Research Article

Meromorphic Parabolic Starlike Functions Associated with $q$-Hypergeometric Series

G. Murugusundaramoorthy and T. Janani

School of Advanced Sciences, VIT University, Vellore 632014, India

Correspondence should be addressed to G. Murugusundaramoorthy; gmsmoorthy@yahoo.com

Received 28 February 2014; Accepted 20 March 2014; Published 7 April 2014

Academic Editors: A. Peris and S. Zhang

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We introduce a new class of meromorphic parabolic starlike functions with a fixed point defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ involving the $q$-hypergeometric functions. We obtained coefficient inequalities, growth and distortion inequalities, and closure results for functions $f \in M_{\lambda}^{(2)}(\lambda, \beta, \gamma)$. We further established some results concerning convolution and the partial sums.

1. Introduction

Let $\xi$ be a fixed point in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $A(\Delta)$ the class of functions which are regular and

$$A(\xi) = \{f \in H(\Delta) : f(\xi) = f'(\xi) - 1 = 0\}. \quad (1)$$

Also denote by $A(\xi) = \{f \in A(\xi) : f$ is univalent in $\Delta\}$, the subclass of $A(\xi)$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n(z - \xi)^n \quad (2)$$

which are analytic in $\Delta$. Note that $A_0 = A$ is subclasses of $A$ consisting of univalent functions in $\Delta$. By $A^*_\beta$ and $A^*_\beta(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions $\Re\left(\frac{(z - \xi)f'(z)/f(z)}{1 + ((z - \xi)f''(z)/f'(z))}\right) > \beta$, and $\Re\left(\frac{(z - w)f'(z)/f(z)}{1 + ((z - \xi)f''(z)/f'(z))}\right) > \beta$, $(z - w) \in \Delta$ for $0 \leq \beta < 1$ introduced and studied by Kanas and Ronning [1]. The class $A^*_\beta(0)$ is defined by geometric property that the image of any circular arc centered at $\xi$ is starlike with respect to $f(\xi)$ and the corresponding class $A^*_\beta(0)$ is defined by the property that the image of any circular arc centered at $\xi$ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [2, 3] for uniformly starlike and convex functions, except that in this case the point $\xi$ is fixed.

In particular, $A = A_0(0)$ and $A^*_\beta = A^*_\beta(0)$, respectively, are the well-known standard classes of convex and starlike functions.

Let $\Sigma$ denote the class of meromorphic functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (3)$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Denote by $\Sigma_\xi$ the subclass of $\Sigma$ consisting of the functions of the form

$$f(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_n(z - \xi)^n, \quad a_n \geq 0; \ z \neq \xi. \quad (4)$$

A function $f$ of the form (4) is in the class of meromorphic starlike of order $\gamma (0 \leq \gamma < 1)$ denoted by $\Sigma^*_\gamma(\gamma)$, if

$$-\Re\left(\frac{(z - \xi)f'(z)/f(z)}{f(z)}\right) > \gamma, \quad z - \xi \in \Delta := \Delta^* \cup \{0\}, \quad (5)$$

and is in the class of meromorphic convex of order \( \gamma \) (0 \leq \gamma < 1) denoted by \( \Sigma^\gamma \), if

\[
-\Re \left(1 + \frac{(z-\xi)}{f''(z)}\right) > \gamma, \quad z - \xi \in \Delta := \Delta^* \cup \{0\}.
\]

(6)

For functions \( f(z) \) given by (4) and \( g(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} b_n(z - \xi)^n \), \( (b_n \geq 0) \) we define the Hadamard product or convolution of \( f \) and \( g \) by

\[
(f \ast g)(z) := \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_n b_n (z - \xi)^n.
\]

(7)

More recently, Purohit and Raina [4] introduced a generalized q-Taylor's formula in fractional q-calculus and derived certain q-generating functions for q-hypergeometric functions. In this work we proceed to derive a generalized differential operator on meromorphic functions in \( \Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\} \) involving these functions and discuss some of their properties.

For complex parameters \( a_1, \ldots, a_l \) and \( b_1, \ldots, b_m \) \( (b_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m) \) the \( q \)-hypergeometric function \( \psi_m(z) \) is defined by

\[
\psi_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z) := \sum_{n=1}^{\infty} \frac{(a_1; q)_n \cdots (a_l; q)_n}{(b_1; q)_n \cdots (b_m; q)_n} \frac{z^n}{n!},
\]

(8)

with \( (a_n) = n(n-1)/2 \) where \( q \neq 0 \) when \( l > m + 1 \) \( (l, m \in \mathbb{N}; b_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}) \).

