Research Article
Spectral Bounds for Polydiagonal Jacobi Matrix Operators

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The research on spectral inequalities for discrete Schrödinger operators has proved fruitful in the last decade. Indeed, several authors analysed the operator’s canonical relation to a tridiagonal Jacobian matrix operator. In this paper, we consider a generalisation of this relation with regard to connecting higher order Schrödinger-type operators with symmetric matrix operators with arbitrarily many nonzero diagonals above and below the main diagonal. We thus obtain spectral bounds for such matrices, similar in nature to the Lieb-Thirring inequalities.

1. Background
Let \( W \) be the self-adjoint Jacobi matrix operator acting on \( \ell^2(\mathbb{Z}) \) as follows:

\[
W = \begin{pmatrix}
..., & \vdots & \vdots & \vdots & \vdots & \\
\vdots & b_{-1} & a_{-1} & 0 & 0 & \vdots \\
\vdots & a_{-1} & b_0 & a_0 & 0 & \vdots \\
\vdots & 0 & a_0 & b_1 & a_1 & \vdots \\
\vdots & 0 & 0 & a_1 & b_2 & \vdots \\
... & \vdots & \vdots & \vdots & \vdots & 
\end{pmatrix},
\]

via

\[
(W\varphi)(n) = a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1),
\]

for \( n \in \mathbb{Z} \),

where \( a_n > 0 \) and \( b_n \in \mathbb{R} \). This operator can be viewed as the one-dimensional discrete Schrödinger operator if \( a_n = 1 \) for all \( n \). A variety of papers examined such operators; for example, we quote the work by Killip and Simon in [1], where they obtained sum rules for such Jacobi matrices. Additionally, Hundertmark and Simonin [2] were able to find spectral bounds for these operators. We thus state their result.

If \( a_n \to 1, b_n \to 0 \) rapidly enough, as \( n \to \pm \infty \), the essential spectrum \( \sigma_{ess}(W) \) of \( W \) is absolutely continuous and coincides with the interval \([-2, 2]\) (see, e.g., [3]). Besides, \( W \) may have simple eigenvalues \( \{E^+_{j}\}_{j=1}^{N_h} \) where \( N_h \in \mathbb{N} \cup \{\infty\} \), and

\[
E_1^+ > E_2^+ > \cdots > 2 > -2 > \cdots > E_2^- > E_1^-.
\]

Indeed, in [2] the authors found the following.

**Theorem 1.** If \( \{b_n\}_{n \in \mathbb{Z}}, \{a_n - 1\}_{n \in \mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z}), \gamma \geq 1/2 \), then

\[
\sum_{j=1}^{N_h} |E_j^+ - 2|^\gamma + \sum_{j=1}^{N_h} |E_j^- + 2|^\gamma
\]

\[
\leq k_\gamma \left[ \sum_{n=-\infty}^{\infty} |b_n|^{\gamma+1/2} + 4 \sum_{n=-\infty}^{\infty} |a_n - 1|^{\gamma+1/2} \right],
\]

where

\[
k_\gamma = 2 \left(3^{\gamma-1/2}\right) L_{\gamma,1}^d, \quad L_{\gamma,1}^d = \frac{\Gamma(\gamma + 1)}{2\sqrt{\pi} \Gamma(\gamma + 3/2)}.
\]

The author (see [4]) then improved their result, achieving the smaller constant: \( k_\gamma = 3^{\gamma-1} n L_{\gamma,1}^c \), by translating a well-known method employed by Dolbeaut et al. in [5] to the discrete scenario. They, in turn, used a simple argument by Eden and Foias (see [6]) to obtain improved constants for Lieb-Thirring inequalities in one dimension.

