Research Article

Conservation Laws for a Degasperis Procesi Equation and a Coupled Variable-Coefficient Modified Korteweg-de Vries System in a Two-Layer Fluid Model via the Multiplier Approach

E. Osman,¹ M. Khalfallah,² and H. Sapoort¹

¹ Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

² Mathematics Department, Faculty of Science, South Valley University, Qena, Egypt

Correspondence should be addressed to H. Sapoort; hussien.abdalnaeem1@science.sohag.edu.eg

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We employ the multiplier approach (variational derivative method) to derive the conservation laws for the Degasperis Procesi equation and a coupled variable-coefficient modified Korteweg-de Vries system in a two-layer fluid model. Firstly, the multipliers are computed and then conserved vectors are obtained for each multiplier.

1. Introduction

The conservation laws are important in the solution and reductions of partial differential equations. Conservation law, also called law of conservation, in physics, is several principles that state that certain physical properties (i.e., measurable quantities) do not change in the course of time within an isolated physical system. In classical physics, laws of this type govern energy, momentum, angular momentum, mass, and electric charge. In particle physics, other conservation laws apply to properties of subatomic particles that are invariant during interactions. An important function of conservation laws is that they make it possible to predict the macroscopic behaviour of a system without having to consider the microscopic details of the course of a physical process or chemical reaction. Many powerful methods have been developed for the construction of conservation laws, such as The Laplace Direct method [1], multiplier approach [2, 3], Kara and Mahomed symmetry condition [4], Wolf [5, 6], Göktaş and Hereman [7], Hereman et al. [8–10], and Cheviakov [11] who developed powerful software packages to compute conservation laws for partial differential equations. Infinitely many conservation laws are obtained based on the Lax pair via the Hirota method and symbolic computation, bilinear forms, bilinear Backlund transformations, and

one- and two-soliton-like solutions are also derived. With different coefficients, bell-shaped, periodic-changing, quadratic-varying, exponential-decreasing, and exponential-increasing soliton-like profiles are obtained in [12]. Also by the spectral analysis the Hamiltonian and periodicity of the qZK equation are investigated by using the Hirota method [13]. The nonautonomous matter waves with time-dependent modulation in a one-dimensional trapped spin-1 Bose-Einstein condensate and the generalized three-coupled Gross-Pitaevskii equations by means of the Hirota bilinear method are studied in [14]. The multiplier approach (also known as variational derivative method) was proposed by Steudel [15] who wrote the conservation law in characteristic form as $D_i T^i = A^\alpha E_\alpha$. Later, Olver [16] modified the method of determining the characteristics (multipliers) by taking the variational derivative of $D_i T^i = Q^\alpha E_\alpha$ not only for the arbitrary functions, but also for solutions of system of partial differential equations. The outline of the paper is as follows. In Section 2, some definitions related to the multiplier approach are given. In Section 3, conservation laws for the Degasperis Procesi equation are derived by first computing the multipliers. The conservation laws for a coupled variable-coefficient modified Korteweg-de Vries system in a two-layer fluid model are derived in Section 4. Finally, conclusions are summarized in Section 5.

2. Necessary Preliminaries

Let $x^i, i = 1, 2, \dots, n$, be n independent variables and let u be the dependent variable.

- (1) The total derivative operator with respect to x^i is

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots \quad i = 1, 2, \dots, n, \quad (1)$$

where u_i denotes the derivative of u with respect to x^i . Similarly u_{ij} denotes the derivative of u with respect to x^i and x^j .

- (2) The Euler operator is defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_{ij} \frac{\partial}{\partial u_{ij}} - D_{ijk} \frac{\partial}{\partial u_{ijk}} + \dots \quad (2)$$

Consider a k th-order partial differential equation of n independent and one dependent variable

$$E(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0. \quad (3)$$

- (3) An n -tuple $T = (T^1, T^2, \dots, T^n)$, $i = 1, 2, \dots, n$, such that

$$D_i T^i = 0 \quad (4)$$

holds for all solutions of (3) is known as the conserved vector of (3).

- (4) The multiplier A of system (3) has the property

$$D_i T^i = AE \quad (5)$$

for arbitrary function $u(x^1, x^2, \dots, x^n)$ [16].

- (5) The determining equations for multipliers are obtained by taking the variational derivative of (5) (see [16]):

$$\frac{\delta}{\delta u} (AE) = 0. \quad (6)$$

Equation (6) holds for arbitrary function $u(x^1, x^2, \dots, x^n)$ not only for solutions of system (3).