The \( q \)-shifted factorial is defined for \( a, q \in \mathbb{C} \) as a product of \( n \) factors by

\[
(a; q)_n = \frac{1}{(1-a) (1-aq) \cdots (1-aq^{n-1})}, \quad n \in \mathbb{N},
\]

(9)

and in terms of basic analogue of the gamma function

\[
(q^n; q)_n = \frac{\Gamma_q (a+n) (1-q)^n}{\Gamma_q (a)}, \quad n > 0.
\]

(10)

It is of interest to note that \( \lim_{n \to \infty} ((q^n; q)_n/(1-q)^n) = (a)_n = a(a+1) \cdots (a+n-1) \) is the familiar Pochhammer symbol and

\[
\psi_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z) := \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n} \frac{z^n}{n!} z^n.
\]

(11)

Now for \( z \in \mathbb{U} \), \( 0 < |q| < 1 \), and \( l = m + 1 \), the basic hypergeometric function defined in (8) takes the form

\[
\psi_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z) = \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n} \frac{z^n}{n!},
\]

(12)

which converges absolutely in the open unit disk \( \mathbb{U} \).

Corresponding to the function \( \psi_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z) \) recently for meromorphic functions \( f \in \Sigma \) consisting of functions of the form (3), Huda and Darus [5] introduce \( q \)-analogue of Liu-Srivastava operator as below:

\[
\mathcal{J}^l_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z) * f(z) := \frac{1}{z} \psi_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z) * f(z)
\]

(13)

\[
= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1} a_n z^n}{(q; q)_{n+1} (b_1; q)_{n+1} \cdots (b_m; q)_{n+1}}
\]

where \( z \in \Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\} \).

In this paper for functions \( f \in \Sigma \) and for real parameters \( a_j, \ldots, a_l \) and \( b_j, \ldots, b_m \) \( (b_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m) \) we define the following new linear operator:

\[
\mathcal{J}^l_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z - \xi) := \sum_{n=1}^{\infty} \mathcal{Y}^l_m(a_1; q)_n (q - \xi)^n,
\]

(14)

as

\[
\mathcal{J}^l_m(a_1; \ldots; a_l; b_1, \ldots, b_m; q, z - \xi) := \sum_{n=1}^{\infty} \mathcal{Y}^l_m(a_1; q)_n (q - \xi)^n,
\]

(15)

where

\[
\mathcal{Y}^l_m(a_1; q)_n = \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1} a_n}{(q; q)_{n+1} (b_1; q)_{n+1} \cdots (b_m; q)_{n+1}}.
\]

(16)

Throughout our study for \( f \in \Sigma \), we let

\[
\mathcal{J}^l_m f(z) = \mathcal{J}^l_m(a_1; q)_n * f(z)
\]

(17)

\[
= \frac{1}{z - \xi} + \sum_{n=1}^{\infty} \mathcal{Y}^l_m(a_1; q)_n (z - \xi)^n
\]

\[
= \mathcal{J}^l_m(a_1; q)_n = \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1} a_n}{(q; q)_{n+1} (b_1; q)_{n+1} \cdots (b_m; q)_{n+1}}
\]

(18)

unless otherwise stated.

Motivated by earlier works on meromorphic functions by function theorists (see [6–14]), we define the following new subclass of functions in \( \Sigma \) by making use of the generalized operator \( \mathcal{J}^l_m \).
For $0 \leq \gamma < 1$ and $0 \leq \lambda < 1/2$, we let $\mathcal{M}^{l}_{m}(\lambda, \beta, \gamma)$ denote a subclass of $\Sigma_{\chi}$ consisting functions of the form (4) satisfying the condition that

$$-\mathfrak{R} \left( \left( z - \xi \right) \left( \mathcal{J}^{l}_{m,f}(z) \right)' \right) + \lambda (z - \xi)^{2} \left( \mathcal{J}^{l}_{m,f}(z) \right)' \right) > 0,$$

where $\mathcal{J}^{l}_{m,f}$ is given by (17).

Further, shortly we can state this condition by

$$-\mathfrak{R} \left( \frac{(z - \xi) G'(z)}{G(z)} \right) > \beta \left( \frac{(z - \xi) G'(z)}{G(z)} \right)' + 1 + \gamma,$$  \hspace{1cm} (20)

where

$$G(z) = (1 - \lambda) \mathcal{J}^{l}_{m,f}(z) + \lambda (z - \xi) \mathcal{J}^{l}_{m,f}(z).$$

(21)

$$a_{n} \geq 0.$$  

It is of interest to note that, on specializing the parameters $\lambda, \beta$ and $l, m$, we can define various new subclasses in $\Sigma_{\chi}$. We illustrate two important subclasses in the following examples.

**Example 1.** For $\lambda = 0$, we let $\mathcal{M}^{l}_{m}(0, \beta, \gamma) = \mathcal{M}^{l}_{m}(\beta, \gamma)$ denote a subclass of $\Sigma_{\chi}$ consisting functions of the form (4) satisfying the condition that

$$-\mathfrak{R} \left( \frac{(z - \xi) \mathcal{J}^{l}_{m,f}(z)}{\mathcal{J}^{l}_{m,f}(z)} \right) \geq \beta \left( \frac{(z - \xi) \mathcal{J}^{l}_{m,f}(z)}{\mathcal{J}^{l}_{m,f}(z)} \right)' + 1 + \gamma,$$  \hspace{1cm} (22)

where $\mathcal{J}^{l}_{m,f}(z)$ is given by (17).

