Research Article

Certain Families of Multivalent Analytic Functions Associated with Iterations of the Owa-Srivastava Fractional Differintegral Operator

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By making use of a multivalent analogue of the Owa-Srivastava fractional differintegral operator and its iterations, certain new families of analytic functions are introduced. Several interesting properties of these function classes, such as convolution theorems, inclusion theorems, and class-preserving transforms, are studied.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\} \quad (1)$$

and let $\mathcal{A}_p$ be the subclass of $\mathcal{A}$ consisting of functions represented by the following *Taylor-Maclaurin's* series:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k+p} \quad (p \in \mathbb{N} := \{1, 2, 3, \dots\}). \quad (2)$$

In a recent paper Patel and Mishra [1] studied several interesting mapping properties of the *fractional differintegral operator*:

$$\Omega_z^{(\lambda, p)} : \mathcal{A}_p \longrightarrow \mathcal{A}_p \quad (p \in \mathbb{N}, -\infty < \lambda < p+1, z \in \mathbb{U}), \quad (3)$$

defined by

$$\begin{aligned} \Omega_z^{(\lambda, p)} f(z) &:= z^p \\ &+ \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} a_k z^{k+p} \quad (4) \\ &(z \in \mathbb{U}), \end{aligned}$$

where $f \in \mathcal{A}_p$ is given by (2). In the particular case $p = 1$ and $-\infty < \lambda < 1$ the *fractional-differintegral operator*

$$\Omega_z^{(\lambda, 1)} := \Omega_z^\lambda \quad (5)$$

was earlier introduced by Owa and Srivastava [2] (see also [3]) and this is popularly known as the *Owa-Srivastava operator* [4–6]. Moreover, for $0 \leq \lambda < 1$ and $p \in \mathbb{N}$, $\Omega_z^{(\lambda, p)}$ was investigated by Srivastava and Aouf [7] which was further extended to the range $-\infty < \lambda < 1$, $p \in \mathbb{N}$ by Srivastava and Mishra [8]. The following are some of the interesting particular cases of $\Omega_z^{(\lambda, p)}$:

$$\Omega_z^{(-1, p)} f(z) = \frac{p+1}{z} \int_0^z f(\xi) d\xi,$$

$$\Omega_z^{(0, p)} f(z) = f(z),$$

$$\Omega_z^{(1, p)} f(z) = \frac{zf'(z)}{p},$$

$$\Omega_z^{(n, p)} f(z) = \frac{(p-n)!z^n f^n(z)}{p!}, \quad (n \in \mathbb{N}; n < p+1),$$

$$\Omega_z^{(-m, p)} f(z) = \frac{p+m}{z^p} \int_0^z t^{m-1} \Omega_z^{(-m+1, p)} f(t) dt \quad (m \in \mathbb{N}). \quad (6)$$

Furthermore

$$\begin{aligned} z(\Omega_z^{(\lambda,p)} f(z))' &= (p - \lambda) \Omega_z^{(\lambda+1,p)} f(z) \\ &\quad + \lambda \Omega_z^{(\lambda,p)} f(z) \quad (-\infty < \lambda < p; z \in \mathbb{U}). \end{aligned} \quad (7)$$

The n -iterates of the operator $\Omega_z^{(\lambda,p)}$ are defined as follows:

$$\Omega_{z,0}^{(\lambda,p)} f(z) = f(z) \quad (8)$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} \Omega_{z,n}^{(\lambda,p)} f(z) &:= \Omega_z^{(\lambda,p)} (\Omega_{z,n-1}^{(\lambda,p)} f(z)) \\ &=: z^p + \sum_{k=1}^{\infty} \left[\frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} \right]^n a_k z^{k+p}. \end{aligned} \quad (9)$$

Similarly, for $f \in \mathcal{A}_p$, represented by (2), let the operator

$$\mathcal{E}_t^m : \mathcal{A}_p \longrightarrow \mathcal{A}_p \quad (m \in \mathbb{N}, t \in \mathbb{C}) \quad (10)$$

be defined by the following:

$$\begin{aligned} \mathcal{E}_t^0 f(z) &= f(z) \\ \mathcal{E}_t^1 f(z) &= (1-t)f(z) + \frac{t}{p} (zf'(z)) \\ &= z^p + \sum_{k=1}^{\infty} \left\{ 1 + \frac{tk}{p} \right\} a_k z^{k+p} \end{aligned} \quad (11)$$

and, for $m \in \mathbb{N} \setminus \{1\}$,

$$\mathcal{E}_t^m f(z) := \mathcal{E}_t^1 (\mathcal{E}_t^{m-1} f(z)) =: z^p + \sum_{k=1}^{\infty} \left\{ 1 + \frac{tk}{p} \right\}^m a_k z^{k+p}. \quad (12)$$

Very recently Srivastava et al. [6] considered the composition of the operators $\mathcal{E}_t^m$ and $\Omega_{z,n}^{(\lambda,p)}$ and introduced the following operator:

$$\mathcal{D}_t^{(\lambda,p)}(n, m) : \mathcal{A}_p \longrightarrow \mathcal{A}_p \quad (n, m \in \mathbb{N} \setminus \{1\}; t \in \mathbb{C}). \quad (13)$$