Once the multipliers are computed from (6), the conserved vectors can be derived systematically using (5) as the determining equation. But in some problems it is not difficult to construct the conserved vectors by elementary manipulations after the determination of the multipliers.

3. Conservation Laws for the Degasperis Procesi Equation

The Degasperis Procesi equation [17] takes the form

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \quad (7)$$

The Degasperis Procesi equation (7) is very interesting as it is an integrable shallow water equation and presents a quite rich structure. Also it can be used to model wave perturbations in relaxing media.

We will derive the conservation laws for (7) by the multiplier approach. The determining equation for multiplier $A(t, x, u)$, from (6), is

$$\frac{\delta}{\delta u} [A(u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx})] = 0. \quad (8)$$

The standard Euler operator $\delta/\delta u$ from (2) can be defined as

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} \\ & + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}} - \dots, \end{aligned} \quad (9)$$

and total derivative operators D_t and D_x using (1) are

$$\begin{aligned} D_t = & \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \dots, \\ D_x = & \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots. \end{aligned} \quad (10)$$

Equation (8) after expansion and simplification takes the following form:

$$\begin{aligned} & u_{xx} (A_{ut} + 3uA_{xu} + 3uu_x A_{uu} - 3u_x A_u + u_t A_{uu}) \\ & + u_x^2 (uA_{uux} + A_{uut} + u_t A_{uuu}) + uu_x^3 A_{uuu} \\ & + u_{xt} (2A_{xu} + 2u_x A_{uu}) + u_x (2A_{xtu} + 3uA_{xxu}) \\ & + u_t (A_{xxu} + 2u_x A_{xuu}) + u (A_{xxx} - 4A_x) \\ & + A_{xxt} - A_t = 0, \end{aligned} \quad (11)$$

which yields

$$A = c_1 + c_2 e^{-2x} + c_3 e^{2x}. \quad (12)$$

From (5) and (12), we have

$$\begin{aligned} & (c_1 + c_2 e^{-2x} + c_3 e^{2x}) \\ & \times (u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx}) \\ = & D_t [c_1 (u - u_{xx}) + c_2 (ue^{-2x} - u_{xx}e^{-2x}) \\ & + c_3 (ue^{2x} - u_{xx}e^{2x})] \\ & + D_x [c_1 (2u^2 - u_x^2 - uu_{xx}) \\ & + c_2 (-2uu_x e^{-2x} - u_x^2 e^{-2x} - uu_{xx} e^{-2x}) \\ & + c_3 (2uu_x e^{2x} - u_x^2 e^{2x} - uu_{xx} e^{2x})] \end{aligned} \quad (13)$$

for arbitrary functions $u(t, x)$. When $u(t, x)$ is solutions of (7) then left hand side of (13) vanishes and we obtain

$$\begin{aligned} D_t [c_1 (u - u_{xx}) + c_2 (ue^{-2x} - u_{xx}e^{-2x}) \\ + c_3 (ue^{2x} - u_{xx}e^{2x})] \\ + D_x [c_1 (2u^2 - u_x^2 - uu_{xx}) \\ + c_2 (-2uu_xe^{-2x} - u_x^2e^{-2x} - uu_{xx}e^{-2x}) \\ + c_3 (2uu_xe^{2x} - u_x^2e^{2x} - uu_{xx}e^{2x})] = 0. \end{aligned} \quad (14)$$

Therefore the conserved vectors for the Degasperis Procesi equation (7) are

$$\begin{aligned} T_1^1 &= u - u_{xx}, & T_1^2 &= 2u^2 - u_x^2 - uu_{xx}, \\ T_2^1 &= ue^{-2x} - u_{xx}e^{-2x}, \\ T_2^2 &= -2uu_xe^{-2x} - u_x^2e^{-2x} - uu_{xx}e^{-2x}, \\ T_3^1 &= ue^{2x} - u_{xx}e^{2x}, \\ T_3^2 &= 2uu_xe^{2x} - u_x^2e^{2x} - uu_{xx}e^{2x}. \end{aligned} \quad (15)$$

The variational derivative approach for the Degasperis Procesi equation gives three multipliers of the form $A(t, x, u)$ and hence three conserved vectors are obtained.

4. Conservation Laws for a Coupled Variable-Coefficient Modified Korteweg-de Vries System in a Two-Layer Fluid Model

In this section we recall some basic definitions related to the multiplier approach.

Let (x, t) be two independent variables and let (u, v) be dependent variables.

