)M^n$ generates the solutions $(-u_{2n+1}, -v_{2n+1})$ for $n \geq 0$,
- (3) $(2t - 1, (1 - t)/2)M^{-n}$ generates the solutions $(u_{2n+1}, -v_{2n+1})$ for $n \geq 0$,
- (4) $(1 - 2t, (t - 1)/2)M^{-n}$ generates the solutions $(-u_{2n+1}, v_{2n+1})$ for $n \geq 0$,
- (5) $(2t - 1, (1 - t)/2)M^n$ generates the solutions (u_{2n}, v_{2n}) for $n \geq 1$,
- (6) $(1 - 2t, (t - 1)/2)M^n$ generates the solutions $(-u_{2n}, -v_{2n})$ for $n \geq 1$,
- (7) $(2t - 1, (t - 1)/2)M^{-n}$ generates the solutions $(u_{2n}, -v_{2n})$ for $n \geq 1$,
- (8) $(1 - 2t, (1 - t)/2)M^{-n}$ generates the solutions $(-u_{2n}, v_{2n})$ for $n \geq 1$,

where $M = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}$ by (11). Thus, we can give the following theorem.

Theorem 1. *The set of all positive integer solutions of $\widetilde{D}^t$ is $\Omega(\widetilde{D}^t) = \{(u_{2n}, v_{2n}), (u_{2n+1}, v_{2n+1})\}$, where*

$$\begin{aligned} (u_{2n+1}, v_{2n+1}) &= \left(2t - 1, \frac{t-1}{2}\right) M^n \quad \text{for } n \geq 0, \\ (u_{2n}, v_{2n}) &= \left(2t - 1, \frac{1-t}{2}\right) M^n \quad \text{for } n \geq 1. \end{aligned} \quad (12)$$

To determine all positive integer solutions of $\widetilde{D}^t$, we have to formulate the n th power of M . So we can give the following theorem which can be proved by induction on n .

Theorem 2. *The n th power of M is*

$$M^n = \begin{pmatrix} C_n & B_n \\ 8B_n & C_n \end{pmatrix}, \quad (13)$$

where B_n is the n th balancing number and C_n is the n th Lucas-balancing number.

Consequently from the above two theorems, we can give the following main theorem.

Theorem 3. *The set of all positive integer solutions of $\widetilde{D}^t$ is $\Omega(\widetilde{D}^t) = \{(u_{2n}, v_{2n}), (u_{2n+1}, v_{2n+1})\}$, where*

$$\begin{aligned} (u_{2n+1}, v_{2n+1}) &= \left((2t - 1)C_n + 4(t - 1)B_n, (2t - 1)B_n + \frac{t-1}{2}C_n \right) \\ &\quad \text{for } n \geq 0 \\ (u_{2n}, v_{2n}) &= \left((2t - 1)C_n + 4(1 - t)B_n, (2t - 1)B_n + \frac{1-t}{2}C_n \right) \\ &\quad \text{for } n \geq 1. \end{aligned} \quad (14)$$

After formulating all integer solutions of $\widetilde{D}^t$, we can determine all integer solutions of D^t . Notice that $x = v - ((t + 1)/2)$ and $y = u$. So applying Theorem 3, we get

$$\begin{aligned} x_{2n} &= (2t - 1)B_n + \frac{1-t}{2}C_n - \frac{t+1}{2}, \\ y_{2n} &= (2t - 1)C_n + 4(1 - t)B_n \end{aligned} \quad (15)$$

for $n \geq 1$ and

$$\begin{aligned} x_{2n+1} &= (2t - 1)B_n + \frac{t-1}{2}C_n - \frac{t+1}{2} \\ y_{2n+1} &= (2t - 1)C_n + 4(t - 1)B_n \end{aligned} \quad (16)$$