**Example 2.** For $\lambda = 0, \beta = 0$ we let $\mathcal{M}^{l}_{m}(0, 0, \gamma) = \mathcal{M}^{l}_{m}(\gamma)$ denote a subclass of $\Sigma_{\chi}$ consisting functions of the form (4) satisfying the condition that

$$-\mathfrak{R} \left( \frac{(z - \xi) \mathcal{J}^{l}_{m,f}(z)}{\mathcal{J}^{l}_{m,f}(z)} \right) \geq \gamma,$$  \hspace{1cm} (23)

where $\mathcal{J}^{l}_{m,f}(z)$ is given by (17).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, and closure results for the function class $\mathcal{M}^{l}_{m}(\lambda, \beta, \gamma)$. Properties of certain integral operator and convolution properties of the new class $\mathcal{M}^{l}_{m}(\lambda, \beta, \gamma)$ are also discussed.

### 2. Coefficients Inequalities

In order to obtain the necessary and sufficient condition for a function, $f \in \mathcal{M}^{l}_{m}(\lambda, \beta, \gamma)$, we recall the following lemmas.

**Lemma 3.** If $\gamma$ is a real number and $w$ is a complex number, then $R(w) \geq \gamma \Leftrightarrow |w + 1 - \gamma| - |w + \gamma| \geq 0$.

**Lemma 4.** If $w$ is a complex number and $\gamma, k$ are real numbers, then

$$R(w) \geq k |w - 1| + \gamma \Leftrightarrow R \left\{ w \left( 1 + k e^{i \theta} \right) - k e^{i \theta} \right\} \geq \gamma, \hspace{1cm} -\pi \leq \theta \leq \pi.$$  \hspace{1cm} (24)

Analogous to the lemma proved by Dziok et al. [8], we state the following lemma without proof.

**Lemma 5.** Suppose that $\gamma \in [0, 1], r \in (0, 1]$, and the function $H \in \Sigma_{\chi}$ is of the form $H(z) = (1/(z - \xi)) + \sum_{n=1}^{\infty} b_{n}(z - \xi)^{n}$, $0 \leq |z - \xi| < r < 1$, with $b_{0} \geq 0$, then

$$\sum_{n=1}^{\infty} (n + \gamma) b_{n} r^{n+1} \leq 1 - \gamma.$$  \hspace{1cm} (25)

**Theorem 6.** Let $f \in \Sigma_{\chi}$ be given by (4). Then $f \in \mathcal{M}^{l}_{m}(\lambda, \beta, \gamma)$ if and only if

$$\sum_{n=1}^{\infty} \left[ (1 + \beta) + (\gamma + \beta) \right] (1 + n \lambda - \lambda) Y^{l}_{m} (n) a_{n} \leq \lambda (1 - 2\lambda) (1 - \gamma).$$  \hspace{1cm} (26)

**Proof.** If $f \in \mathcal{M}^{l}_{m}(\lambda, \beta, \gamma)$, then by (20) we have

$$-\mathfrak{R} \left( \frac{(z - \xi) G'(z)}{G(z)} \right) > \beta \left( \frac{(z - \xi) G'(z)}{G(z)} \right)' + 1 + \gamma.$$  \hspace{1cm} (27)

Making use of Lemma 4,

$$-\mathfrak{R} \left( \frac{(z - \xi)(1 + \beta e^{i \theta}) G'(z) + \beta e^{i \theta} G(z)}{G(z)} \right) > \gamma.$$  \hspace{1cm} (28)

where $G(z)$ is given by (21). Substituting $G(z)$, $G'(z)$ and letting $|z - \xi| < r \rightarrow 1^{-}$, we have

$$\left\{ (1 - 2\lambda) (1 - \gamma) - \sum_{n=1}^{\infty} \left[ (1 + \beta) + (\gamma + \beta) \right] \right\} \cdot \left( 1 + n \lambda - \lambda) Y^{l}_{m} (n) a_{n} \right) \times \left( 1 + 2\lambda) \sum_{n=1}^{\infty} \left( 1 + n \lambda - \lambda) \right) Y^{l}_{m} (n) a_{n} \right)^{-1} > 0.$$  \hspace{1cm} (29)

This shows that (26) holds.
Conversely, assume that (26) holds. Since $-\Re(w) > \gamma$, if and only if $|w + 1| < |w - (1 - 2\gamma)|$, it is sufficient to show that
\[
\frac{|w + 1|}{|w - (1 - 2\gamma)|} < 1, \quad |w - (1 - 2\gamma)| \neq 0
\]
for $|z - \xi| < r \leq 1$, $(z - \xi) \in \Delta$.

