Research Article

A Local Classification of Some Special (α, β) -Metrics of Constant Flag Curvature

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We classify some special Finsler metrics of constant flag curvature on a manifold of dimension $n > 2$.

1. Introduction

One of the important problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature, which is the generalization of sectional curvature in Riemannian geometry. The local structure of Finsler metrics of constant flag curvature has been historically mysterious and their classification seems to be far from being solved.

The (α, β) -metrics are an important class of Finsler metrics including Randers metrics as the simplest class. By making use of navigation problem, Bao et al. gave a local classification of Randers metrics with constant flag curvature [1]. Recently, Zhou has classified that the (α, β) -metrics with constant flag curvature in the following form,

$$F = \frac{(\alpha + \beta)^2}{\alpha}, \quad (1)$$

are locally projectively flat [2].

Lately, Shen and Zhao have studied projectively flat (α, β) -metrics

$$F = \alpha \left(1 + \varepsilon s + 2ks^2 - \frac{k^2}{3}s^4 \right), \quad s = \frac{\beta}{\alpha}, \quad (2)$$

where ε is a constant and k is a nonzero constant, and they proved such projectively flat Finsler metrics with constant flag curvature must be locally Minkowskian [3]. Hence, one natural problem is to consider the classification of such metrics with constant flag curvature. In this paper, we prove the following rigidity result.

Theorem 1. *Let $F = \alpha + \varepsilon\beta + (2k\beta^2/\alpha) - (k^2\beta^4/3\alpha^3)$, where the 1-form β is nonzero, ε is a constant, and k is a nonzero constant, be an (α, β) -metric on a manifold of dimension $n > 2$. Suppose that F is of constant flag curvature; then it must be locally Minkowskian.*

2. Preliminaries

Let F be a Finsler metric on an n -dimensional manifold M and G^i the geodesic coefficients of F , which are defined by

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}. \quad (3)$$

For any $x \in M$ and $y \in T_x M \setminus \{0\}$, the Riemann curvature $R_y = R_j^i(\partial/\partial x^i) \otimes dx^j$ is defined by

$$R_j^i = 2 \frac{\partial G^i}{\partial x^j} - \frac{\partial^2 G^i}{\partial x^k \partial y^j} y^k + 2G^k \frac{\partial^2 G^i}{\partial y^k \partial y^j} - \frac{\partial G^i}{\partial y^k} \frac{\partial G^k}{\partial y^j}. \quad (4)$$

(α, β) -metrics were first introduced by Matsumoto [4]. They are expressed in the following form:

$$F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}, \quad (5)$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. $\phi = \phi(s)$ is a smooth positive function satisfying

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (6)$$

$$\forall |s| \leq b < b_0.$$

For any flag (P, y) , where $P = \text{span}\{y, u\} \subset T_x M$, the flag curvature is defined by

$$K(P, y) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - [g_y(y, u)]^2}. \quad (7)$$

When F is Riemannian, $K(P, y) = K(P)$ is independent of $y \in P$. It is just the sectional curvature of P in Riemannian geometry. F is said to be of scalar curvature if, for any $y \in T_p M$, the flag curvature $K(P, y) = K(y)$ is independent of P containing $y \in T_p M$ that is equivalent to the following system of equations in a local coordinate system (x^i, y^i) in TM ,

$$R_j^i = KF^2 (\delta_j^i - F^{-1}F_{y^j}y^i). \quad (8)$$

If K is a constant, then F is said to be of constant flag curvature.