That is, for $f \in \mathcal{A}_p$, given by (2), we know that

$$\begin{aligned} \mathcal{D}_t^{(\lambda,p)}(n, m) f(z) &:= \Omega_{z,n}^{(\lambda,p)} (\mathcal{E}_t^m f(z)) \\ &=: \mathcal{E}_t^m (\Omega_{z,n}^{(\lambda,p)} f(z)) \\ &=: z^p + \sum_{k=1}^{\infty} \left[\frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} \right]^n \\ &\quad \times \left\{ 1 + \frac{tk}{p} \right\}^m a_k z^{k+p} \\ &\quad (n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, -\infty < \lambda < p+1, t \geq 0; z \in \mathbb{U}). \end{aligned} \quad (14)$$

The transformation $\mathcal{D}_t^{(\lambda,p)}(n, m)$ includes, among many, the following two previously studied interesting operators as particular cases.

- (i) For $n = m \in \mathbb{N}$, $p = 1$, $0 \leq \lambda < 1$, the fractional derivative operator $\mathcal{D}_t^{(\lambda,1)}(n, n) := \mathcal{D}_t^{n,\lambda}$ was recently introduced and investigated by Al-Oboudi and Al-Amoudi [9, 10], in the context of functions represented by conical domains.
- (ii) For $n = m \in \mathbb{N}$, $p = 1$, $\lambda = 0$, $t = 1$, $\mathcal{D}_1^{(0,1)}(n, n)$ is the Sălăgean operator [11], which is, in fact, the n -iterates of the popular Alexander's differential transform $f(z) \rightarrow zf'(z)$ [12].

We next recall the definition of subordination. Suppose that $f \in \mathcal{A}$ and $g \in \mathcal{A}$ is univalent in $\mathbb{U}$. We say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. Considering the function $w(z) = g^{-1}(f(z))$, it is readily checked that $w(z)$ satisfies the conditions of the Schwarz lemma and

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \quad (15)$$

In a broader sense the function $f \in \mathcal{A}$ is said to be subordinate to the function $g \in \mathcal{A}$ (g need not be univalent in $\mathbb{U}$), written as

$$f < g \text{ in } \mathbb{U} \quad \text{or} \quad f(z) < g(z) \quad (z \in \mathbb{U}), \quad (16)$$

if condition (15) holds for some Schwarz function $w(z)$ (see [12] for details). We also need the following definition of Hadamard product (or convolution). For the functions f and g in $\mathcal{A}_p$, given by the following Taylor-Maclaurin's series

$$\begin{aligned} f(z) &= z^p + \sum_{k=1}^{\infty} a_k z^{k+p}, \\ g(z) &= z^p + \sum_{k=1}^{\infty} b_k z^{k+p} \end{aligned} \quad (17)$$

$$(z \in \mathbb{U}),$$

their Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_k b_k z^{k+p} \quad (z \in \mathbb{U}). \quad (18)$$

It is easy to see that $f * g \in \mathcal{A}_p$.

The study of iterations of entire and meromorphic functions, as the number of iterations tends to infinity, is a popular topic in complex analysis. However, investigations have been initiated only recently regarding iterations of certain transforms defined on classes of analytic and meromorphic functions. For example, Al-Oboudi and Al-Amoudi [9, 10] investigated properties of certain classes of analytic functions associated with conical domains, by making use of the operator $\mathcal{D}_t^{n,\lambda}$. Their work generalized several earlier results of Srivastava and Mishra [13]. This theme has been further pursued in our more recent papers [6, 14, 15]. In the sequel to

these current investigations, in the present paper, we define the following subclass of $\mathcal{A}_p$ associated with the iterated operator $\mathcal{D}_t^{(\lambda,p)}(n,m)$ and investigate its several interesting properties. Our work is also motivated by earlier works in [16–20], connecting subordination and Hadamard product.

Definition 1. The function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{H}_p^{n,m}(\lambda, t; h)$ if the following subordination condition is satisfied:

$$(1-t)z^{-p}\mathcal{D}_t^{(\lambda,p)}(n,m)f(z) + \frac{t}{p}z^{-p+1}\left(\mathcal{D}_t^{(\lambda,p)}(n,m)f(z)\right)' < h(z) \quad (19)$$

$$(z \in \mathbb{U}),$$

where t is a complex number and h is an analytic convex univalent function in $\mathbb{U}$.