Using (26) and taking $w(z) = \frac{(z - \xi)(1 + \beta e^{i\theta})G'(z) + \beta e^{i\theta}G(z)}{G(z)}$, we get
\[
\left| \frac{w + 1}{w - (1 - 2\gamma)} \right| < 1, \quad |w - (1 - 2\gamma)| \neq 0
\]
for $|z - \xi| < r \leq 1$, $(z - \xi) \in \Delta$.

Since $f_n(z) = \frac{1}{z - \xi} + \frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma)Y_m^l(n)(z - \xi)^n}$ (37) satisfies the conditions of Theorem 6, $f_n(z) \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ and the equality is attained for this function. □

**Theorem 7.** Suppose that there exists a positive number $\nu$:
\[
\nu = \inf_{n \in \mathbb{N}} \left\{ d_n(\lambda, \beta, \gamma)Y_m^l(n) \right\}.
\]
If $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$, then
\[
\left| \frac{1}{r} - \frac{(1 - \gamma)(1 - 2\lambda)}{\nu} \right| r \leq |f(z)| \leq \left| \frac{1}{r} + \frac{(1 - \gamma)(1 - 2\lambda)}{\nu} \right| r, \quad (|z - \xi| = r),
\]
\[
\left| \frac{1}{r^2} - \frac{(1 - \gamma)(1 - 2\lambda)}{\nu} \right| r \leq |f'(z)| \leq \left| \frac{1}{r^2} + \frac{(1 - \gamma)(1 - 2\lambda)}{\nu} \right| r.
\]

The result is sharp for function (40) with
\[
v = d_1(\lambda, \beta, \gamma)Y_m^l(1) = (1 + \gamma + 2\beta)Y_m^l(1).
\]

Proof. Let $f \in \sum$ and be given by (4)
\[
|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n.
\]
Since $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$, and by Theorem 6,
\[
\sum_{n=1}^{\infty} a_n \leq \frac{(1 - \gamma)(1 - 2\lambda)}{\nu}.
\]
Using this, we have
\[
|f(z)| \geq \frac{1}{r} + \frac{(1 - \gamma)(1 - 2\lambda)}{\nu} r.
\]
Similarly
\[
|f'(z)| \geq \left| \frac{1}{r} - \frac{(1 - \gamma)(1 - 2\lambda)}{\nu} r \right|.
\]
The result is sharp for function (40) with
\[
v = d_1(\lambda, \beta, \gamma)Y_m^l(1) = (1 + \gamma + 2\beta)Y_m^l(1).
\]
Similarly we can prove the other inequality $|f'(z)|$. □
3. Order of Starlikeness

In the following theorem we obtain the order of starlikeness for the class $\mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$. We say that $f$ given by (4) is meromorphically starlike of order $\rho$, $0 \leq \rho < 1$, in $|z - \xi| < r$ when it satisfies condition (5) in $|z - \xi| < r$.

**Theorem 9.** Let the function $f$ given by (4) be in the class $\mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$. Then, if there exists

$$r = r_1(\lambda, \gamma, \rho) = \inf_{n \geq 1} \left[ \frac{(1 - \rho)d_n(\lambda, \beta, \gamma)Y_m^l(n)}{(n + \rho)(1 - \gamma)(1 - 2\lambda)} \right]^{1/(n+1)}$$

(46)

and it is positive, then $f$ is meromorphically starlike of order $\rho$ in $|z - \xi| < r \leq r_1(\lambda, \gamma, \rho)$.

**Proof.** Let the function $f \in \mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$ be of the form (4). If

$$0 < r \leq r_1(\lambda, \gamma, \rho),$$

then by (46)

$$r^{n+1} \leq \frac{(1 - \rho)d_n(\lambda, \beta, \gamma)Y_m^l(n)}{(n + \rho)(1 - \gamma)(1 - 2\lambda)},$$

(47)

for all $n \in \mathbb{N}$. From (47) we get

$$n + \rho r^{n+1} \leq \frac{d_n(\lambda, \beta, \gamma)Y_m^l(n)}{(1 - \gamma)(1 - 2\lambda)},$$

(48)

for all $n \in \mathbb{N}$, and thus

$$\sum_{n=1}^{\infty} n + \rho a_{n+1}n^{n+1} \leq \sum_{n=1}^{\infty} \frac{d_n(\lambda, \beta, \gamma)Y_m^l(n)}{(1 - \gamma)(1 - 2\lambda)}a_n \leq 1,$$

(49)

because of (26). Hence, from (49) and (25), $f$ is meromorphically starlike of order $\rho$ in $|z - \xi| < r \leq r_1(\lambda, \gamma, \rho)$.

