Hindawi Publishing Corporation
ISRN Mathematical Analysis
Volume 2014, Article ID 923607, 9 pages
http://dx.doi.org/10.1155/2014/923607

Research Article

Meromorphic Parabolic Starlike Functions Associated with q-Hypergeometric Series

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Received 28 February 2014; Accepted 20 March 2014; Published 7 April 2014

Academic Editors: A. Peris and S. Zhang

We introduce a new class of meromorphic parabolic starlike functions with a fixed point defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ involving the q-hypergeometric functions. We obtained coefficient inequalities, growth and distortion inequalities, and closure results for functions $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$. We further established some results concerning convolution and the partial sums.

1. Introduction

Let ξ be a fixed point in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{H}(\Delta)$ the class of functions which are regular and

$$\mathcal{A}(\xi) = \left\{f \in H(\Delta) : f(\xi) = f'(\xi) - 1 = 0\right\}. \tag{1}$$

Also denote by $\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is univalent in } \Delta\}$, the subclass of $\mathcal{A}(\xi)$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n \tag{2}$$

which are analytic in Δ. Note that $\mathcal{S}_0 = \mathcal{S}$ is subclasses of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{S}_w^*(\beta)$ and $\mathcal{K}_w(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions $\Re\{(z-\xi)f'(z)/f(z)\} > \beta$, and $\Re\{1 + ((z-\xi)f''(z)/f'(z))\} > \beta$, $(z - w) \in \Delta$ for $0 \leqq \beta < 1$ introduced and studied by Kanas and Ronning [1]. The class $\mathcal{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}_\xi^*(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [2, 3] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K} = \mathcal{K}_0(0)$ and $\mathcal{S}_0^* = \mathcal{S}^*(0)$, respectively, are the well-known standard classes of convex and starlike functions.

Let Σ denote the class of meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{3}$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Denote by Σ_ξ the subclass of Σ consisting of the functions of the form

$$f(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_n (z - \xi)^n, \quad a_n \geq 0; \ z \neq \xi. \tag{4}$$

A function f of the form (4) is in the class of *meromorphic starlike of order* γ $(0 \leq \gamma < 1)$ denoted by $\Sigma_\xi^*(\gamma)$, if

$$-\Re\left(\frac{(z-\xi) f'(z)}{f(z)}\right) > \gamma, \qquad z - \xi \in \Delta := \Delta^* \cup \{0\}, \tag{5}$$

and is in the class of *meromorphic convex of order* γ $(0 \le \gamma < 1)$ denoted by $\Sigma_\xi^K(\gamma)$, if

$$-\Re\left(1+\frac{(z-\xi)f''(z)}{f'(z)}\right)>\gamma, \quad z-\xi\in\Delta:=\Delta^*\cup\{0\}. \tag{6}$$

For functions $f(z)$ given by (4) and $g(z)=(1/(z-\xi))+\sum_{n=1}^{\infty}b_n(z-\xi)^n$, $(b_n\ge 0)$ we define the Hadamard product or convolution of f and g by

$$(f*g)(z):=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_nb_n(z-\xi)^n. \tag{7}$$

More recently, Purohit and Raina [4] introduced a generalized q-Taylor's formula in fractional q-calculus and derived certain q-generating functions for q-hypergeometric functions. In this work we proceed to derive a generalized differential operator on meromorphic functions in $\Delta^*=\{z\in\mathbb{C}:0<|z|<1\}$ involving these functions and discuss some of their properties.

For complex parameters $a_1,\dots,a_l$ and $b_1,\dots,b_m$ $(b_j\ne 0,-1,\dots;\ j=1,2,\dots,m)$ the q-hypergeometric function ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned}{}_l\Psi_m&(a_1,\dots a_l;b_1,\dots,b_m;q,z)\\ &:=\sum_{n=0}^{\infty}\frac{(a_1,q)_n\cdots(a_l,q)_n}{(q,q)_n(b_1,q)_n\cdots(b_m,q)_n}\\ &\quad\times\left[(-1)^nq^{\binom{n}{2}}\right]^{1+m-l}z^n,\end{aligned} \tag{8}$$

with $\binom{n}{2}=n(n-1)/2$ where $q\ne 0$ when $l>m+1$ $(l,m\in\mathbb{N}_0=\mathbb{N}\cup\{0\};z\in\mathbb{U})$.