The aim of this paper is to answer the natural question of whether these methods can be generalised to give bounds
2. Notation and Preliminary Material

For a sequence \( \{ \varphi(n) \}_{n \in \mathbb{Z}} \), let \( D \) and \( D^* \) be the difference operator and its adjoint, respectively, denoted by \( D \varphi(n) = \varphi(n + 1) - \varphi(n) \) and \( D^* \varphi(n) = \varphi(n) - \varphi(n - 1) \). We then denote the discrete one-dimensional Laplacian by \( \Delta_D := (D^* D \varphi)(n) = -\varphi(n + 1) + 2 \varphi(n) - \varphi(n - 1) \). For \( \sigma \in \mathbb{N}, n \in \mathbb{Z} \), and a sequence \( \varphi \in \ell^2(\mathbb{Z}) \), with \( \Delta_D^\sigma = \Delta_D \), we define \( \Delta_D^\sigma \) by

\[
(\Delta_D^\sigma \varphi)(n) := (\Delta_D \left(\Delta_D^{\sigma - 1} \varphi\right))(n).
\]

We note that \( \Delta_D^\sigma \) being self-adjoint immediately implies that \( \Delta_D^\sigma \) is also self-adjoint.

Finding an explicit formula for \( \Delta_D^\sigma \) requires a few combinatorial techniques, all of which are standard. Let \( a_C \) denote the discrete one-dimensional Laplacian by

\[
\Delta(x) := \sum_{k=1}^{N} (x_k - x_{k+1})^2.
\]

Finally, we will apply these results to obtain spectral bounds for the following operator.

Let \( W_o \) be a polydiagonal self-adjoint Jacobi-type matrix operator as follows:

\[
W_o := \begin{pmatrix}
    b_1 & a_1^0 & 0 & 0 \\
    a_1^1 & b_0 & a_0^0 & 0 \\
    0 & a_2^1 & b_1 & a_1^0 \\
    0 & 0 & a_2^2 & b_2
\end{pmatrix}
\]

viewed as an operator acting on \( \ell^2(\mathbb{Z}) \) as follows: for \( n \in \mathbb{Z}, i \in \{1, \ldots, \sigma\} \),

\[
(W_o \varphi)(n) = \sum_{i=1}^{\sigma} a_i \varphi(n-i) + b_i \varphi(n) + \sum_{i=1}^{\sigma} a_i^i \varphi(n+i)
\]

where \( a_i, b_i \in \mathbb{R} \), for all \( i \in \{1, \ldots, \sigma\} \). We denote \( (W_o^\gamma \{a_1^1, \ldots, a_\sigma^\gamma, b_\gamma\})(n) := (W_o \varphi)(n) \) where we understand \( \{\} \) to mean \( \{0\}_{n \in \mathbb{Z}} \). We are then interested in perturbations of the following special case:

\[
(W_o^\gamma \varphi)(n) := \left(W_o \left\{ a_1^1 \equiv \omega_1, \ldots, a_\sigma^\gamma \equiv \omega_\gamma \right\} \right)(n),
\]

Theorem 2. Let \( b_n \geq 0, \{b_n\}_{n \in \mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z}), \gamma \geq 1 \). Then the negative eigenvalues \( \{e_j\}_{j=1}^{N} \) of the operator \( H_o^\gamma \) satisfy the inequality

\[
\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_\gamma^\gamma \sum_{n \in \mathbb{Z}} b_n^{\gamma+1/2\sigma},
\]

where

\[
\eta_\gamma^\gamma := \frac{2\sigma}{(2\sigma + 1)2^{(2\sigma+1)/2\sigma}} \left( \frac{\Gamma((4\sigma + 1)/2\sigma)}{\Gamma(\gamma + 1)} \right)^{\sigma}(\frac{\Gamma(\gamma + (2\sigma + 1)/2\sigma)}{\Gamma(\gamma + 1)}).
\]

Remark 3. As the discrete spectrum of \( H_o^\gamma \) lies in \([-\infty, 0] \) and \([4\gamma, \infty] \), we shift our operator to the left by \( 4\gamma \) and by analogy have an estimate for the positive eigenvalues of that operator, thus immediately obtaining Corollary 4.

Corollary 4. Let \( b_n \geq 0, \{b_n\}_{n \in \mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z}), \gamma \geq 1 \). Then the positive eigenvalues \( \{e_j\}_{j=1}^{N} \) of the operator \( \Delta_D^\sigma - 4\gamma b \) satisfy the inequality:

\[
\sum_{j=1}^{N} e_j^\gamma \leq \eta_\gamma^\gamma \sum_{n \in \mathbb{Z}} b_n^{\gamma+1/2\sigma}, \quad \text{with } \eta_\gamma^\gamma \text{ given above.}
\]
where \( \omega_i := 3\alpha C_\sigma (-1)^i \), and explicitly
\[
(W_0^0 \varphi)(n) = \left( (\Delta^\sigma_D - 2\alpha C_\sigma) \varphi \right)(n) = \sum_{k=0}^{2\alpha} C_\sigma (-1)^k \varphi(n - \sigma + k),
\]
(17)
called the free Jacobi-type matrix of order \( \sigma \). In particular, we examine the case where \( W_\sigma - W_0^0 \) is compact. Thus in what follows we assume that our sequences tend to the operator coefficients rapidly enough; that is, \( a_i^k \to \omega_i, b_n \to 0 \), as \( n \to \pm \infty \). Then the essential spectrum \( \kappa_{es} \) is given by \( \kappa_{es}(W_\sigma) = \kappa_{es}(W_0^0) = [2\alpha^2 C_\sigma, 4^\sigma - 2\alpha C_\sigma] \) and \( W_\sigma \) may have simple eigenvalues \( \{E_j^k\}_{j=1}^N \) where \( N_k \in \mathbb{N} \), and
\[
E_1^+ > E_2^+ > \cdots > 4^\sigma - 2\alpha C_\sigma > \cdots > E_n^+ > E_n^-.
\]
(18)

**Theorem 5.** Let \( \gamma \geq 1, \{a_i^k\}_{i \in \mathbb{Z}} \), and \( \{b_n - \omega_n\}_{n \in \mathbb{Z}} \in \ell^{\gamma + 1/2\sigma}(\mathbb{Z}) \) for all \( i \in \{1, \ldots, \sigma \} \). Then for the eigenvalues \( \{E_j^k\}_{j=1}^N \) of the operator \( W_\sigma \) we have
\[
\sum_{j=1}^N |E_j^+ + 2\alpha C_\sigma|^\gamma + \sum_{j=1}^N |E_j^- - (4^\sigma - 2\alpha C_\sigma)|^\gamma \leq \nu^\gamma \left( \sum_{n \in \mathbb{Z}} |b_n|^{\gamma + 1/2\sigma} + 4 \sum_{n \in \mathbb{Z}} \sum_{k=1}^\sigma |a_n - \omega_k|^{\gamma + 1/2\sigma} \right),
\]
where
\[
\nu^\gamma = 2\sigma(2\sigma + 1)^{\gamma - 1/2} \Gamma{(4\sigma + 1)}/\Gamma{(\gamma + 1)}. \quad (19)
\]

**4. Auxiliary Results**

We require the following discrete Kolmogorov-type inequality.

**Lemma 6.** For a sequence \( \varphi \in \ell^1(\mathbb{Z}) \), and for \( n > k \geq 1 \), we have the following inequality:
\[
\|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})}. \quad (20)
\]

**Proof.** We proceed by induction, where we note that the initial case, \( k = 1, n = 2 \), holds true as the inequality
\[
\|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})}. \quad (21)
\]
is in fact the simple inequality found by Copson in [7]. This case in turn, if used repeatedly, shows that the inequality holds true for all \( k, \) if \( k = k+1 \). We then take the inductive step on the variable \( n \). Hence we assume that we have the required inequality for \( k < n \leq m \), given a fixed \( k \), and proceed to prove the statement for \( n = m + 1 \). Thus
\[
\|D_m^{\varphi}\|_{\ell^1(\mathbb{Z})} = \langle D_m^{\varphi}, D_m^{\varphi} \rangle = \langle D^* D_m^{\varphi}, D_m^{m-1} \varphi \rangle \leq \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})} \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})},
\]
(22)

We thus apply our induction hypothesis and set \( k = m - 1 \) and \( n = m \) as follows:
\[
\|D_m^{\varphi}\|_{\ell^1(\mathbb{Z})} \leq \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})} \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})} \leq \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})} \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})} \leq \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})} \|D_m^{m-1} \varphi\|_{\ell^1(\mathbb{Z})},
\]
(23)

We now return to the induction hypothesis as follows:
\[
\|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})} \|D_k^{\varphi}\|_{\ell^1(\mathbb{Z})},
\]
(24)

We are now equipped to prove an Agmon-Kolmogorov-type inequality.