Suppose that there exists a number $\tilde{r}, \tilde{r} > r_1(\lambda, \gamma, \rho)$, such that each $f \in \mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$ is meromorphically starlike of order $\rho$ in $|z - \xi| < \tilde{r} \leq 1$. The function

$$f(z) = \frac{1}{z - \xi} + \frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma)Y_m^l(n)}(z - \xi)^n$$

(50)

is in the class $\mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$; thus it should satisfy (25) with $\tilde{r}$:

$$\sum_{n=1}^{\infty} n + \rho a_{n+1}n^{n+1} \leq 1 - \rho,$$

(51)

while the left–hand side of (51) becomes

$$(n + \rho)\frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma)Y_m^l(n)}r^{n+1}$$

$$> (n + \rho)\frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma)Y_m^l(n)}(n + \rho)(1 - \gamma)(1 - 2\lambda)$$

$$= 1 - \rho,$$

(52)

which contradicts (51). Therefore the number $r_1(\lambda, \gamma, \rho)$ in Theorem 9 cannot be replaced with a greater number. This means that $r_1(\lambda, \gamma, \rho)$ is called radius of meromorphically starlikeness of order $\rho$ for the class $\mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$.

4. Results Involving Modified Hadamard Products

For functions

$$f_j(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_{n,j}(z - \xi)^n, \quad a_{n,j} \geq 0,$$

(53)

we define the Hadamard product or convolution of $f_1$ and $f_2$ by

$$(f_1 * f_2)(z) := \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}(z - \xi)^n.$$  

(54)

Let

$$\Psi(n, \lambda) = \frac{(n\lambda - \lambda + 1)}{(1 - 2\lambda)} Y_m^l(n).$$

(55)

**Theorem 10.** For functions $f_j (j = 1, 2)$ defined by (53), let $f_1 \in \mathcal{M}_m^{(l)}(\lambda, \beta, \gamma)$ and $f_2 \in \mathcal{M}_m^{(l)}(\lambda, \beta, \delta)$. Then $f_1 * f_2 \in \mathcal{M}_m^{(l)}(\lambda, \beta, \eta)$ where

$$\eta = 1 - \frac{(1 - \gamma)(1 - \delta)(3 + \beta)}{(1 + \gamma + 2\beta)(1 + \delta + 2\beta)\Psi(1, \lambda) - 2(1 - \gamma)(1 - \delta)},$$

(56)

and $\Psi(1, \lambda) = Y_m^l(1)/(1 - 2\lambda)$. The results are the best possible for

$$f_1(z) = \frac{1}{z - \xi} + \frac{1 - \gamma}{(1 + \gamma + 2\beta)\Psi(1, \lambda)}(z - \xi),$$

(57)

$$f_2(z) = \frac{1}{z - \xi} + \frac{1 - \delta}{(1 + \delta + 2\beta)\Psi(1, \lambda)}(z - \xi),$$

where $\Psi(1, \lambda) = Y_m^l(1)/(1 - 2\lambda)$.

**Proof.** In view of Theorem 6, it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{n(1 + \beta)(n + \eta + \beta)}{(1 - \eta)}\Psi(n, \lambda)a_{n,1}a_{n,2} \leq 1,$$

(58)

where $\eta$ is defined by (56) under the hypothesis. It follows from (26) and the Cauchy–Schwarz inequality that

$$\sum_{n=1}^{\infty} \frac{n(1 + \beta)(\gamma + \beta)^{1/2}[n(1 + \beta) + (\delta + \beta)]^{1/2}}{(1 - \gamma)(1 - \delta)}$$

$$\times \Psi(n, \lambda)\sqrt{a_{n,1}a_{n,2}} \leq 1.$$  

(59)
Thus we need to find the largest $\eta$ such that
\[
\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)} \Psi(n,\lambda) a_{n,1} a_{n,2} \\
\leq \sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{1-\gamma}(1-\delta)}

\times \frac{1}{(1-\gamma)(1-\delta)}

\times \Psi(n,\lambda) \sqrt{a_{n,1} a_{n,2}} \\
\leq 1.
\]

(60)

By virtue of (59) it is sufficient to find the largest $\eta$ such that
\[
\frac{1}{(1-\gamma)(1-\delta)}

\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{1-\gamma}(1-\delta)}

\times \Psi(n,\lambda) \times (1-\eta) (1-\gamma) (1-\delta) \Psi(n,\lambda)

\leq 1.
\]

(61)

which yields
\[
\eta \leq 1 - \left( (1-\gamma)(1-\delta)(2n+1+\beta) \right)

\times \left( [(n(1+\beta)+(\gamma+\beta)] [n(1+\beta)+(\delta+\beta)] \right)

\times \Psi(n,\lambda) \times (1-\gamma)(1-\delta)(n+1)^{-1},
\]

(62)

for $n \geq 1$ where $\Psi(n,\lambda)$ is given by (55) and, since $\Psi(n,\lambda)$ is a decreasing function of $n$ ($n \geq 1$), we have
\[
\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)} \Psi(1,\lambda) - 2(1-\gamma)(1-\delta),
\]

(63)

and $\Psi(1,\lambda) = \Psi_{m}(1)/(1-2\lambda)$, which completes the proof.