The q-shifted factorial is defined for $a,q\in\mathbb{C}$ as a product of n factors by

$$(a;q)_n=\begin{cases}1 & n=0\\ (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right) & n\in\mathbb{N},\end{cases} \tag{9}$$

and in terms of basic analogue of the gamma function

$$(q^a;q)_n=\frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n>0. \tag{10}$$

It is of interest to note that $\lim_{q\to 1^-}((q^a;q)_n/(1-q)^n)=(a)_n=a(a+1)\cdots(a+n-1)$ is the familiar Pochhammer symbol and

$${}_l\Psi_m(a_1,\dots,a_l;b_1,\dots,b_m;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_l)_n}{(b_1)_n\cdots(b_m)_n}\frac{z^n}{n!}. \tag{11}$$

Now for $z\in\mathbb{U}$, $0<|q|<1$, and $l=m+1$, the basic hypergeometric function defined in (8) takes the form

$$\begin{aligned}{}_l\Psi_m&(a_1;\dots a_l;b_1,\dots,b_m;q,z)\\ &=\sum_{n=0}^{\infty}\frac{(a_1,q)_n\cdots(a_l,q)_n}{(q,q)_n(b_1,q)_n\cdots(b_m,q)_n}z^n,\end{aligned} \tag{12}$$

which converges absolutely in the open unitdisk $\mathbb{U}$.

Corresponding to the function ${}_l\Psi_m(a_1;\dots a_l;b_1,\dots,b_m;q,z)$ recently for meromorphic functions $f\in\Sigma_0$ consisting functions of the form (3), Huda and Darus [5] introduce q-analogue of Liu-Srivastava operator as below:

$$\begin{aligned}{}_l\Psi_m&(a_1;\dots a_l;b_1,\dots,b_m;q,z)*f(z)\\ &=\frac{1}{z}\,{}_l\Psi_m(a_1;\dots a_l;b_1,\dots,b_m;q,z)*f(z)\\ &=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{(a_1;q)_{n+1}\cdots(a_l;q)_{n+1}}{(q;q)_{n+1}(b_1;q)_{n+1}\cdots(b_m,q)_{n+1}}a_nz^n,\end{aligned} \tag{13}$$

where $z\in\Delta^*:=\{z\in\mathbb{C}:0<|z|<1\}$.

In this paper for functions $f\in\Sigma_\xi$ and for real parameters $a_1,\dots,a_l$ and $b_1,\dots,b_m$ $(b_j\ne 0,-1,\dots;\ j=1,2,\dots,m)$ we define the following new linear operator:

$$\mathcal{I}_m^l(a_1;\dots a_l;b_1,\dots,b_m;q,z-\xi):\Sigma_\xi\longrightarrow\Sigma_\xi, \tag{14}$$

as

$$\begin{aligned}\mathcal{I}_m^l&(a_1;\dots a_l;b_1,\dots,b_m;q,z-\xi)\\ &=\frac{1}{z-\xi}\,{}_l\Psi_m(a_1;\dots a_l;b_1,\dots,b_m;q,z-\xi)\\ \mathcal{I}_m^l&[a_l,q]=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}\Upsilon_n^{l,m}[a_1,q](z-\xi)^n,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\Upsilon_n^{l,m}&[a_1,q]\\ &=\frac{(a_1;q)_{n+1}\cdots(a_l;q)_{n+1}}{(q;q)_{n+1}(b_1;q)_{n+1}\cdots(b_m,q)_{n+1}}.\end{aligned} \tag{16}$$

Throughout our study for $f\in\Sigma_\xi$, we let

$$\begin{aligned}\mathcal{I}_m^lf(z)&=\mathcal{I}_m^l[a_l,q]*f(z)\\ &=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}\Upsilon_m^l(n)\,a_n(z-\xi)^n,\end{aligned} \tag{17}$$

$$\begin{aligned}\Upsilon_m^l(n)&=\Upsilon_n^{l,m}[a_1,q]\\ &=\frac{(a_1;q)_{n+1}\cdots(a_l;q)_{n+1}}{(q;q)_{n+1}(b_1;q)_{n+1}\cdots(b_m,q)_{n+1}},\end{aligned} \tag{18}$$

unless otherwise stated.

Motivated by earlier works on meromorphic functions by function theorists (see [6–14]), we define the following new subclass of functions in Σ_ξ by making use of the generalized operator $\mathcal{I}_m^l$.