for $n \geq 0$. Here we note that (x_{2n}, y_{2n}) and (x_{2n+1}, y_{2n+1}) are positive integer solutions. By symmetry, $(x_{2n}, -y_{2n})$ and $(x_{2n+1}, -y_{2n+1})$ are also integer solutions. Apart from them, we get from $(1 - 2t, (1 - t)/2)M^n$ generates the solutions $(-u_{2n+1}, -v_{2n+1})$ for $n \geq 0$ that

$$X_{2n} = (1 - 2t)B_n + \frac{t-1}{2}C_n - \frac{t+1}{2} \quad (17)$$

for $n \geq 1$ and from $(1 - 2t, (t - 1)/2)M^n$ generates the solutions $(-u_{2n}, -v_{2n})$ for $n \geq 1$ that

$$X_{2n+1} = (1 - 2t)B_n + \frac{1-t}{2}C_n - \frac{t+1}{2} \quad (18)$$

for $n \geq 0$. So $(X_{2n}, \pm Y_{2n})$ and $(X_{2n+1}, \pm Y_{2n+1})$ are also integer solutions, where $Y_{2n} = y_{2n}$ and $Y_{2n+1} = y_{2n+1}$. Thus, we proved the following theorem.

Theorem 4. *The set of all integer solutions of D^t is $\Omega(D^t) = \{(x_{2n}, \pm y_{2n}), (x_{2n+1}, \pm y_{2n+1}), (X_{2n}, \pm Y_{2n}), (X_{2n+1}, \pm Y_{2n+1})\}$, where*

$$\begin{aligned} x_{2n+1} &= (2t - 1)B_n + \frac{t-1}{2}C_n - \frac{t+1}{2}, \\ y_{2n+1} &= (2t - 1)C_n + 4(t - 1)B_n, \\ X_{2n+1} &= (1 - 2t)B_n + \frac{1-t}{2}C_n - \frac{t+1}{2}, \\ Y_{2n+1} &= y_{2n+1} \end{aligned} \quad (19)$$

for $n \geq 0$ and

$$\begin{aligned} x_{2n} &= (2t-1)B_n + \frac{1-t}{2}C_n - \frac{t+1}{2}, \\ y_{2n} &= (2t-1)C_n + 4(1-t)B_n, \\ X_{2n} &= (1-2t)B_n + \frac{t-1}{2}C_n - \frac{t+1}{2}, \\ Y_{2n} &= y_{2n} \end{aligned} \quad (20)$$

for $n \geq 1$.

Example 5. Let $t = 3$. Then we get

$$\begin{aligned} x_{2n+1} &\in \{-1, 6, 45, 272, 1595, \dots\}, \\ y_{2n+1} &\in \{5, 23, 133, 775, 4517, \dots\}, \\ X_{2n+1} &\in \{-3, -10, -49, -276, -1599, \dots\}, \\ x_{2n} &\in \{0, 11, 74, 441, 2580, \dots\}, \\ y_{2n} &\in \{7, 37, 215, 1253, 7303, \dots\}, \\ X_{2n} &\in \{-4, -15, -78, -445, -2584, \dots\}. \end{aligned} \quad (21)$$

So the set of all integer solutions of $D^3 : 8x^2 - y^2 + 32x + 49 = 0$ is

$$\begin{aligned} \Omega(D^3) &= \{\dots, (-2584, \pm 7303), (-1599, \pm 4517), \\ &(-445, \pm 1253), (-276, \pm 775), (-78, \pm 215), \\ &(-49, \pm 133), (-15, \pm 37), (-10, \pm 23), \\ &(-4, \pm 7), (-3, \pm 5), (-1, \pm 5), (0, \pm 7), \\ &(6, \pm 23), (11, \pm 37), (45, \pm 133), (74, \pm 215), \\ &(272, \pm 775), (441, \pm 1253), (1595, \pm 4517), \\ &(2580, \pm 7303), \dots\}. \end{aligned} \quad (22)$$

After determining all integer solutions of D^t , we can determine all t -balancing numbers.