Theorem 11. Let the functions $f_{j}$, $(j = 1, 2)$, defined by (53) be in the class $M_{m}(\lambda, \beta, \gamma)$. Then $(f_{j} \ast f_{j})(z) \in M_{m}(\lambda, \beta, \eta)$ where
\[
\eta = 1 - \frac{(1-\gamma)^{2}(3+\beta)}{(1+\gamma+2\beta)^{2}} \Psi(1,\lambda) - 2(1-\gamma)^{2}
\]

(64)

with $\Psi(1,\lambda) = \Psi_{m}(1)/(1-2\lambda)$.

Proof. By taking $\delta = \gamma$ in the above theorem, the results follow.

For functions in the class $M_{m}(\lambda, \beta, \gamma)$, we can prove the following inclusion property.

Theorem 12. Let the functions $f_{j}$, $(j = 1, 2)$, defined by (53) be in the class $M_{m}(\lambda, \beta, \gamma)$. Then the function $h$, defined by
\[
h(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} \left( a_{n,1}^{2} + a_{n,2}^{2} \right) (z - \xi)^{n},
\]

(65)

is in the class $M_{m}(\lambda, \beta, \delta)$ where
\[
\delta \leq 1 - \frac{4(1-\gamma)^{2}(1+\beta)}{[1+\gamma+2\beta]^{2} \Psi(1,\lambda) + 2(1-\gamma)^{2}},
\]

(66)

and $\Psi(1,\lambda) = \Psi_{m}(1)/(1-2\lambda)$.

Proof. In view of Theorem 6, it is sufficient to prove that
\[
\sum_{n=2}^{\infty} \Psi(n,\lambda) \frac{n(1+\beta)+(\delta+\beta)}{(1-\delta)} \left( a_{n,1}^{2} + a_{n,2}^{2} \right) \leq 1,
\]

(67)

where $f_{j} \in M_{m}(\lambda, \beta, \gamma)$ ($j = 1, 2$); we find from (53) and Theorem 6 that
\[
\sum_{n=1}^{\infty} \Psi(n,\lambda) \frac{n(1+\beta)+(\gamma+\beta)}{1-\gamma} \left( a_{n,1}^{2} + a_{n,2}^{2} \right) \leq 1,
\]

(68)

which would yield
\[
\sum_{n=2}^{\infty} \frac{1}{\Psi(n,\lambda)} \frac{n(1+\beta)+(\gamma+\beta)}{1-\gamma} \left( a_{n,1}^{2} + a_{n,2}^{2} \right) \leq 1.
\]

(69)

On comparing (67) and (69) it can be seen that inequality (66) will be satisfied if
\[
\Psi(n,\lambda) \frac{n(1+\beta)+(\gamma+\beta)}{1-\gamma} \left( a_{n,1}^{2} + a_{n,2}^{2} \right) \leq 1 - \frac{2(1-\gamma)^{2}}{[n(1+\beta)+(\gamma+\beta)]^{2} \Psi(n,\lambda) + 2(1-\gamma)^{2}},
\]

(70)

That is, if
\[
\delta \leq 1 - \frac{2(1-\gamma)^{2}}{[n(1+\beta)+(\gamma+\beta)]^{2} \Psi(n,\lambda) + 2(1-\gamma)^{2}},
\]

(71)

where $\Psi(n,\lambda)$ is given by (55) and $\Psi(n,\lambda)$ is a decreasing function of $n$ ($n \geq 1$), we get (66), which completes the proof.
5. Closure Theorems

We state the following closure theorems for \( f \in \mathcal{M}_m(\lambda, \beta, \gamma) \) without proof (see [8–10]).

**Theorem 13.** Let the function \( f_k(z) = (1/(z - \xi)) + \sum_{n=1}^\infty a_{nk}(z - \xi)^n \) in the class \( \mathcal{M}_m(\lambda, \beta, \gamma) \) for every \( k = 1, 2, \ldots, m \). Then the function \( f \) defined by

\[
f(z) = \frac{1}{z - \xi} + \frac{1}{n!} \sum_{n=1}^\infty a_{nk}(z - \xi)^n, \quad (a_{nk} \geq 0)
\]

(72)

belongs to the class \( \mathcal{M}_m^l(\lambda, \beta, \gamma) \), where \( a_{nk} = (1/m) \sum_{k=1}^m a_{nk} \).

**Theorem 14.** Let \( f_n(z) = 1/(z - \xi) \) and \( f_n(z) = (1/(z - \xi)) + ((1 - \gamma)(1 - 2\lambda) + d_n(\lambda, \beta, \gamma) \gamma_n^l(n)) + \xi, \) for \( n = 1, 2, \ldots \). Then \( f \in \mathcal{M}_m^l(\lambda, \beta, \gamma) \) if and only if \( f \) can be expressed in the form \( f(z) = \sum_{n=0}^\infty h_n f_n(z) \) where \( h_n \geq 0 \) and \( \sum_{n=0}^\infty h_n = 1 \).