**Proposition 7.** For a sequence \( \varphi \in \ell^1(\mathbb{Z}) \), we have for any \( \sigma \in \mathbb{N} \)
\[
\|\varphi\|_{\ell^\infty(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})}^{1/\sigma} \|D^\sigma \varphi\|_{\ell^1(\mathbb{Z})}^{1/\sigma}. \quad (25)
\]

**Proof.** First we use Lemma 6 with \( k = 1, n = \sigma \) as follows:
\[
\|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})},
\]
(26)

and we apply this estimate to the well-known discrete Agmon inequality (see [4]):
\[
\|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \leq \|\varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})} \|D \varphi\|_{\ell^1(\mathbb{Z})},
\]
(27)

**Proposition 8.** Let \( \{\psi_j\}_{j=1}^N \) be an orthonormal system of sequences in \( \ell^2(\mathbb{Z}) \); that is, \( \langle \psi_j, \psi_k \rangle = \delta_{jk} \), and let \( \rho(n) := \sum_{j=1}^N |\psi_j(n)|^2 \). Then
\[
\sum_{n \in \mathbb{Z}} \rho^{2\alpha + 1}(n) \leq \sum_{j=1}^N \sum_{n \in \mathbb{Z}} |D^\alpha \psi_j(n)|^2. \quad (29)
\]

**Proof.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{C}^N \). By Proposition 7, we have
\[
\sum_{j=1}^N \xi_j \psi_j(n) \leq \sum_{j=1}^N \xi_j \psi_j(n) \leq |D^\alpha \sum_{j=1}^N \xi_j \psi_j(n)|^{1/\alpha} \leq |\sum_{j=1}^N \xi_j \psi_j(n)|^{2\alpha + 1/\alpha} = \left( \sum_{j,k=1}^N \xi_j \xi_k \langle \psi_j, \psi_k \rangle \right)^{(2\alpha + 1)/2\alpha},
\]
(28)
\[
\times \left( \sum_{j=1}^{N} \xi_{j} \xi_{k} \left( D^\sigma \psi_{j}, D^\sigma \psi_{k} \right) \right)^{1/2\sigma} \leq \left( \sum_{j=1}^{N} \xi_{j}^2 \right)^{(2\sigma-1)/2\sigma} \left( \sum_{j=1}^{N} \xi_{j} \xi_{k} \left( D^\sigma \psi_{j}, D^\sigma \psi_{k} \right) \right)^{1/2\sigma} \times \left( \sum_{j=1}^{N} \xi_{j} \xi_{k} \left( D^\sigma \psi_{j}, D^\sigma \psi_{k} \right) \right)^{1/2\sigma} .
\]

Let \( \xi_{j} := \psi_{j}(n) \) and as \( \rho(n) = \sum_{j=1}^{N} |\psi_{j}(n)|^2 \),
\[
\rho^2(n) \leq \rho^{(2\sigma-1)/2\sigma}(n)
\times \left( \sum_{j=1}^{N} \psi_{j}(n) \bar{\psi}_{k}(n) \left( D^\sigma \psi_{j}, D^\sigma \psi_{k} \right) \right)^{1/2\sigma} \Rightarrow \rho^{2\sigma+1}(n) \leq \sum_{j,k=1}^{N} \psi_{j}(n) \bar{\psi}_{k}(n) \left( D^\sigma \psi_{j}, D^\sigma \psi_{k} \right)
\Rightarrow \sum_{n \in \mathbb{Z}} \rho^{2\sigma+1}(n) \leq \sum_{j=1}^{N} \left( \sum_{n \in \mathbb{Z}} |D^\sigma \psi_{j}(n)|^2 \right) .
\]

5. Proof of Theorem 2

We take the inner product with \( j \) sides of the equation with respect to \( j \). We obtain
\[
\sum_{j=1}^{N} e_{j} = \sum_{j=1}^{N} \left( \sum_{n \in \mathbb{Z}} |D^\sigma \psi_{j}(n)|^2 \right) - \sum_{j=1}^{N} \left( \sum_{n \in \mathbb{Z}} b_{n} |\psi_{j}(n)|^2 \right) .
\]

We now use Proposition 8 and apply the appropriate Hölder's inequality; that is,
\[
\sum_{j=1}^{N} e_{j} \geq \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_{j}(n)|^2 \right)^{2\sigma+1} - \left( \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} \right)^{2\sigma/(2\sigma+1)} \times \left( \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_{j}(n)|^2 \right)^2 \right)^{1/(2\sigma+1)} .
\]