For $0 \le \gamma < 1$ and $0 \le \lambda < 1/2$, we let $\mathcal{M}_m^l(\lambda, \beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$-\Re\left(\frac{(z-\xi)\left(\mathcal{I}_m^l f(z)\right)' + \lambda(z-\xi)^2\left(\mathcal{I}_m^l f(z)\right)''}{(1-\lambda)\mathcal{I}_m^l f(z) + \lambda(z-\xi)\left(\mathcal{I}_m^l f(z)\right)'}\right) > \beta\left|\frac{(z-\xi)\left(\mathcal{I}_m^l f(z)\right)' + \lambda(z-\xi)^2\left(\mathcal{I}_m^l f(z)\right)''}{(1-\lambda)\mathcal{I}_m^l f(z) + \lambda(z-\xi)\left(\mathcal{I}_m^l f(z)\right)'} + 1\right| + \gamma, \tag{19}$$

where $\mathcal{I}_m^l f$ is given by (17).

Further, shortly we can state this condition by

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)G'(z)}{G(z)} + 1\right| + \gamma, \tag{20}$$

where

$$G(z) = (1-\lambda)\mathcal{I}_m^l f(z) + \lambda(z-\xi)\left(\mathcal{I}_m^l f(z)\right)' = \frac{1-2\lambda}{z-\xi} + \sum_{n=1}^{\infty}(n\lambda - \lambda + 1)\Upsilon_m^l(n)a_n(z-\xi)^n, \quad a_n \ge 0. \tag{21}$$

It is of interest to note that, on specializing the parameters λ, β and l, m, we can define various new subclasses of Σ_ξ. We illustrate two important subclasses in the following examples.

Example 1. For $\lambda = 0$, we let $\mathcal{M}_m^l(0, \beta, \gamma) = \mathcal{M}_m^l(\beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$-\Re\left(\frac{(z-\xi)\left(\mathcal{I}_m^l f(z)\right)'}{\mathcal{I}_m^l f(z)}\right) > \beta\left|\frac{(z-\xi)\left(\mathcal{I}_m^l f(z)\right)'}{\mathcal{I}_m^l f(z)} + 1\right| + \gamma, \tag{22}$$

where $\mathcal{I}_m^l f(z)$ is given by (17).

Example 2. For $\lambda = 0$, $\beta = 0$ we let $\mathcal{M}_m^l(0, 0, \gamma) = \mathcal{M}_m^l(\gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$-\Re\left(\frac{(z-\xi)\left(\mathcal{I}_m^l f(z)\right)'}{\mathcal{I}_m^l f(z)}\right) > \gamma, \tag{23}$$

where $\mathcal{I}_m^l f(z)$ is given by (17).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, and closure results for the function class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$. Properties of certain integral operator and convolution properties of the new class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$ are also discussed.

2. Coefficients Inequalities

In order to obtain the necessary and sufficient condition for a function, $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$, we recall the following lemmas.

Lemma 3. *If γ is a real number and w is a complex number, then $\Re(w) \ge \gamma \Leftrightarrow |w + (1-\gamma)| - |w - (1+\gamma)| \ge 0$.*

Lemma 4. *If w is a complex number and γ, k are real numbers, then*

$$\Re(w) \ge k|w-1| + \gamma \Longleftrightarrow \Re\left\{w\left(1 + ke^{i\theta}\right) - ke^{i\theta}\right\} \ge \gamma, \quad -\pi \le \theta \le \pi. \tag{24}$$

Analogous to the lemma proved by Dziok et al. [8], we state the following lemma without proof.

Lemma 5. *Suppose that $\gamma \in [0, 1)$, $r \in (0, 1]$, and the function $H \in \Sigma_\xi(\gamma)$ is of the form $H(z) = (1/(z-\xi)) + \sum_{n=1}^{\infty} b_n(z-\xi)^n$, $0 < |z-\xi| < r < 1$, with $b_n \ge 0$, then*

$$\sum_{n=1}^{\infty}(n+\gamma)b_n r^{n+1} \le 1 - \gamma. \tag{25}$$

Theorem 6. *Let $f \in \Sigma_\xi$ be given by (4). Then $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\left[n(1+\beta) + (\gamma+\beta)\right](1 + n\lambda - \lambda)\Upsilon_m^l(n)a_n \le (1-2\lambda)(1-\gamma). \tag{26}$$

Proof. If $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$, then by (20) we have

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)G'(z)}{G(z)} + 1\right| + \gamma. \tag{27}$$

Making use of Lemma 4,

$$-\Re\left(\frac{(z-\xi)\left(1+\beta e^{i\theta}\right)G'(z) + \beta e^{i\theta}G(z)}{G(z)}\right) > \gamma, \tag{28}$$

where $G(z)$ is given by (21). Substituting $G(z)$, $G'(z)$ and letting $|z-\xi| < r \to 1^-$, we have

$$\left\{\left((1-2\lambda)(1-\gamma) - \sum_{n=1}^{\infty}\left[n(1+\beta) + (\gamma+\beta)\right] \times (1+n\lambda-\lambda)\Upsilon_m^l(n)a_n\right) \times \left((1-2\lambda) - \sum_{n=1}^{\infty}(1+n\lambda-\lambda)\Upsilon_m^l(n)a_n\right)^{-1}\right\} > 0. \tag{29}$$

This shows that (26) holds.