**Theorem 15.** The class \( \mathcal{M}_m^l(\lambda, \beta, \gamma) \) is closed under convex linear combination.

Now, we prove that the class is \( \mathcal{M}_m^l(\lambda, \beta, \gamma) \) closed under integral transforms.

**Theorem 16.** Let the function \( f(z) \) given by (4) be in \( \mathcal{M}_m^l(\lambda, \beta, \gamma) \). Then the integral operator

\[
F(z) = c \int_0^1 u^c f(uz) \, du \quad (0 < u \leq 1, 0 < c < \infty)
\]

(73)
is in \( \mathcal{M}_m^l(\lambda, \beta, \gamma) \), where

\[
\delta \leq \left( n^2 (1 + \beta) + n \left[ (1 + \gamma) + (1 + \beta) (1 + cy) \right] + (c + 1) (\gamma + \beta) + c\beta (1 - \gamma) \right) \\
\times \left( n^2 (1 + \beta) + (1 + c) \left[ (\gamma + \beta) + (1 + c) (1 + \beta) \right] + (1 + c) (\gamma + \beta) + c (1 - \gamma) \right)^{-1}.
\]

The result is sharp for the function \( f(z) = (1/(z - \xi)) + ((1 - \gamma)(1 - 2\lambda)) + (1 + \gamma + 2\beta) \gamma_n^l(1)(z - \xi) \).

Proof. Let \( f(z) \in \mathcal{M}_m^l(\lambda, \beta, \gamma) \). Then

\[
F(z) = c \int_0^1 u^c f(uz) \, du = \frac{1}{z - w} + \sum_{n=1}^\infty c_n (z - \xi)^n.
\]

(75)

It is sufficient to show that

\[
\sum_{n=1}^\infty c_n (\lambda, \beta, \gamma) \gamma_n^l(n) \\
\leq \frac{1}{(c + n + 1)(1 - \delta)}.
\]

(76)

Since \( f \in \mathcal{M}_m^l(\lambda, \beta, \gamma) \), we have

\[
\sum_{n=1}^\infty d_n (\lambda, \beta, \gamma) \gamma_n^l(n) \\
(1 - \gamma)(1 - 2\lambda) a_n \leq 1.
\]

(77)

Note that (76) is satisfied if

\[
\frac{cd_n(\lambda, \beta, \delta) \gamma_n^l(n)}{(c + n + 1)(1 - \delta)} \leq \frac{d_n(\lambda, \beta, \gamma) \gamma_n^l(n)}{(1 - \gamma)(1 - 2\lambda)}.
\]

(78)

From (78), we have

\[
\delta \leq \left( \frac{1 + \beta} {n^2} + n \left[ (\gamma + \beta) + (1 + \beta) (1 + cy) \right] + (c + 1) (\gamma + \beta) + c\beta (1 - \gamma) \right) \\
\times \left( (1 + \beta) + n \left[ (\gamma + \beta) + (1 + c) (1 + \beta) \right] + (1 + c) (\gamma + \beta) + c (1 - \gamma) \right)^{-1}.
\]

(79)

A simple computation will show that \( \Phi(n) \) is increasing and \( \Phi(n) \geq \Phi(1) \). Using this, the results follow.

6. Partial Sums

Silverman [15] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested in searching results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [15] and Cho and Owa [16], we will investigate the ratio of a function of the form (4) to its sequence of partial sums. Consider

\[
f_k(z) = \frac{1}{z - \xi} + \sum_{n=1}^k a_n (z - \xi)^n,
\]

(80)

when the coefficients are sufficiently small to satisfy the condition analogous to

\[
\sum_{n=1}^\infty d_n(\lambda, \beta, \gamma) \gamma_n^l(m) a_n \leq (1 - \gamma)(1 - 2\lambda).
\]

(81)

More precisely we will determine sharp lower bounds for \( \Re(f(z)/f_k(z)) \) and \( \Re(f_k(z)/f(z)) \). In this connection we make use of the well-known results that \( \Re((1 + w(\zeta))/(1 - w(z))) > 0, \) if and only if \( w(z) = \sum_{n=1}^\infty c_n(z - \xi)^n \) satisfies the inequality \( |w(z)| \leq |z - \xi| \).

Unless otherwise stated, we will assume that \( f \) is of the form (4) and its sequence of partial sums is denoted by (80).