We define
\[
\chi := \left( \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_{j}(n)|^2 \right)^{2\sigma+1} \right)^{1/(2\sigma+1)} ,
\]
\[
\zeta := \left( \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} \right)^{2\sigma/(2\sigma+1)} .
\]

The latter inequality can be written as
\[
\chi^{2\sigma+1} - \zeta \chi \leq \sum_{j=1}^{N} e_{j} .
\]

The LHS is maximal when
\[
\chi = \frac{1}{(2\sigma+1)^{1/2\sigma}} \left( \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} \right)^{1/(2\sigma+1)} .
\]

Substituting this into (33), we obtain
\[
\sum_{j=1}^{N} e_{j} \geq \left( \frac{1}{(2\sigma+1)^{1/2\sigma}} \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} \right)
- \frac{1}{(2\sigma+1)^{1/2\sigma}} \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} .
\]

Therefore,
\[
\sum_{j=1}^{N} e_{j} \leq \frac{2\sigma}{(2\sigma+1)^{1/2\sigma}} \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} .
\]

We lift this bound now with regard to moments by using the standard Aizenman-Lieb procedure (see [8]). We let \( \{e_{j}(\tau)\}_{j=1}^{N} \) be the negative eigenvalues of the operator \( \Delta^\sigma_D - (b_{n} - \tau) \). By the variational principle for the negative eigenvalues \( \{-|e_{j}| - \tau\}_{j=1}^{N} \) of the operator \( \Delta^\sigma_D - (b_{n} - \tau) \) we have
\[
(|e_{j}| - \tau)_{+} \leq |e_{j}(\tau)| .
\]

By this estimate, we find that
\[
\sum_{j=1}^{N} |e_{j}|^{\gamma} \leq \frac{1}{B(y - 1, 2)} \int_{0}^{\infty} \tau^{\gamma/2} \sum_{j=1}^{N} |e_{j}| (\tau)_{+} d\tau
\leq \frac{1}{B(y - 1, 2)} \int_{0}^{\infty} \tau^{\gamma/2} \sum_{j=1}^{N} |e_{j}| (\tau) d\tau
\leq \frac{2\sigma}{(2\sigma+1)^{1/2\sigma}} \frac{1}{B(y - 1, 2)} \int_{0}^{\infty} \tau^{\gamma/2} \sum_{n \in \mathbb{Z}} (b_{n} - \tau)^{(2\sigma+1)/2\sigma} d\tau ,
\]

by (38) above, where \( B(x, y) = \Gamma(x) \Gamma(y)/\Gamma(x + y) \) is the well-known Beta function. Thus, after a change of variable,
\[
\sum_{j=1}^{N} |e_{j}|^{\gamma} \leq \frac{2\sigma}{(2\sigma+1)^{1/2\sigma}} \Gamma((4\sigma+1)/2\sigma) \Gamma(y + 1) \sum_{n \in \mathbb{Z}} b_{n}^{(2\sigma+1)/2\sigma} ,
\]
completing our proof.
6. Proof of Theorem 5

We have the following matrix bounds for square, \( m \times m \) matrices, as given in [2]. For \( a^m_n, \omega_m \in \mathbb{R} \), we have

\[
\begin{pmatrix}
-|a^m_n - \omega_m| & 0 & 0 & \cdots & 0 & \omega_m \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\omega_m & 0 & 0 & \cdots & 0 & |a^m_n - \omega_m| \\
\end{pmatrix} \leq \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\omega_m & 0 & 0 & \cdots & 0 & |a^m_n - \omega_m| \\
\end{pmatrix}
\]

(42)

Now \( (E_j^+ - (4^\sigma - 2^\alpha C_\sigma)) \) are positive eigenvalues of \( W_\sigma ([a^\sigma_n], \ldots, [a^\sigma_n], b_n - (4^\sigma - 2^\alpha C_\sigma)) \). Thus by using (43) and the variational principle, we have

\[
W_\sigma ([a^\sigma_n], \ldots, [a^\sigma_n], b_n - (4^\sigma - 2^\alpha C_\sigma)) \leq W_\sigma ([a^\sigma_n \equiv \omega_1], \ldots, [a^\sigma_n \equiv \omega_\sigma], \{b_n^{(+)} - (4^\sigma - 2^\alpha C_\sigma)\})
\]

\[
\Rightarrow |E_j^+ - (4^\sigma - 2^\alpha C_\sigma)| \leq e_j^+ ,
\]