Let

$$r_{ij} := \frac{1}{2}(b_{ilj} + b_{jli}), \quad s_{ij} := \frac{1}{2}(b_{ilj} - b_{jli}), \quad (9)$$

where “ $|$ ” denotes the covariant derivative with respect to the Levi-Civita connection of α . Clearly β is closed if and only if $s_{ij} = 0$. Moreover, we denote

$$\begin{aligned} r_j^i &:= a^{ik}r_{kj}, & s_j^i &:= a^{ik}s_{kj}, & r_{00} &:= r_{ij}y^i y^j, \\ r_{i0} &:= r_{ij}y^j, & s_i &:= b_j s_i^j, & s_0 &:= s_i y^i, \\ r &:= r_{ij}b^i b^j, & s_{i0} &:= s_{ij}y^j, & r_i &:= b^j r_{ji}, \\ r^i &:= a^{ij}r_j, & s^i &:= a^{ij}s_j, \end{aligned} \quad (10)$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. Let G^i and ${}^\alpha G^i$ be the geodesic coefficients of F and α , respectively. Then we have the following.

Lemma 2 (see [5]). *For an (α, β) -metric $F = \alpha\phi(s)$, $s = (\beta/\alpha)$, the geodesic coefficients G^i are given by*

$$G^i = {}^\alpha G^i + \alpha Q s_0^i + \Theta \{r_{00} - 2Q\alpha s_0\} l^i + \Psi \{r_{00} - 2Q\alpha s_0\} b^i, \quad (11)$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi') + (B - s^2)\phi'']}, \\ \Psi &= \frac{\phi''}{2[(\phi - s\phi') + (B - s^2)\phi'']}. \end{aligned} \quad (12)$$

Here $l^i := (y^i/\alpha)$ and $B := b^2$.

Let $u(s, B) := -2\Theta Q$, $v(s, B) := -2\Psi Q$; one has the following.

Proposition 3 (see [6]). *For any (α, β) -metric $F = \alpha\phi(s)$, $s = (\beta/\alpha)$, the Riemannian curvature is given by*

$$R_j^i = {}^\alpha R_j^i + T_j^i, \quad (13)$$

where

$$\begin{aligned} T_j^i &= \Delta\delta_j^i + C_{11}l^i l_j + C_{12}l^i b_j + C_{13}l^i s_j + C_{14}l^i r_j \\ &\quad + C_{130}l^i s_{0j} + C_{140}l^i r_{j0} - (u\alpha)l^i s_{j|0} + (2u\alpha)l^i s_{0|j} \\ &\quad - (uQ\alpha^2)l^i s_k s_j^k + (2\Theta)l^i r_{00|j} - (2\Theta)l^i r_{j0|0} \\ &\quad - (2Q\alpha\Theta)l^i r_{k0} s_j^k + (4Q\alpha\Theta)l^i r_{jk} s_0^k + C_{21}b^i l_j \\ &\quad + C_{22}b^i b_j + C_{23}b^i s_j + C_{24}b^i r_j + C_{230}b^i s_{0j} + C_{240}b^i r_{j0} \\ &\quad - (v\alpha)b^i s_{j|0} + (2v\alpha)b^i s_{0|j} - (vQ\alpha^2)b^i s_k s_j^k \\ &\quad + (2\Psi)b^i r_{00|j} - (2\Psi)b^i r_{j0|0} - (2Q\alpha\Psi)b^i r_{k0} s_j^k \\ &\quad + (4Q\alpha\Psi)b^i r_{jk} s_0^k + C_{33}s_j^i \\ &\quad + [C_{31}s^i l_j + C_{32}s^i b_j + (vQ\alpha^2)s^i s_j + (2Q\alpha\Psi)s^i r_{j0}] \\ &\quad + [C_{310}s_0^i l_j + C_{320}s_0^i b_j + C_{330}s_0^i s_j \\ &\quad + C_{331}s_0^i s_{0j} + C_{340}s_0^i r_{j0}] \\ &\quad + C_4 r_j^i + [C_{410}r_0^i l_j + C_{420}r_0^i b_j \\ &\quad - (v\alpha)r_0^i s_j - (2\Psi)r_0^i r_{j0}] \\ &\quad + [C_{332}s_k^i s_0^k l_j + C_{333}s_k^i s_0^k b_j - (Q^2\alpha^2)s_k^i s_j^k] \\ &\quad + (2Q\alpha)s_{0|j} - (Q\alpha)s_{j|0} + [C_{311}s_{0|0}^i l_j - Qs_{0|0}^i b_j], \\ \Delta &= \frac{r_{00}^2}{\alpha^2} [2\Theta_s \Psi (B - s^2) - \Theta_s + \Theta^2 - 2s\Theta\Psi] \\ &\quad + \frac{r_{00}s_0}{\alpha} [2(v\Theta_s + u_s\Psi)(B - s^2) + 2u\Theta \\ &\quad - 2vs\Theta - u_s + 2Q\Theta_s - 2\Theta_B] \\ &\quad + \frac{2r_{00}r_0}{\alpha} (2\Theta\Psi - \Theta_B) \\ &\quad - \frac{r_{00|0}}{\alpha} \Theta + s_0^2 [-2u_B + u^2 + 2u_s v (B - s^2) + 2Qu_s] \end{aligned}$$