- (1) The total derivative operators D_t and D_x are

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} \\ &\quad + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} \\ &\quad + v_{xx} \frac{\partial}{\partial v_x} + u_{tx} \frac{\partial}{\partial u_t} + v_{tx} \frac{\partial}{\partial v_t} + \dots. \end{aligned} \quad (16)$$

- (2) The standard Euler operators $\delta/\delta u$ and $\delta/\delta v$ are

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} \\ &\quad + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}} - \dots, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t^2 \frac{\partial}{\partial v_{tt}} \\ &\quad + D_x^2 \frac{\partial}{\partial v_{xx}} + D_x D_t \frac{\partial}{\partial v_{tx}} - \dots. \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t^2 \frac{\partial}{\partial v_{tt}} \\ &\quad + D_x^2 \frac{\partial}{\partial v_{xx}} + D_x D_t \frac{\partial}{\partial v_{tx}} - \dots. \end{aligned} \quad (18)$$

Consider a k th-order system of two partial differential equations of two independent and two dependent variables

$$\begin{aligned} E_1(t, x, u, v, u_t, v_t, \dots), \\ E_2(t, x, u, v, u_t, v_t, \dots). \end{aligned} \quad (19)$$

- (3) A vector $T = (T^1, T^2)$ satisfying

$$D_t T^1 + D_x T^2 = 0 \quad (20)$$

for all solutions of (19) is known as the conserved vector of (19).

- (4) The multipliers A_1, A_2 of system (19) have the property

$$D_t T^1 + D_x T^2 = A_1 E_1 + A_2 E_2, \quad (21)$$

for the arbitrary functions $u(x, t), v(x, t)$.

- (5) The determining equations for the multipliers are obtained by taking variational derivative of (21):

$$\begin{aligned} \frac{\delta}{\delta u} [A_1 E_1 + A_2 E_2] &= 0, \\ \frac{\delta}{\delta v} [A_1 E_1 + A_2 E_2] &= 0. \end{aligned} \quad (22)$$

Equation (22) holds for the arbitrary functions $u(x, t), v(x, t)$ not only for the solutions of system (19). Equation (22) yields multipliers for all local conservation laws. Then conserved vectors can be derived systematically using (21) as the determining equation. But in some problems it is not difficult to construct the conserved vectors by elementary manipulations once the multiplier has been determined.

Example 1. Consider a coupled variable-coefficient modified Korteweg-de Vries system in a two-layer fluid model

$$\begin{aligned} u_t - \alpha(t) [u_{xxx} + 6(u^2 - v^2)u_x - 12uvv_x] - 4\beta(t)u_x &= 0, \\ v_t - \alpha(t) [v_{xxx} + 6(u^2 - v^2)v_x + 12uvu_x] - 4\beta(t)v_x &= 0. \end{aligned} \quad (23)$$

System (23) was proposed in [18] as an important particular case of the formidable generalized coupled variable-coefficient modified Korteweg-de Vries (CVmKdV) system. The (CVmKdV) system was derived by Gao and Tang [19] as a two-layer model describing atmospheric and oceanic phenomena like interactions between the atmosphere and ocean, atmospheric blocking, oceanic circulations, hurricanes, typhoons, and so forth.

The determining equations for multipliers of the forms $A_1(x, t, u, v)$ and $A_2(x, t, u, v)$ from (22) are

$$\begin{aligned} & \frac{\delta}{\delta u} \left[A_1 \left(u_t - \alpha(t) \left[u_{xxx} + 6(u^2 - v^2) u_x - 12uvv_x \right] \right. \right. \\ & \quad \left. \left. - 4\beta(t) u_x \right) \right. \\ & \quad \left. + A_2 \left(v_t - \alpha(t) \left[v_{xxx} + 6(u^2 - v^2) v_x + 12uvu_x \right] \right. \right. \\ & \quad \left. \left. - 4\beta(t) v_x \right) \right] = 0, \\ & \frac{\delta}{\delta v} \left[A_1 \left(u_t - \alpha(t) \left[u_{xxx} + 6(u^2 - v^2) u_x - 12uvv_x \right] \right. \right. \\ & \quad \left. \left. - 4\beta(t) u_x \right) \right. \\ & \quad \left. + A_2 \left(v_t - \alpha(t) \left[v_{xxx} + 6(u^2 - v^2) v_x + 12uvu_x \right] \right. \right. \\ & \quad \left. \left. - 4\beta(t) v_x \right) \right] = 0, \end{aligned} \quad (24)$$

where the standard Euler operators $\delta/\delta u$ and $\delta/\delta v$ are defined in (17) and (18), respectively. Expansion of (24) yields