(48)

where \( e_j^+ \) are the positive eigenvalues of

\[
W_\sigma ([a^\sigma_n \equiv \omega_1], \ldots, [a^\sigma_n \equiv \omega_\sigma], \{b_n^{(+)} - (4^\sigma - 2^\alpha C_\sigma)\}) = \Delta_\sigma^D - 4^\sigma + b_n^{(+)}.
\]

(49)

Let us now define \( (b_{h}) := \max(b_h, 0), (b_{h}) := -\min(b_h, 0). \) Then, by Corollary 4 for the positive eigenvalues of our operator, we have

\[
\sum_{j=1}^{N_\sigma} (e_j^+)^{y} \leq \eta_0^{y} \sum_{n \in Z} (b_n^{(+)} + 1/2^\alpha).
\]

(50)

Thus, applying (48),

\[
\sum_{j=1}^{N_\sigma} |E_j^+ - (4^\sigma - 2^\alpha C_\sigma)|^y \leq \eta_0^{y} \sum_{n \in Z} \left( (b_n^{(+)} + 1/2^\alpha \right)^{y + 1/2^\alpha}
\]

(51)

where

\[
\eta_0^{y} := \frac{2^\alpha}{(2^\sigma + 1)^{(2^\alpha + 1)/2^\sigma}} \Gamma\left((4^\sigma + 1)/2^\sigma\right) \Gamma\left(y + 1\right) \Gamma\left(y + (2^\sigma + 1)/2^\sigma\right).
\]

(52)

Similarly, using Theorem 2 on (47),

\[
\sum_{j=1}^{N_\sigma} |E_j^+ + 2^\alpha C_\sigma|^y \leq \eta_0^{y} \sum_{n \in Z} \left( (b_n^{(-)} + 1/2^\alpha \right)^{y + 1/2^\alpha}
\]

(53)

Using the following application of Jensen’s inequality, that is, for \( i \in \{1, \ldots, 2^\alpha + 1\} \), let \( \alpha_i, q \in \mathbb{R} \), with \( q \geq 1 \),

\[
\left( \sum_{i=1}^{2^\alpha + 1} \alpha_i \right)^q \leq (2^\sigma + 1)^{q-1} \left( \sum_{i=1}^{2^\alpha + 1} \alpha_i^q \right).
\]

(54)
to each of (51) and (53), we have

\[
\left( (b^*_n) + \sum_{k=1}^\sigma |d^k_{n-k} - \omega_k| + |a^k_n - \omega_k| \right)^{\gamma+1/2\sigma} \\
\leq (2\sigma + 1)^{-(2\sigma-1)/2\sigma} \times \left( (b^*_n)^{\gamma+1/2\sigma} + \sum_{k=1}^\sigma |d^k_{n-k} - \omega_k|^{\gamma+1/2\sigma} + |a^k_n - \omega_k|^{\gamma+1/2\sigma} \right). 
\]

(55)

Summing these two inequalities, we arrive at

\[
\sum_{j=1}^N |E_j + 2\sigma C_\sigma|^\gamma + \sum_{j=1}^N |E_j - (4\sigma - 2\sigma C_\sigma)|^\gamma \\
\leq \nu_\sigma^\gamma \left( \sum_{n \in \mathbb{Z}} |b^*_n|^{\gamma+1/2\sigma} + 4 \sum_{n \in \mathbb{Z}} \sum_{k=1}^\sigma |d^k_{n-k} - \omega_k|^{\gamma+1/2\sigma} \right), 
\]

(56)

where

\[
\nu_\sigma^\gamma = 2\sigma(2\sigma + 1)^{\gamma-2} \frac{\Gamma \left( (4\sigma + 1) / 2\sigma \right) \Gamma \left( \gamma + 1 \right)}{\Gamma \left( \gamma + (2\sigma + 1) / 2\sigma \right)}.
\]

(57)

and the proof of Theorem 5 is complete.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


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