**Theorem 17.** Let \( f(z) \in \mathcal{M}_m^l(\lambda, \beta, \gamma) \) be given by (4) which satisfies condition (26) and suppose that all of its partial sums (80) do not vanish in \( \Delta \). Moreover, suppose that

\[
2 - 2 \sum_{n=1}^k |a_n| - \frac{d_{k+1}(\lambda, \beta, \gamma) \gamma_n^l(k + 1)}{(1 - \gamma)(1 - 2\lambda)} \sum_{n=1}^{k+1} |a_n| > 0, \quad \forall k \in \mathbb{N}.
\]

(82)

Then,

\[
\left| \frac{f(z)}{f_k(z)} \right| \geq 1 - \frac{(1 - \gamma)(1 - 2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \gamma_n^l(k + 1)} (z - \xi) \in \Delta,
\]

(83)
where
\[
d_n(\lambda, \beta, \gamma) \geq \begin{cases} 
(1-\gamma)(1-2\lambda), & \text{if } n = 1, 2, 3, \ldots, k \\
d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1), & \text{if } n = k+1, k+2, \ldots
\end{cases}
\tag{84}
\]

The result (83) is sharp with the function given by
\[
f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}(z-\xi)^{k+1}.
\tag{85}
\]

Proof. Define the function \( w(z) \) by
\[
w(z) = \frac{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)} \times \left[ f(z) - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)} \right] = 1
\tag{86}
\]
\[
+ \left( \left( d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right) \times \sqrt{(1-\gamma)(1-2\lambda)} \right)^{-1}
\times \sum_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}.
\times \left( 1 + \sum_{n=1}^{k} a_n(z-\xi)^{n+1} \right)^{-1}.
\]

It suffices to show that \( \Re(w(z)) > 0 \); hence we find that
\[
\frac{1 + w(z)}{1 - w(z)} \leq \left( \left( d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right) \times \sqrt{(1-\gamma)(1-2\lambda)} \right)^{-1}
\times \sum_{n=k+1}^{\infty} |a_n| \times \left( 1 + \sum_{n=1}^{k} |a_n| + \sum_{n=1}^{\infty} |a_n| \right) \leq 1.
\tag{87}
\]

From condition (26), it readily yields the assertion (83) of Theorem 17.

To see that the function given by (85) gives the sharp result, we observe that for \( z = r e^{i\pi/(k+2)} \)
\[
\frac{f(z)}{f_k(z)} = 1 + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}(z-\xi)^n
\tag{88}
\]
\[
\longrightarrow 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}
\]
when \( r \to 1^- \) which shows that the bound (83) is the best possible for each \( k \in \mathbb{N} \).

We next determine bounds for \( f_k(z)/f(z) \).

**Theorem 18.** Under the assumptions of Theorem 17, we have
\[
\Re \left( \frac{f_k(z)}{f(z)} \right) \geq \frac{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)} \times \frac{1 + \sum_{n=k+1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |a_n|}{1 + (d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)/\sqrt{(1-\gamma)(1-2\lambda)})}
\tag{89}
\]

The result (89) is sharp with the function given by (85).

Proof. By setting
\[
w(z) = \left( \left( d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right) \times \sqrt{(1-\gamma)(1-2\lambda)} \right)^{-1}
\times \sum_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}.
\times \left( 1 + \sum_{n=1}^{k} a_n(z-\xi)^{n+1} \right)^{-1}.
\]
\[
\frac{w(z)}{1 + w(z)} \leq \left( \left( d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right) \times \sqrt{(1-\gamma)(1-2\lambda)} \right)^{-1}
\times \sum_{n=k+1}^{\infty} |a_n| \times \left( 1 + \sum_{n=1}^{k} |a_n| + \sum_{n=1}^{\infty} |a_n| \right) \leq 1.
\tag{87}
\]

and proceeding as in Theorem 17, we get the desired result and so we omit the details.

**Concluding Remark.** We observe that, if we specialize the parameters \( \lambda \) and \( \beta \) as mentioned in Examples 1 and 2, we obtain the analogous results for the classes \( \mathcal{M}_m(\beta, \gamma) \) and \( \mathcal{M}_m(\gamma) \). Further specializing the parameters \( l, m \) various other interesting results (as in Theorems 6–18) can be derived easily for the function class based on interesting differential operators as illustrated below.

1. For \( a_i = q^i, b_j = q^j, a_i > 0, b_j > 0, (i = 1, \ldots, k; j = 1, \ldots, m,l = m + 1), q \to 1 \), the operator \( \mathcal{P}_m f(z) = \mathcal{P}_m[a_i] f(z) \) defined by Liu and Srivastava [10].
2. For \( l = 2, m = 1, a_1 = q, b_1 = q \to 1 \), the operator \( \mathcal{L}_1[a_1; q, b_1] f(z) = \mathcal{L}_1[a_1; b_1] f(z) \) was introduced and studied by Liu and Srivastava [9].
3. For \( l = 1, m = 0, a_1 = \delta + 1, q \to 1 \), the operator \( \mathcal{L}[a_1; b_1] f(z) = D^2 f(z) = (1/(z-\delta+1)^{-1}) \ast f(z), (\delta > -1) \)
where $D^\delta$ is the differential operator which was introduced by Ganigi and Uralegaddi [17].

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgment**

The authors thank the referee for their valuable suggestions.

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