$$\begin{aligned}
& + 2r_0s_0(-u_B + 2v\Theta) + 2uQ\alpha(s_k s_0^k) + 4Q\Theta(r_{k0}s_0^k) \\
& - u s_{0|0}, \\
C_{11} = & \frac{r_{00}^2}{\alpha^2} \left[(s\Theta_s \Psi_s - 4\Theta_s \Psi - 2s\Theta_{ss} \Psi)(B - s^2) + 6s\Theta \Psi \right. \\
& + \Theta^2 + 6s^2 \Theta_s \Psi + s\Theta \Theta_s + s\Theta_{ss} + 2\Theta_s \\
& \left. - s^2 \Theta \Psi_s \right] \\
& + \frac{r_{00}s_0}{\alpha} \left[(-2s\Psi u_{ss} + su_s \Psi_s + sv_s \Theta_s - 2sv\Theta_{ss} \right. \\
& - 5v\Theta_s - 2\Psi u_s)(B - s^2) + 2s\Theta_{Bs} + \Theta u \\
& + su\Theta_s + 4s^2 \Psi u_s - 5Q\Theta_s + su_{ss} \\
& + s\Theta u_s + sQ_s \Theta_s - 2sQ_s \Theta_{ss} - s^2 \Theta v_s \\
& \left. + 6s^2 v\Theta_s + 7sv\Theta + u_s + 2\Theta_B \right] \\
& + \frac{2r_{00}r_0}{\alpha} (\Theta_B + s\Theta_{Bs} + s\Theta \Psi_s - 2\Theta \Psi - 2s\Theta_s \Psi) \\
& + \frac{r_{00|0}}{\alpha} (\Theta + s\Theta_s) \\
& + s_0^2 \left[(su_s v_s - 3u_s v - 2su_{ss} v) \right. \\
& \times (B - s^2) + 4s^2 u_s v - 2Qu_{ss} s \\
& \left. + suu_s - 3Qu_s + 2su_{Bs} + su_s Q_s \right] \\
& + 2r_0s_0(-2sv\Theta_s + s\Theta v_s + su_{Bs} - 3v\Theta) \\
& + \alpha s_k s_0^k (sQ_s u - Qu - 2sQu_s) \\
& + s_{0|0}(su_s) + 2r_{k0}s_0^k (sQ_s \Theta - 2sQ\Theta_s - 3Q\Theta), \\
C_{12} = & \frac{r_{00}^2}{\alpha^2} \left[(2\Theta_{ss} \Psi - \Theta_s \Psi_s)(B - s^2) \right. \\
& \left. - 2\Theta \Psi - \Theta \Theta_s + s\Theta \Psi_s - 6s\Theta_s \Psi - \Theta_{ss} \right] \\
& + \frac{r_{00}s_0}{\alpha} \left[(2v\Theta_{ss} + 2\Psi u_{ss} - u_s \Psi_s - v_s \Theta_s)(B - s^2) \right. \\
& - (Q_s + u + 6sv) \Theta_s + 2Q\Theta_{ss} - 2\Theta_{Bs} \\
& \left. - 4su_s \Psi + sv_s \Theta - u_{ss} - \Theta u_s - 2v\Theta \right] \\
& + \frac{2r_{00}r_0}{\alpha} (-\Theta \Psi_s + 2\Theta_s \Psi - \Theta_{Bs}) \\
& + s_0^2 \left[(2u_{ss} v - u_s v_s)(B - s^2) - uu_s \right. \\
& \left. - Q_s u_s - 2u_{Bs} - 4su_s v + 2Qu_{ss} \right] \\
& - 2r_0s_0(u_{Bs} + \Theta v_s - 2v\Theta_s) \\
& + 2r_{k0}s_0^k (2Q\Theta_s - Q_s \Theta) + \alpha s_k s_0^k (2Qu_s - Q_s u)
\end{aligned}$$