The function class $\mathcal{H}_p^{n,m}(\lambda, t; h)$ includes several previously studied subclasses of $\mathcal{A}$ as particular cases. For example,

- (i) for $n = 1, m = 0$ the class $\mathcal{H}_p^{1,0}(\lambda, t; h) := \mathcal{H}_p(\lambda, t; h)$ was recently studied by Liu [17];
- (ii) for $n = 1; m = 0; p = 1; \lambda = 1; t = 1$; and $h(z) = (1+z)/(1-z)$, the class $\mathcal{H}_1^{1,0}(1, 1; (1+z)/(1-z))$ was earlier investigated by R. Singh and S. Singh [21];
- (iii) for $n = 1; m = 0; p = 1; \lambda = 1$; and $h(z) = (1+az)/(1+bz)$ ($-1 \leq b < a \leq 1$), $\mathcal{H}_1^{1,0}(1, t; (1+az)/(1+bz))$ reduces to $\mathcal{H}(t, a, b)$ which was investigated by Yang [22];
- (iv) for $n = 1; m = 0; p = 1; \lambda = 1$; and $h(z) = 1+Mz$ ($M > 0$), $\mathcal{H}_1^{1,0}(1, t; 1+Mz)$ reduces to $\mathcal{S}(t, M)$, the class introduced and studied by Zhongzhu and Owa [23] and Jinlin [24].

In the present paper we primarily focus on a variety of convolution theorems for the class $\mathcal{H}_p^{n,m}(\lambda, t; h)$. We also find inclusion theorems and study behavior of the Libera-Livingston integral operator.

2. Some More Definitions and Preliminary Lemmas

We need the following definitions and results for the presentation of our results. Let $\mathcal{CV}(\rho)$ and $\mathcal{S}^*(\rho)$ ($0 \leq \rho < 1$) denote, respectively, the classes of *univalent convex functions of order ρ* and *starlike functions of order ρ* (see [12] for details). The function $f \in \mathcal{A}_1$ is said to be in the class $\mathcal{PS}^*(\rho)$ consisting of *prestarlike functions of order ρ* [25] if

$$\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in \mathcal{S}^*(\rho). \quad (20)$$

It is readily seen that

$$\mathcal{PS}^*(0) =: \mathcal{CV}(0) := \mathcal{CV}, \quad \mathcal{PS}^*\left(\frac{1}{2}\right) =: \mathcal{S}^*\left(\frac{1}{2}\right). \quad (21)$$

Furthermore, it is well known [25] that

$$\mathcal{PS}^*(\mu) \subset \mathcal{PS}^*(\lambda) \quad (\mu \leq \lambda). \quad (22)$$

We will also need the following lemmas in order to derive our main results.

Lemma 2 (see [26]; also see [27]). *Let g be analytic in $\mathbb{U}$ and let h be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu}zg'(z) < h(z) \quad (z \in \mathbb{U}), \quad (23)$$

where $\Re(\mu) \geq 0$ and $\mu \neq 0$, then

$$g(z) < \tilde{h}(z) = \mu z^{-\mu} \int_0^z \xi^{\mu-1} h(\xi) d\xi < h(z) \quad (z \in \mathbb{U}) \quad (24)$$

and $\tilde{h}$ is the best dominant of (24).

Lemma 3 (see [25]). *Let $\rho < 1$, $f \in \mathcal{S}^*(\rho)$, and $g \in \mathcal{PS}^*(\rho)$. Then, for any analytic function $\mathcal{F}$ in $\mathbb{U}$,*

$$\frac{g * (f\mathcal{F})}{g * f}(\mathbb{U}) \subset \overline{CO}(\mathcal{F}(\mathbb{U})), \quad (25)$$

where $\overline{CO}(\mathcal{F}(\mathbb{U}))$ denotes the closed convex hull of $\mathcal{F}(\mathbb{U})$.

Lemma 4 (see [28]). *Let H and G be univalent convex functions in $\mathbb{U}$, and let h and g be functions in $\mathcal{A}$. Suppose that $h < H$ and $g < G$ in $\mathbb{U}$. Then $h * g < H * G$ in $\mathbb{U}$.*

The following well known result is a consequence of the principle of subordination and can be found, for example, in [12, 29].

Lemma 5. *Let the function $g \in \mathcal{A}$ satisfy $g(0) = 1$ and $\Re(g(z)) > \beta_0$ ($0 \leq \beta_0 < 1, z \in \mathbb{U}$). Then*

$$\Re(g(z)) > \beta_0 + (1 - \beta_0) \frac{1 - |z|}{1 + |z|} \quad (z \in \mathbb{U}). \quad (26)$$

3. Convolution Results

We state and prove the following convolution results.

Theorem 6. *Let $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$, and suppose that g in $\mathcal{A}_p$ satisfies the following:*

$$\Re\{z^{-p}g(z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \quad (27)$$

Then

$$f * g \in \mathcal{H}_p^{n,m}(\lambda, t; h). \quad (28)$$

Proof. For every f and g in $\mathcal{A}_p$, we have

$$\begin{aligned} & (1-t) z^{-P} \mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \\ & + \frac{t}{p} z^{-P+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \right)' \\ & = (1-t) (z^{-P} g(z)) * \left(z^{-P} \mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \right) \\ & + \frac{t}{p} (z^{-P} g(z)) * \left(z^{-P+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \right)' \right) \\ & = (z^{-P} g(z)) * \Psi(z), \end{aligned} \quad (29)$$

where

$$\begin{aligned} \Psi(z) & = (1-t) z^{-P} \mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \\ & + \frac{t}{p} z^{-P+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \right)'. \end{aligned} \quad (30)$$