Conversely, assume that (26) holds. Since $-\Re(w) > \gamma$, if and only if $|w+1| < |w-(1-2\gamma)|$, it is sufficient to show that

$$\left|\frac{w+1}{w-(1-2\gamma)}\right| < 1, \qquad |w-(1-2\gamma)| \neq 0 \quad \text{for } |z-\xi| < r \leq 1,\ (z-\xi) \in \Delta. \tag{30}$$

Using (26) and taking $w(z) = ((z-\xi)(1+\beta e^{i\theta})G'(z) + \beta e^{i\theta}G(z))/G(z)$, we get

$$\begin{aligned}\left|\frac{w+1}{w-(1-2\gamma)}\right| &\leq \left(\left(\sum_{n=1}^{\infty}(1+n\lambda-\lambda)\left[(n+1)(1+\beta)\right]\Upsilon_m^l(n)a_n\right)\right.\\ &\quad\times\left(2(1-\gamma)(1-2\lambda)-\sum_{n=1}^{\infty}(1+n\lambda-\lambda)\right.\\ &\quad\left.\left.\times\left[n(1+\beta)+(\beta+2\gamma-1)\right]\Upsilon_m^l(n)a_n\right)^{-1}\right)\\ &\leq 1.\end{aligned} \tag{31}$$

Thus we have $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$. □

For the sake of brevity throughout this paper we let

$$\begin{aligned} d_n(\lambda,\beta,\gamma) &= \left[n(1+\beta)+(\gamma+\beta)\right](1+n\lambda-\lambda),\\ d_1(\lambda,\beta,\gamma) &= (1+\gamma+2\beta),\end{aligned} \tag{32}$$

unless otherwise stated.

Our next result gives the coefficient estimates for functions in $\mathcal{M}_m^l(\lambda,\beta,\gamma)$.

Theorem 7. *If $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, then*

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}, \quad n = 1,2,3,\ldots. \tag{33}$$

The result is sharp for the functions $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-\xi} + \frac{1-\gamma}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}(z-\xi)^n, \quad n = 1,2,3,\ldots. \tag{34}$$

Proof. If $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, then we have, for each n,

$$\begin{aligned} d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)a_n &\leq \sum_{n=1}^{\infty} d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)a_n\\ &\leq (1-\gamma)(1-2\lambda).\end{aligned} \tag{35}$$

Therefore we have

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}. \tag{36}$$

Since

$$f_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}(z-\xi)^n \tag{37}$$

satisfies the conditions of Theorem 6, $f_n(z) \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ and the equality is attained for this function. □

Theorem 8. *Suppose that there exists a positive number ν:*

$$\nu = \inf_{n\in\mathbb{N}}\left\{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)\right\}. \tag{38}$$

If $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, then

$$\begin{aligned}\left|\frac{1}{r}-\frac{(1-\gamma)(1-2\lambda)}{\nu}r\right| &\leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r, \quad (|z-\xi| = r),\\ \left|\frac{1}{r^2}-\frac{(1-\gamma)(1-2\lambda)}{\nu}\right| &\leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)}{\nu} \quad (|z-\xi| = r).\end{aligned} \tag{39}$$

If $\nu = d_1(\lambda,\beta,\gamma)\Upsilon_m^l(1) = (1+\gamma+2\beta)\Upsilon_m^l(1)$, then the result is sharp for

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)r^2\Upsilon_m^l(1)}(z-\xi). \tag{40}$$

Proof. Let $f \in \sum_\xi$ and be given by (4)

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty}a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty}a_n. \tag{41}$$

Since $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, and by Theorem 6,

$$\sum_{n=1}^{\infty}a_n \leq \frac{(1-\gamma)(1-2\lambda)}{\nu}. \tag{42}$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r. \tag{43}$$

Similarly

$$|f(z)| \geq \left|\frac{1}{r}-\frac{(1-\gamma)(1-2\lambda)}{\nu}r\right|. \tag{44}$$

The result is sharp for function (40) with

$$\nu = d_1(\lambda,\beta,\gamma)\Upsilon_m^l(1) = (1+\gamma+2\beta)\Upsilon_m^l(1). \tag{45}$$

Similarly we can prove the otherinequality $|f'(z)|$. □

3. Order of Starlikeness

In the following theorem we obtain the order of starlikeness for the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. We say that f given by (4) is meromorphically starlike of order ρ, $(0 \le \rho < 1)$, in $|z-\xi| < r$ when it satisfies condition (5) in $|z-\xi| < r$.