$$\begin{aligned} & A_{1u} \left[u_t - \alpha(t) \left[u_{xxx} + 6(u^2 - v^2) u_x - 12uvv_x \right] \right. \\ & \quad \left. - 4\beta(t) u_x \right] \\ & + A_{2u} \left[v_t - \alpha(t) \left[v_{xxx} + 6(u^2 - v^2) v_x + 12uvu_x \right] \right. \\ & \quad \left. - 4\beta(t) v_x \right] \\ & + (12\alpha v v_x - 12\alpha u u_x) A_1 - (12\alpha u v_x + 12\alpha v u_x) A_2 \\ & - D_t(A_1) + D_x \left[(6\alpha(u^2 - v^2) + 4\beta) A_1 + 12\alpha u v A_2 \right] \\ & + D_x^3(\alpha A_1), \end{aligned} \quad (25)$$

$$\begin{aligned} & A_{1v} \left[u_t - \alpha(t) \left[u_{xxx} + 6(u^2 - v^2) u_x - 12uvv_x \right] \right. \\ & \quad \left. - 4\beta(t) u_x \right] \\ & + A_{2v} \left[v_t - \alpha(t) \left[v_{xxx} + 6(u^2 - v^2) v_x + 12uvu_x \right] \right. \\ & \quad \left. - 4\beta(t) v_x \right] \\ & + (12\alpha v v_x - 12\alpha u u_x) A_2 + (12\alpha u v_x + 12\alpha v u_x) A_1 \\ & - D_t(A_2) + D_x \left[(6\alpha(u^2 - v^2) + 4\beta) A_2 - 12\alpha u v A_1 \right] \\ & + D_x^3(\alpha A_2). \end{aligned} \quad (26)$$

Equations (25) and (26) are separated according to different combinations of derivatives of u and v and, after some

simplification following system of equations for A_1, A_2 is obtained:

$$\begin{aligned} & A_{1uu} = 0, \quad A_{1vv} = 0, \quad A_{1ux} = 0, \\ & A_{1vx} = 0, \quad A_{1uv} = 0, \\ & A_{1v} - A_{2u} = 0, \quad A_{1u} + A_{2v} = 0, \\ & (6\alpha(u^2 - v^2) + 4\beta) A_{1x} \\ & + 12\alpha u v A_{2x} - A_{1t} + \alpha A_{1xxx} = 0, \\ & A_{2uu} = 0, \quad A_{2vv} = 0, \quad A_{2ux} = 0, \\ & A_{2vx} = 0, \quad A_{2uv} = 0, \\ & (6\alpha(u^2 - v^2) + 4\beta) A_{2x} \\ & - 12\alpha u v A_{1x} - A_{2t} + \alpha A_{2xxx} = 0. \end{aligned} \quad (27)$$

Solution of system (27) yields

$$\begin{aligned} A_1 &= c_1 + c_3 u + c_4 v, \\ A_2 &= c_2 - c_3 v + c_4 u, \end{aligned} \quad (28)$$

where c_1, c_2, c_3 , and c_4 are constants.

Equations (21) and (28) give the following conserved vectors satisfying (20):

$$\begin{aligned} T_1^1 &= u, \quad T_1^2 = -2\alpha u^3 - \alpha u_{xx} + 6\alpha u v^2 - 4\beta u, \\ T_2^1 &= v, \quad T_2^2 = 2\alpha v^3 - \alpha v_{xx} - 6\alpha u^2 v - 4\beta v, \\ T_3^1 &= \frac{u^2}{2} - \frac{v^2}{2}, \\ T_3^2 &= \alpha(v v_{xx} - u u_{xx}) + 9\alpha u^2 v^2 + \alpha \left(\frac{u_x^2}{2} - \frac{v_x^2}{2} \right) \\ &+ v^2 \left(2\beta - \frac{3}{2}\alpha v^2 \right) - u^2 \left(\frac{3}{2}\alpha u^2 + 2\beta \right), \\ T_4^1 &= uv, \\ T_4^2 &= -6\alpha u^3 v + \alpha(u_x v_x - v u_{xx}) \\ &+ u(6\alpha v^3 - 4v\beta - \alpha v_{xx}). \end{aligned} \quad (29)$$

5. Conclusions

The conservation laws for the Degasperis Procesi equation and a coupled variable-coefficient modified Korteweg-de Vries system in a two-layer fluid model were established with the help of the multiplier approach. The multiplier approach on the Degasperis Procesi equation yielded three multipliers and thus three local conserved vectors were obtained in each case. The multiplier approach when applied to a coupled variable-coefficient modified Korteweg-de Vries system in a two-layer fluid model gave four multipliers of form $A(x, t, u, v)$. Each multiplier corresponds to a conserved vector and thus four local conserved vectors were obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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