Now, if $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$, then

$$\Psi(z) < h(z) \quad (z \in \mathbb{U}). \quad (31)$$

Furthermore, condition (27) is equivalent to

$$z^{-P} g(z) < \frac{1}{1-z} \quad (z \in \mathbb{U}). \quad (32)$$

Therefore, an application of Lemma 4 in (29) yields

$$\begin{aligned} & (1-t) z^{-P} \mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \\ & + \frac{t}{p} z^{-P+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \right)' < \frac{1}{1-z} * h(z) \\ & = h(z). \end{aligned} \quad (33)$$

This shows that $f * g \in \mathcal{H}_p^{n,m}(\lambda, t; h)$. The proof of Theorem 6 is completed. $\square$

Remark 7. Taking $n = 1$, $m = 0$, in Theorem 6 we get a recent result of Liu ([17], Theorem 3). The choices $n = 1$, $m = 0$, $p = 1$, $\lambda = 1$, and $h(z) = (1 + az)/(1 + bz)$ ($-1 \leq b < 1$; $a > b$) yield a result of Yang ([22], Theorem 4).

Corollary 8. Let the function f given by (2) be a member of $\mathcal{H}_p^{n,m}(\lambda, t; h)$ and

$$s_r(z) = z^P + \sum_{k=1}^{r-1} a_k z^{k+P} \quad (r \in \mathbb{N} \setminus \{1\}; z \in \mathbb{U}). \quad (34)$$

Then the function

$$\sigma_r(z) = \int_0^1 t^{-P} s_r(tz) dt \quad (35)$$

is in the class $\mathcal{H}_p^{n,m}(\lambda, t; h)$.

Proof. Let $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$. We note that

$$\sigma_r(z) = z^P + \sum_{k=1}^{r-1} \frac{a_k}{k+1} z^{k+P} = (f * g_r)(z) \quad (r \in \mathbb{N} \setminus \{1\}), \quad (36)$$

where

$$\begin{aligned} f(z) & = z^P + \sum_{k=1}^{\infty} a_k z^{k+P}, \\ g_r(z) & = z^P + \sum_{k=1}^{r-1} \frac{z^{k+P}}{k+1} \in \mathcal{A}_p. \end{aligned} \quad (37)$$

Also, for $r \in \mathbb{N} \setminus \{1\}$, it is well known [18] that

$$\Re \{z^{-P} g_r(z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \quad (38)$$

In view of (36) and (38), an application of Theorem 6 gives

$$\sigma_r \in \mathcal{H}_p^{n,m}(\lambda, t; h). \quad (39)$$

The proof of Corollary 8 is completed. $\square$

Theorem 9. Let the function g in $\mathcal{A}_p$ be such that $z^{-P+1}g(z)$ is a prestarlike function of order ρ ($\rho < 1$). If $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$, then

$$f * g \in \mathcal{H}_p^{n,m}(\lambda, t; h). \quad (40)$$

Proof. Let $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$ and $g \in \mathcal{A}_p$. Then (29) gives

$$\begin{aligned} & (1-t) z^{-P} \mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \\ & + \frac{t}{p} z^{-P+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \right)' \\ & = \frac{(z^{-P+1}g(z)) * (z\Psi(z))}{(z^{-P+1}g(z)) * z} \quad (z \in \mathbb{U}), \end{aligned} \quad (41)$$

where $\Psi(z)$ is defined as in (30). We noted in the proof of Theorem 6 that $\Psi(z) < h(z)$. Since $z^{-P+1}g(z) \in \mathcal{PS}^*(\rho)$, $z \in \mathcal{S}^*(\rho)$, and $h(z)$ is convex univalent in $\mathbb{U}$, an application of Lemma 3 in (41) yields the following:

$$\begin{aligned} & (1-t) z^{-P} \mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \\ & + \frac{t}{p} z^{-P+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) (f * g)(z) \right)' < h(z) \quad (z \in \mathbb{U}). \end{aligned} \quad (42)$$

Therefore, $f * g \in \mathcal{H}_p^{n,m}(\lambda, t; h)$. The proof Theorem 9 is completed. $\square$

Taking $\rho = 1/2$ in Theorem 9 we get the following.

Corollary 10. Let $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$ and suppose that g in $\mathcal{A}_p$ is such that $z^{-P+1}g \in \mathcal{S}^*(1/2)$. Then

$$f * g \in \mathcal{H}_p^{n,m}(\lambda, t; h). \quad (43)$$

In particular, if $z^{-P+1}g$ is univalent convex then $f * g \in \mathcal{H}_p^{n,m}(\lambda, t; h)$.

In the following theorem we discuss convolution properties of the function class $\mathcal{H}(\lambda, t; h)$ when h is a right half plane mapping.