Theorem 9. *Let the function f given by (4) be in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. Then, if there exists*

$$r = r_1(\lambda,\gamma,\rho) = \inf_{n\ge1}\left[\frac{(1-\rho)d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}\right]^{1/(n+1)} \tag{46}$$

and it is positive, then f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho)$.

Proof. Let the function $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ be of the form (4). If $0 < r \le r_1(\lambda,\gamma,\rho)$, then by (46)

$$r^{n+1} \le \frac{(1-\rho)\,d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}, \tag{47}$$

for all $n \in \mathbb{N}$. From (47) we get

$$\frac{n+\rho}{1-\rho}r^{n+1} \le \frac{d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}{(1-\gamma)(1-2\lambda)}, \tag{48}$$

for all $n \in \mathbb{N}$, and thus

$$\sum_{n=1}^{\infty}\frac{n+\rho}{1-\rho}a_n r^{n+1} \le \sum_{n=1}^{\infty}\frac{d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}{(1-\gamma)(1-2\lambda)}a_n \le 1, \tag{49}$$

because of (26). Hence, from (49) and (25), f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho) = r$. □

Suppose that there exists a number $\tilde{r}$, $\tilde{r} > r_1(\lambda,\gamma,\rho)$, such that each $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ is meromorphically starlike of order ρ in $|z-\xi| < \tilde{r} \le 1$. The function

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}(z-\xi)^n \tag{50}$$

is in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$; thus it should satisfy (25) with $\tilde{r}$:

$$\sum_{n=1}^{\infty}(n+\rho)\,a_n\tilde{r}^{n+1} \le 1-\rho, \tag{51}$$

while the left–hand side of (51) becomes

$$\begin{aligned}&(n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}\tilde{r}^{n+1}\\ &\quad> (n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}\frac{(1-\rho)\,d_n(\lambda,\beta,\gamma)\,\Upsilon_m^l(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}\\ &\quad= 1-\rho,\end{aligned} \tag{52}$$

which contradicts (51). Therefore the number $r_1(\lambda,\gamma,\rho)$ in Theorem 9 cannot be replaced with a greater number. This means that $r_1(\lambda,\gamma,\rho)$ is called radius of meromorphically starlikeness of order ρ for the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$.

4. Results Involving Modified Hadamard Products

For functions

$$f_j(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,j}(z-\xi)^n, \quad a_{n,j} \ge 0, \tag{53}$$

we define the Hadamard product or convolution of f_1 and f_2 by

$$(f_1 * f_2)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,1}a_{n,2}(z-\xi)^n. \tag{54}$$

Let

$$\Psi(n,\lambda) = \frac{(n\lambda-\lambda+1)}{(1-2\lambda)}\Upsilon_m^l(n). \tag{55}$$

Theorem 10. *For functions f_j $(j = 1,2)$ defined by (53), let $f_1 \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ and $f_2 \in \mathcal{M}_m^l(\lambda,\beta,\delta)$. Then $f_1 * f_2 \in \mathcal{M}_m^l(\lambda,\beta,\eta)$ where*

$$\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)}, \tag{56}$$

and $\Psi(1,\lambda) = \Upsilon_m^l(1)/(1-2\lambda)$. The results are the best possible for

$$\begin{aligned}f_1(z) &= \frac{1}{z-\xi} + \frac{1-\gamma}{(1+\gamma+2\beta)\Psi(1,\lambda)}(z-\xi),\\ f_2(z) &= \frac{1}{z-\xi} + \frac{1-\delta}{(1+\delta+2\beta)\Psi(1,\lambda)}(z-\xi),\end{aligned} \tag{57}$$

where $\Psi(1,\lambda) = \Upsilon_m^l(1)/(1-2\lambda)$.