Theorem 6. The general terms of all t -balancing numbers are

$$\begin{aligned} B_{2n-1}^t &= (B_n + b_{n+1})t - (b_{n+1} + 1), \\ b_{2n-1}^t &= (B_n + b_n)t - B_n, \\ C_{2n-1}^t &= (4b_{n+1} + 2)t - c_{n+1}, \\ c_{2n-1}^t &= 4tB_n - C_n \end{aligned} \quad (23)$$

for $n \geq 1$ and

$$\begin{aligned} B_{2n-2}^t &= (B_{n-1} + b_n)t + b_n, \\ b_{2n-2}^t &= (B_{n-1} + b_{n-1})t + B_{n-1}, \\ C_{2n-2}^t &= (4b_n + 2)t + c_n, \\ c_{2n-2}^t &= 4tB_{n-1} + C_{n-1} \end{aligned} \quad (24)$$

for $n \geq 2$.

Proof. Since $x = B_n^t$, we get from Theorem 4 that

$$\begin{aligned} B_{2n-1}^t &= (2t-1)B_n + \frac{t-1}{2}C_n - \frac{t+1}{2} \\ &= (B_n + b_{n+1})t - (b_{n+1} + 1) \end{aligned} \quad (25)$$

for $n \geq 1$ since $2B_n + ((C_n - 1)/2) = B_n + b_{n+1}$ and $B_n + ((C_n + 1)/2) = b_{n+1} + 1$.

Note that $b_n = ((\alpha^{2n-1} - \beta^{2n-1})/4\sqrt{2}) - (1/2)$ and $c_n = (\alpha^{2n-1} + \beta^{2n-1})/2$. So we get from (7) that

$$\begin{aligned} C_{2n-1}^t &= \sqrt{8(B_{2n-1}^t)^2 + 8B_{2n-1}^t(1+t) + (2t+1)^2} \\ &= \sqrt{8[(B_n + b_{n+1})t - (b_{n+1} + 1)]^2 + 8[(B_n + b_{n+1})t - (b_{n+1} + 1)](1+t) + (2t+1)^2} \\ &= \sqrt{\left(\frac{\alpha^{2n+1}(\sqrt{2}t-1) - \beta^{2n+1}(\sqrt{2}t+1)}{2}\right)^2} \\ &= \left[4\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}\right) + 2\right]t - \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right) \\ &= (4b_{n+1} + 2)t - c_{n+1}. \end{aligned} \quad (26)$$

Thus from (6), we obtain

$$b_{2n-1}^t = (B_n + b_n)t - B_n \quad (27)$$

and hence

$$\begin{aligned} c_{2n-1}^t &= \sqrt{8(b_{2n-1}^t)^2 + 8tb_{2n-1}^t + 1} \\ &= \sqrt{8[(B_n + b_n)t - B_n]^2 + 8t[(B_n + b_n)t - B_n] + 1} \\ &= \sqrt{\left(\frac{\alpha^{2n}(\sqrt{2}t-1) - \beta^{2n}(\sqrt{2}t+1)}{2}\right)^2} \\ &= \frac{\alpha^{2n}(\sqrt{2}t-1) - \beta^{2n}(\sqrt{2}t+1)}{2} \\ &= 4t\left(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}\right) - \left(\frac{\alpha^{2n} + \beta^{2n}}{2}\right) \\ &= 4tB_n - C_n \end{aligned} \quad (28)$$

since $B_n = (\alpha^{2n} - \beta^{2n})/4\sqrt{2}$ and $C_n = (\alpha^{2n} + \beta^{2n})/2$.

Similarly we deduce that

$$\begin{aligned} B_{2n-2}^t &= (B_{n-1} + b_n)t + b_n, \\ C_{2n-2}^t &= (4b_n + 2)t + c_n, \\ b_{2n-2}^t &= (B_{n-1} + b_{n-1})t + B_{n-1}, \\ c_{2n-2}^t &= 4tB_{n-1} + C_{n-1} \end{aligned} \quad (29)$$

for $n \geq 2$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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