Theorem 11. Let $t \geq 0$ and suppose that each of the functions f_j ($j = 1, 2$) is a member of the class $\mathcal{H}_p^{n,m}(\lambda, t; h_j)$, where

$$h_j(z) = \frac{1 + (1 - 2\beta_j)z}{1 - z} \quad (0 \leq \beta_j < 1). \quad (44)$$

If $f \in \mathcal{A}_p$ is defined by the following

$$f(z) = \mathcal{D}_t^{(\lambda,p)}(n, m)(f_1 * f_2)(z) \quad (z \in \mathbb{U}), \quad (45)$$

then $f \in \mathcal{H}_p^{n,m}(\lambda, t; h)$, where

$$h(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (46)$$

and β is given by

$$\beta = \begin{cases} 1 - 4(1 - \beta_1)(1 - \beta_2) \\ \quad \times \left(1 - \frac{p}{t} \int_0^1 \frac{u^{(p/t)-1}}{1 + u} du\right) & (\lambda > 0) \\ 1 - 2(1 - \beta_1)(1 - \beta_2) & (\lambda = 0). \end{cases} \quad (47)$$

The bound on β is the best possible.

Proof. We consider the case $t > 0$. Suppose that $f_j \in \mathcal{H}_p^{n,m}(\lambda, t; h_j)$, ($j = 1, 2$), where $h_j(z)$ is given by (44). By setting

$$\begin{aligned} \Phi_j(z) &= (1 - t)z^{-p}\mathcal{D}_t^{(\lambda,p)}(n, m)f_j(z) \\ &\quad + \frac{t}{p}z^{-p+1}\left(\mathcal{D}_t^{(\lambda,p)}(n, m)f_j(z)\right)' \end{aligned} \quad (48)$$

we see, in the light of Definition 1, that

$$\Phi_j(z) < h_j(z) \quad (j = 1, 2). \quad (49)$$

A routine calculation yields the following:

$$\begin{aligned} \mathcal{D}_t^{(\lambda,p)}(n, m)f_j(z) &= \frac{p}{t}z^{-p(1-t)/t} \int_0^z \xi^{(p/t)-1} \Phi_j(\xi) d\xi \\ &\quad (j = 1, 2). \end{aligned} \quad (50)$$

Now, if $f(z)$ is defined by (45), then using (50) we get that

$$\begin{aligned} \mathcal{D}_t^{(\lambda,p)}(n, m)f(z) &= \mathcal{D}_t^{(\lambda,p)}(n, m)f_1(z) * \mathcal{D}_t^{(\lambda,p)}(n, m)f_2(z) \\ &= \left(\frac{p}{t}z^p \int_0^1 u^{(p/t)-1} \Phi_1(uz) du\right) \\ &\quad * \left(\frac{p}{t}z^p \int_0^1 u^{(p/t)-1} \Phi_2(uz) du\right) \\ &= \frac{p}{t}z^p \int_0^1 u^{(p/t)-1} \Phi_0(uz) du, \end{aligned} \quad (51)$$

where

$$\Phi_0(z) = \frac{p}{t}z^p \int_0^1 u^{(p/t)-1} (\Phi_1 * \Phi_2)(uz) du. \quad (52)$$

Since

$$\begin{aligned} \frac{\Phi_1(z) - \beta_1}{1 - \beta_1} &< \frac{1 + z}{1 - z}, \\ \frac{1}{2} + \frac{\Phi_2(z) - \beta_2}{2(1 - \beta_2)} &< \frac{1}{1 - z}, \end{aligned} \quad (53)$$

by using Lemma 4, we get

$$\left(\frac{\Phi_1(z) - \beta_1}{1 - \beta_1}\right) * \left(\frac{1}{2} + \frac{\Phi_2(z) - \beta_2}{2(1 - \beta_2)}\right) < \frac{1 + z}{1 - z}. \quad (54)$$

A simple calculation gives that

$$\Phi_1(z) * \Phi_2(z) < 1 \pm 4(1 - \beta_1)(1 - \beta_2) \frac{z}{1 - z}. \quad (55)$$

Therefore, by the Lindelöf principle of subordination we have

$$\begin{aligned} \Re\{\Phi_1(z) * \Phi_2(z)\} \\ > \beta_0 := 1 - 2(1 - \beta_1)(1 - \beta_2) \quad (z \in \mathbb{U}). \end{aligned} \quad (56)$$

Since

$$\begin{aligned} (1 - t)z^{-p}\mathcal{D}_t^{(\lambda,p)}(n, m)f(z) \\ + \frac{t}{p}z^{-p+1}\left(\mathcal{D}_t^{(\lambda,p)}(n, m)f(z)\right)' = \Phi_0(z), \end{aligned} \quad (57)$$

by using (52) in conjunction with (56) and Lemma 4 we get the following:

$$\begin{aligned} \Re\left\{(1 - t)z^{-p}\mathcal{D}_t^{(\lambda,p)}(n, m)f(z) \right. \\ \left. + \frac{t}{p}z^{-p+1}\left(\mathcal{D}_t^{(\lambda,p)}(n, m)f(z)\right)'\right\} &= \Re\{\Phi_0(z)\} \\ &= \frac{p}{t} \int_0^1 u^{(p/t)-1} \Re\{(\Phi_1 * \Phi_2)(uz)\} du \\ &\geq \frac{p}{t} \int_0^1 u^{(p/t)-1} \left(\beta_0 + (1 - \beta_0) \frac{1 - u|z|}{1 + u|z|}\right) du \\ &> \beta_0 + \frac{p(1 - \beta_0)}{t} \int_0^1 u^{(p/t)-1} \frac{1 - u}{1 + u} du \\ &= 1 - 4(1 - \beta_1)(1 - \beta_2) \left(1 - \frac{p}{t} \int_0^1 \frac{u^{(p/t)-1}}{1 + u} du\right) \\ &= \beta \quad (z \in \mathbb{U}). \end{aligned} \quad (58)$$

This proves that $f(z) \in \mathcal{H}_p^{n,m}(\lambda, t; h)$, where the function $h(z)$ is given by (46).