Proof. In view of Theorem 6, it suffices to prove that

$$\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)}\Psi(n,\lambda)\,a_{n,1}a_{n,2} \le 1, \tag{58}$$

where η is defined by (56) under the hypothesis. It follows from (26) and the Cauchy-Schwarz inequality that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}\\ &\qquad\times\Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \le 1.\end{aligned} \tag{59}$$

Thus we need to find the largest η such that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)}\Psi(n,\lambda)\,a_{n,1}a_{n,2}\\&\quad\le\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}\\&\qquad\times\Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}}\\&\quad\le 1.\end{aligned}\tag{60}$$

By virtue of (59) it is sufficient to find the largest η, such that

$$\begin{aligned}&\frac{\sqrt{(1-\gamma)(1-\delta)}}{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}\Psi(n,\lambda)}\\&\quad\le\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}\\&\qquad\times\frac{1-\eta}{[n(1+\beta)+(\eta+\beta)]},\end{aligned}\tag{61}$$

which yields

$$\begin{aligned}\eta\le 1-\Big(&((1-\gamma)(1-\delta)(2n+1+\beta))\\&\times([n(1+\beta)+(\gamma+\beta)][n(1+\beta)+(\delta+\beta)]\\&\times\Psi(n,\lambda)-(1-\gamma)(1-\delta)(n+1))^{-1}\Big),\end{aligned}\tag{62}$$

for $n\ge 1$ where $\Psi(n,\lambda)$ is given by (55) and, since $\Psi(n,\lambda)$ is a decreasing function of n $(n\ge 1)$, we have

$$\begin{aligned}\eta=&\,1\\&-\frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda)-2(1-\gamma)(1-\delta)},\end{aligned}\tag{63}$$

and $\Psi(1,\lambda)=\Upsilon_m^l(1)/(1-2\lambda)$, which completes the proof. □

Theorem 11. *Let the functions f_j, $(j=1,2)$, defined by* (53) *be in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. Then $(f_1*f_2)(z)\in\mathcal{M}_m^l(\lambda,\beta,\eta)$ where*

$$\eta=1-\frac{(1-\gamma)^2(3+\beta)}{(1+\gamma+2\beta)^2\Psi(1,\lambda)-2(1-\gamma)^2}\tag{64}$$

with $\Psi(1,\lambda)=\Upsilon_m^l(1)/(1-2\lambda)$.

Proof. By taking $\delta=\gamma$ in the above theorem, the results follow. □

For functions in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$, we can prove the following inclusion property.

Theorem 12. *Let the functions f_j $(j=1,2)$ defined by* (53) *be in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. Then the function h, defined by*

$$h(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)(z-\xi)^n,\tag{65}$$

is in the class $\mathcal{M}_m^l(\lambda,\beta,\delta)$ where

$$\delta\le 1-\frac{4(1-\gamma)^2(1+\beta)}{[1+\gamma+2\beta]^2\Psi(1,\lambda)+2(1-\gamma)^2},\tag{66}$$

and $\Psi(1,\lambda)=\Upsilon_m^l(1)/(1-2\lambda)$.

Proof. In view of Theorem 6, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{(1-\delta)}\left(a_{n,1}^2+a_{n,2}^2\right)\le 1,\tag{67}$$

where $f_j\in\mathcal{M}_m^l(\lambda,\beta,\gamma)$ $(j=1,2)$; we find from (53) and Theorem 6 that

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2a_{n,j}^2\\&\quad\le\sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}a_{n,j}\right]^2\le 1,\end{aligned}\tag{68}$$

which would yield

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2\left(a_{n,1}^2+a_{n,2}^2\right)\le 1.\tag{69}$$

On comparing (67) and (69) it can be seen that inequality (66) will be satisfied if

$$\begin{aligned}&\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{1-\delta}\left(a_{n,1}^2+a_{n,2}^2\right)\\&\quad\le\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2\\&\qquad\times\left(a_{n,1}^2+a_{n,2}^2\right).\end{aligned}\tag{70}$$

That is, if

$$\delta\le 1-\frac{2(1-\gamma)^2[(n+1)(1+\beta)]}{[n(1+\beta)+(\gamma+\beta)]^2\Psi(n,\lambda)+2(1-\gamma)^2},\tag{71}$$

where $\Psi(n,\lambda)$ is given by (55) and $\Psi(n,\lambda)$ is a decreasing function of n $(n\ge 1)$, we get (66), which completes the proof. □

5. Closure Theorems

We state the following closure theorems for $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ without proof (see [8–10]).

Theorem 13. *Let the function* $f_k(z) = (1/(z-\xi)) + \sum_{n=1}^{\infty} a_{n,k} (z-\xi)^n$ *be in the class* $\mathcal{M}_m^l(\lambda, \beta, \gamma)$ *for every* $k = 1, 2, \ldots, m$. *Then the function* f *defined by*

$$f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_{n,k}(z-\xi)^n, \quad (a_{n,k} \geq 0) \tag{72}$$

belongs to the class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$, *where* $a_{n,k} = (1/m) \sum_{k=1}^{m} a_{n,k}$, $(n = 1, 2, \ldots)$.

Theorem 14. *Let* $f_0(z) = 1/(z-\xi)$ *and* $f_n(z) = (1/(z-\xi)) + ((1-\gamma)(1-2\lambda)/d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n))(z-\xi)^n$ *for* $n = 1, 2, \ldots$. *Then* $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ *if and only if* f *can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z)$ *where* $\eta_n \geq 0$ *and* $\sum_{n=0}^{\infty} \eta_n = 1$.

Theorem 15. *The class* $\mathcal{M}_m^l(\lambda, \beta, \gamma)$ *is closed under convex linear combination.*

Now, we prove that the class is $\mathcal{M}_m^l(\lambda, \beta, \gamma)$ closed under integral transforms.

Theorem 16. *Let the function* $f(z)$ *given by* (4) *be in* $\mathcal{M}_m^l(\lambda, \beta, \gamma)$. *Then the integral operator*

$$F(z) = c \int_0^1 u^c f(uz)\, du \quad (0 < u \leq 1, 0 < c < \infty) \tag{73}$$

is in $\mathcal{M}_m^l(\lambda, \beta, \delta)$, *where*

$$\begin{aligned} \delta \leq {} & \left(n^2(1+\beta) + n[(\gamma+\beta) + (1+\beta)(1+c\gamma)]\right. \\ & \left. + (c+1)(\gamma+\beta) + c\beta(1-\gamma)\right) \\ & \times \left(n^2(1+\beta) + n[(\gamma+\beta) + (1+c)(1+\beta)]\right. \\ & \left. + (1+c)(\gamma+\beta) + c(1-\gamma)\right)^{-1}. \end{aligned} \tag{74}$$

The result is sharp for the function $f(z) = (1/(z-\xi)) + ((1-\gamma)(1-2\lambda)/(1+\gamma+2\beta)\Upsilon_m^l(1))(z-\xi)$.

Proof. Let $f(z) \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$. Then

$$F(z) = c \int_0^1 u^c f(uz)\, du = \frac{1}{z-w} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n (z-\xi)^n. \tag{75}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c d_n(\lambda, \beta, \delta)\Upsilon_m^l(n)}{(c+n+1)(1-\delta)} a_n \leq 1. \tag{76}$$

Since $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$, we have

$$\sum_{n=1}^{\infty} \frac{d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n)}{(1-\gamma)(1-2\lambda)} a_n \leq 1. \tag{77}$$

Note that (76) is satisfied if

$$\frac{c d_n(\lambda, \beta, \delta)\Upsilon_m^l(n)}{(c+n+1)(1-\delta)} \leq \frac{d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n)}{(1-\gamma)(1-2\lambda)}. \tag{78}$$

From (78), we have

$$\begin{aligned} \delta \leq {} & \left(\left(n^2(1+\beta) + n[(\gamma+\beta) + (1+\beta)(1+c\gamma)]\right.\right. \\ & \left. + (c+1)(\gamma+\beta) + c\beta(1-\gamma)\right) \\ & \times \left(n^2(1+\beta) + n[(\gamma+\beta) + (1+c)(1+\beta)]\right. \\ & \left.\left. + (1+c)(\gamma+\beta) + c(1-\gamma)\right)^{-1}\right) = \Phi(n). \end{aligned} \tag{79}$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Using this, the results follow. □

6. Partial Sums

Silverman [15] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested in searching results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [15] and Cho and Owa [16], we will investigate the ratio of a function of the form (4) to its sequence of partial sums. Consider

$$f_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{k} a_n (z-\xi)^n, \tag{80}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n) a_n \leq (1-\gamma)(1-2\lambda). \tag{81}$$

More precisely we will determine sharp lower bounds for $\Re(f(z)/f_k(z))$ and $\Re(f_k(z)/f(z))$. In this connection we make use of the well-known results that $\Re((1+w(z))/(1-w(z))) > 0$, $(z-\xi \in \Delta)$, if and only if $w(z) = \sum_{n=1}^{\infty} c_n (z-\xi)^n$ satisfies the inequality $|w(z)| \leq |z-\xi|$.

Unless otherwise stated, we will assume that f is of the form (4) and its sequence of partial sums is denoted by (80).