In order to show that the value of β is the least possible, we take the functions $f_j(z) \in \mathcal{A}_p$ ($j = 1, 2$) defined by

$$\begin{aligned} \mathcal{D}_t^{(\lambda, p)}(n, m) f_j(z) &= \frac{p}{t} z^{-p(1-t)/t} \int_0^z \xi^{(p/t)-1} \\ &\times \left(\beta_j + (1 - \beta_j) \frac{1 + \xi}{1 - \xi} \right) d\xi, \end{aligned} \quad (59)$$

for which we have

$$\begin{aligned} \Phi_j(z) &= (1 - t) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f_j(z) \\ &+ \frac{p}{t} z^{-p+1} \left(\mathcal{D}_t^{(\lambda, p)}(n, m) f_j(z) \right)' \\ &= \beta_j + (1 - \beta_j) \frac{1 + z}{1 - z} \quad (j = 1, 2), \end{aligned} \quad (60)$$

$$(\Phi_1 * \Phi_2)(z) = 1 + 4(1 - \beta_1)(1 - \beta_2) \frac{z}{1 - z}. \quad (61)$$

Hence, for $f \in \mathcal{A}_p$, given by (45), we obtain

$$\begin{aligned} (1 - t) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \\ &+ \frac{p}{t} z^{-p+1} \left(\mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \right)' \\ &= \frac{p}{t} \int_0^1 u^{(p/t)-1} \left(1 + 4(1 - \beta_1)(1 - \beta_2) \frac{uz}{1 - uz} \right) du \\ &\longrightarrow \beta \quad (\text{as } z \longrightarrow -1). \end{aligned} \quad (62)$$

Finally, for the case $t = 0$, the proof of Theorem 11 is simple, so we choose to omit the details involved. $\square$

4. Properties of the Libera-Livingston Transform

For the function $f \in \mathcal{A}_p$, the function $\mathcal{F}$ defined by

$$\mathcal{F}(z) = \frac{\mu + p}{z^\mu} \int_0^z \xi^{\mu-1} f(\xi) d\xi \quad (\Re(\mu) > -p; z \in \mathbb{U}) \quad (63)$$

is popularly known as the *Libera-Livingston transform* of f . We state and prove the following.

Theorem 12. Let $f \in \mathcal{H}_p^{n, m}(\lambda, t; h)$. Then the function $\mathcal{F}$ defined by (63) is in the class $\mathcal{H}_p^{n, m}(\lambda, t; \tilde{h})$, where

$$\tilde{h}(z) = (\mu + p) z^{-(\mu+p)} \int_0^z \xi^{\mu+p-1} h(\xi) d\xi < h(z) \quad (z \in \mathbb{U}). \quad (64)$$

Consequently, $\mathcal{F} \in \mathcal{H}_p^{n, m}(\lambda, t; h)$. The function $\tilde{h}(z)$ is the best dominant in (64).

Proof. We define the function K on $\mathbb{U}$ by

$$\begin{aligned} z^p K(z) &= (1 - t) \mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \\ &+ \frac{t}{p} z \left(\mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \right)'. \end{aligned} \quad (65)$$

Differentiating both the sides of (65) with respect to z , we get

$$\begin{aligned} pK(z) + zK'(z) \\ &= p(1 - t) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \left(\frac{z\mathcal{F}'(z)}{p} \right) \\ &+ tz^{-p+1} \left(\mathcal{D}_t^{(\lambda, p)}(n, m) \left\{ \frac{z\mathcal{F}'(z)}{p} \right\}' \right). \end{aligned} \quad (66)$$

Also, the defining relation (63) yields

$$(\mu + p) f(z) = \mu \mathcal{F}(z) + z\mathcal{F}'(z). \quad (67)$$

Now, a routine calculation using (65), (66), and (67) gives

$$\begin{aligned} (1 - t) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \\ &+ \frac{t}{p} z^{-p+1} \left(\mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \right)' \\ &= K(z) + \frac{zK'(z)}{\mu + p} \quad (z \in \mathbb{U}). \end{aligned} \quad (68)$$

Since $f \in \mathcal{H}_p^{n, m}(\lambda, t; h)$, we get the following from the preceding equation (68):

$$K(z) + \frac{zK'(z)}{\mu + p} < h(z). \quad (69)$$

Therefore, by applying Lemma 2 we have

$$K(z) < \tilde{h}(z) = (\mu + p) z^{-(\mu+p)} \int_0^z t^{\mu+p-1} h(t) dt < h(z). \quad (70)$$

This last subordination (70) is equivalent to

$$\mathcal{F} \in \mathcal{H}_p^{n, m}(\lambda, t; \tilde{h}) \subset \mathcal{H}_p^{n, m}(\lambda, t; h). \quad (71)$$

The proof of Theorem 12 is completed. $\square$

Theorem 13. Let $f \in \mathcal{A}_p$ and suppose that the function $\mathcal{F}$ is defined as in (63). If

$$\begin{aligned} (1 - \gamma) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \\ &+ \gamma z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) < h(z) \quad (\gamma > 0), \end{aligned} \quad (72)$$

then $\mathcal{F} \in \mathcal{H}_p^{n, m}(\lambda, 0; \tilde{h})$, where

$$\begin{aligned} \tilde{h}(z) &= \frac{\mu + p}{\gamma} z^{-(\mu+p)/\gamma} \\ &\times \int_0^z \xi^{((\mu+p)/\gamma)-1} h(\xi) d\xi < h(z) \quad (\Re(\mu) > -p). \end{aligned} \quad (73)$$

The function $\tilde{h}(z)$ is the best dominant in (73).

Proof. We define the function H on $\mathbb{U}$ by

$$H(z) = z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z). \quad (74)$$

Differentiation of both the sides of (74) combined with the identity

$$\frac{z \left(\mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \right)'}{p} = \mathcal{D}_t^{(\lambda, p)}(n, m) \left(\frac{z \mathcal{F}'(z)}{p} \right) \quad (75)$$

gives

$$z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \left(\frac{z \mathcal{F}'(z)}{p} \right) = H(z) + \frac{z H'(z)}{p}. \quad (76)$$

By making use of (67) we simplify the subordinate of (72) as follows:

$$\begin{aligned} & (1 - \gamma) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \\ & + \gamma z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \\ & = (1 - \gamma) H(z) + \gamma z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \\ & \times \left(\frac{\mu}{\mu + p} \mathcal{F}(z) + \frac{p}{\mu + p} \frac{z \mathcal{F}'(z)}{p} \right) \\ & = (1 - \gamma) H(z) + \frac{\mu \gamma}{\mu + p} z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \\ & + \frac{p \gamma}{\mu + p} z^{-p} \mathcal{D}_t^{(\lambda, p)} \left(\frac{z \mathcal{F}'(z)}{p} \right). \end{aligned} \quad (77)$$

Next, by using (74) and (76), the above identity further simplifies to the following:

$$\begin{aligned} & (1 - \gamma) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) \mathcal{F}(z) \\ & + \gamma z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \\ & = \left(1 - \frac{p \gamma}{\mu + p} \right) H(z) + \frac{p \gamma}{\mu + p} \left(H(z) + \frac{z H'(z)}{p} \right) \\ & = H(z) + \frac{\gamma}{\mu + p} z H'(z). \end{aligned} \quad (78)$$

The subordination (72) is thus equivalent to

$$\begin{aligned} & H(z) + \frac{\gamma}{\mu + p} z H'(z) < h(z), \\ & (\Re(\mu) > -p, \gamma > 0; z \in \mathbb{U}). \end{aligned} \quad (79)$$

Therefore, an application of Lemma 2 yields the assertion of Theorem 13. The proof of Theorem 13 is completed. $\square$

5. Inclusion Theorems

Theorem 14. Let $t > 0$, $\gamma > 0$, and $f \in \mathcal{H}_p^{n, m}(\lambda, t; \gamma h + 1 - \gamma)$. If $\gamma \leq \gamma_0$, then $f(z) \in \mathcal{H}_p^{n, m}(\lambda, 0; h)$, where

$$\gamma_0 = \frac{1}{2} \left(1 - \frac{p}{t} \int_0^1 \frac{u^{(p/t)-1}}{1+u} du \right)^{-1}. \quad (80)$$

The bound γ_0 is sharp when $h(z) = 1/(1-z)$.

Proof. Let $f \in \mathcal{H}_p^{n, m}(\lambda, t; \gamma h + 1 - \gamma)$. By setting

$$g(z) = z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \quad (z \in \mathbb{U}), \quad (81)$$

we have

$$\begin{aligned} & (1 - t) z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \\ & + \frac{t}{p} z^{-p+1} \left(\mathcal{D}_t^{(\lambda, p)}(n, m) f(z) \right)' \\ & = g(z) + \frac{t}{p} z g'(z). \end{aligned} \quad (82)$$

Therefore, by using Definition 1, we get

$$g(z) + \frac{t}{p} z g'(z) < \gamma h(z) + 1 - \gamma \quad (83)$$

and an application of Lemma 2 yields

$$g(z) < \frac{\gamma p}{t} z^{-p/t} \int_0^z \xi^{(p/t)-1} h(\xi) d\xi + 1 - \gamma = (h * \psi)(z), \quad (84)$$

where

$$\psi(z) = \frac{\gamma p}{t} z^{-p/t} \int_0^z \frac{\xi^{(p/t)-1}}{1-\xi} d\xi + 1 - \gamma. \quad (85)$$