Theorem 17. *Let* $f(z) \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ *be given by* (4) *which satisfies condition* (26) *and suppose that all of its partial sums* (80) *do not vanish in* Δ. *Moreover, suppose that*

$$2 - 2\sum_{n=1}^{k} |a_n| - \frac{d_{k+1}(\lambda, \beta, \gamma)\Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)} \sum_{n=k+1}^{\infty} |a_n| > 0, \quad \forall k \in \mathbb{N}. \tag{82}$$

Then,

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma)\Upsilon_m^l(k+1)} \quad (z-\xi \in \Delta), \tag{83}$$

where

$$d_n(\lambda,\beta,\gamma) \geq \begin{cases} (1-\gamma)(1-2\lambda), & \text{if } n=1,2,3,\ldots,k \\ d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1), & \text{if } n=k+1,k+2,\ldots. \end{cases} \tag{84}$$

The result (83) *is sharp with the function given by*

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}(z-\xi)^{k+1}. \tag{85}$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)} \times \left[\frac{f(z)}{f_k(z)} - \left(1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}\right)\right] \\ &= 1 + \left(\left(\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)\right) \times ((1-\gamma)(1-2\lambda))^{-1} \times \sum_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}\right) \times \left(1 + \sum_{n=1}^{k} a_n(z-\xi)^{n+1}\right)^{-1}\right). \end{aligned} \tag{86}$$

It suffices to show that $\Re(w(z)) > 0$; hence we find that

$$\begin{aligned} \left|\frac{1+w(z)}{1-w(z)}\right| &\leq \left(\left(\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)\right) \times ((1-\gamma)(1-2\lambda))^{-1} \times \sum_{n=k+1}^{\infty} |a_n|\right) \times \left(2 - 2\sum_{n=1}^{k} |a_n| - \left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)\right) \times ((1-\gamma)(1-2\lambda))^{-1} \times \sum_{n=k+1}^{\infty} |a_n|\right)^{-1}\right) \leq 1. \end{aligned} \tag{87}$$

From condition (26), it readily yields the assertion (83) of Theorem 17.

To see that the function given by (85) gives the sharp result, we observe that for $z = re^{i\pi/(k+2)}$

$$\begin{aligned} \frac{f(z)}{f_k(z)} &= 1 + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}(z-\xi)^n \\ &\longrightarrow 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}, \end{aligned} \tag{88}$$

when $r \to 1^-$ which shows that the bound (83) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 18. *Under the assumptions of Theorem 17, we have*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1) + (1-\gamma)(1-2\lambda)} \quad (z-w \in \Delta), \tag{89}$$

The result (89) *is sharp with the function given by* (85).

Proof. By setting

$$\begin{aligned} w(z) &= \left(1 + \frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)}\right) \times \left[\frac{f_k(z)}{f(z)} - \frac{\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)/(1-\gamma)(1-2\lambda)\right)}{1 + \left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)/(1-\gamma)(1-2\lambda)\right)}\right] \end{aligned} \tag{90}$$

and proceeding as in Theorem 17, we get the desired result and so we omit the details. □

Concluding Remark. We observe that, if we specialize the parameters λ and β as mentioned in Examples 1 and 2, we obtain the analogous results for the classes $\mathscr{M}_m^l(\beta,\gamma)$ and $\mathscr{M}_m^l(\gamma)$. Further specializing the parameters l, m various other interesting results (as in Theorems 6–18) can be derived easily for the function class based on interesting differential operators as illustrated below.

(1) For $a_i = q^{a_i}$, $b_j = q^{b_j}$, $a_i > 0$, $b_j > 0$, $(i = 1,\ldots,l;\ j = 1,\ldots,m,\ l = m+1)$, $q \to 1$, the operator $\mathscr{I}_m^l f(z) = \mathscr{H}_m^l[a_1]f(z)$ defined by Liu and Srivastava [10].

(2) For $l = 2$, $m = 1$, $a_2 = q$, $q \to 1$, the operator $\mathscr{L}_1^2[a_1,q,b_1,q]f(z) = \mathscr{L}[a_1;b_1]f(z)$ was introduced and studied by Liu and Srivastava [9].

(3) For $l = 1$, $m = 0$, $a_1 = \delta + 1$, $q \to 1$, the operator $\mathscr{L}[a_1;b_1]f(z) = D^\delta f(z) = (1/z(1-z)^{\delta+1}) * f(z)$, $(\delta > -1)$

where D^{δ} is the differential operator which was introduced by Ganigi and Uralegaddi [17].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors thank the referee for their valuable suggestions.

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