It follows from (85) that if $0 < \gamma \leq \gamma_0$, where $\gamma_0 > 1$ is given by (80), then

$$\begin{aligned} \Re(\psi(z)) &= \frac{\gamma p}{t} \int_0^1 u^{(p/t)-1} \Re\left(\frac{1}{1-uz}\right) du + 1 - \gamma \\ &> \frac{\gamma p}{t} \int_0^1 \frac{u^{(p/t)-1}}{1+u} du + 1 - \gamma \\ &\geq \frac{1}{2} \quad (z \in \mathbb{U}). \end{aligned} \quad (86)$$

Equivalently,

$$\psi(z) < \frac{1}{1-z}, \quad (z \in \mathbb{U}). \quad (87)$$

Since $h(z)$ and $\psi(z)$ are both convex univalent functions in $\mathbb{U}$, using Lemma 4, we obtain from (84) that

$$g(z) < (h * \psi)(z) < h(z). \quad (88)$$

Therefore in view of (81) we have

$$z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) < h(z). \quad (89)$$

Or equivalently, $f(z) \in \mathcal{H}_p^{n, m}(\lambda, 0; h)$.

In order to prove that the bound on γ_0 is the best possible, set $h(z) = 1/(1-z)$ and let the function f be defined on $\mathbb{U}$ by

$$z^{-p} \mathcal{D}_t^{(\lambda, p)}(n, m) f(z) = \frac{\gamma p}{t} z^{-p/t} \int_0^z \frac{\xi^{(p/t)-1}}{1-\xi} d\xi + 1 - \gamma. \quad (90)$$

It can be readily verified that $f \in \mathcal{A}_p$ and

$$\begin{aligned} & (1-t) z^{-p} \mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \\ & + \frac{t}{p} z^{-p+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \right)' \\ & = \gamma h(z) + 1 - \gamma. \end{aligned} \quad (91)$$

Thus $f(z) \in \mathcal{H}_p^{n,m}(\lambda, t; \gamma h + 1 - \gamma)$. Also, for $\gamma > \gamma_0$, and $z \rightarrow -1$, we have

$$\begin{aligned} \Re \left\{ z^{-p} \mathcal{D}_t^{(\lambda,p)}(n,m) f(z) \right\} & \rightarrow \frac{\gamma p}{t} \int_0^1 \frac{u^{(p/t)-1}}{1+u} du \\ & + 1 - \gamma < \frac{1}{2} \end{aligned} \quad (92)$$

which shows that

$$f(z) \notin \mathcal{H}_p^{n,m}(\lambda, 0; h). \quad (93)$$

Hence the bound γ_0 cannot be increased when $h(z) = 1/(1-z)$. The proof of Theorem 14 is completed. $\square$

Taking $m = 0$, we use the notation

$$\mathcal{H}_p^{n,0}(\lambda, t; h) := \mathcal{H}_p^n(\lambda, t; h) \quad (94)$$

in the following theorem.

Theorem 15. Let $0 \leq t_1 < t_2$. Then

$$\mathcal{H}_p^n(\lambda, t_2; h) \subset \mathcal{H}_p^n(\lambda, t_1; h). \quad (95)$$

Proof. Let $0 \leq t_1 < t_2$ and suppose that $f \in \mathcal{H}_p^n(\lambda, t_2; h)$. We define

$$g(z) = z^{-p} \mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \quad (z \in \mathbb{U}). \quad (96)$$

The function g is analytic in $\mathbb{U}$ with $g(0) = 1$. Differentiating the expressions on both the sides of (96) with respect to z and using Definition 1, we have

$$\begin{aligned} & (1-t_2) z^{-p} \mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \\ & + \frac{p}{t_2} z^{-p+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \right)' \\ & = g(z) + \frac{t_2}{p} z g'(z) < h(z) \quad (z \in \mathbb{U}). \end{aligned} \quad (97)$$

Hence an application of Lemma 2 yields

$$g(z) < h(z) \quad (z \in \mathbb{U}). \quad (98)$$

Noting that $0 \leq t_1/t_2 < 1$ and $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (96), (97), and (98) that

$$\begin{aligned} & (1-t_1) z^{-p} \mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \\ & + \frac{p}{t_1} z^{-p+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \right)' \\ & = \frac{t_1}{t_2} \left\{ (1-t_2) z^{-p} \mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \right. \\ & \quad \left. + \frac{p}{t_2} z^{-p+1} \left(\mathcal{D}_t^{(\lambda,p)}(n,0) f(z) \right)' \right\} + \left(1 - \frac{t_1}{t_2} \right) g(z) \\ & < h(z) \quad (z \in \mathbb{U}). \end{aligned} \quad (99)$$

Thus $f \in \mathcal{H}_p^n(\lambda, t_1; h)$. Consequently,

$$\mathcal{H}_p^n(\lambda, t_2; h) \subset \mathcal{H}_p^n(\lambda, t_1; h). \quad (100)$$

The proof of Theorem 15 is completed. $\square$

Remark 16. The particular case $n = 1$ of Theorem 15 gives a result of Liu ([17], Theorem 1).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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