$$\begin{aligned}
& - s_{0|0}u_s - \frac{r_{00|0}}{\alpha} \Theta_s, \\
C_{13} = & r_{00} \left[(2u_s \Psi - \Theta_s v)(B - s^2) - u\Theta \right. \\
& \left. + 4\Theta_B + sv\Theta - u_s - Q\Theta_s \right] \\
& + \alpha s_0 \left[2u_B - u^2 + Qu_s + u_s v(B - s^2) \right] \\
& - \alpha r_0(2v\Theta + 2u_B), \\
C_{14} = & 4r_{00}(\Theta_B + \Theta \Psi) + 4\alpha s_0(u_B + v\Theta), \\
C_{130} = & \frac{3r_{00}}{\alpha} [Q\Theta + (1 + sQ)\Theta_s] + 3s_0(sQ + 1)u_s, \\
C_{140} = & \frac{r_{00}}{\alpha} \left[2\Theta_s \Psi(B - s^2) - 2s\Theta \Psi - \Theta_s - 2\Theta^2 \right] \\
& + s_0 \left[(4v\Theta_s - 2\Psi u_s)(B - s^2) \right. \\
& \left. + 4Q\Theta_s + u_s - 2u\Theta - 4\Theta_B - 4sv\Theta \right] \\
& - 4r_0(\Theta_B + \Theta \Psi), \\
C_{21} = & \frac{r_{00}^2}{\alpha^2} \left[(s\Psi_s^2 - 2s\Psi \Psi_{ss} - 2\Psi \Psi_s)(B - s^2) \right. \\
& \left. + s\Psi_{ss} + \Psi_s + 4s^2 \Psi \Psi_s \right] \\
& + \frac{r_{00}s_0}{\alpha} \left[(2s\Psi_s v_s - 3v\Psi_s - 2sv\Psi_{ss} - 2s\Psi v_{ss})(B - s^2) \right. \\
& + sv_{ss} + 2s\Psi_{Bs} - 2Qs\Psi_{ss} + 5s^2 v\Psi_s \\
& \left. + sQ_s \Psi_s + 2s^2 v_s \Psi - 2sv\Psi - 3Q\Psi_s \right] \\
& + \frac{2r_{00}r_0}{\alpha} (-s\Psi \Psi_s + s\Psi_{Bs}) \\
& + s_0^2 \left[(sv_s^2 - 2svv_{ss} - vv_s)(B - s^2) + 3s^2 vv_s \right. \\
& \left. - 2sQv_{ss} - 3sv^2 - Qv_s - 2v_B + sQ_s v_s + 2sv_{Bs} \right] \\
& + 2s_0r_0(sv_{Bs} - \Psi v + s\Psi v_s - 2sv\Psi_s - v_B) \\
& + \frac{r_{00|0}}{\alpha} (s\Psi_s) + 2r_{k0}s_0^k (-Q\Psi + s\Psi Q_s - 2sQ\Psi_s) \\
& + s_{0|0}(sv_s - v) + \alpha s_k s_0^k (sQ_s v + Qv - 2sQv_s), \\
C_{22} = & \frac{r_{00}^2}{\alpha^2} \left[(2\Psi \Psi_{ss} - \Psi_s^2)(B - s^2) - \Psi_{ss} - 4s\Psi \Psi_s \right] \\
& + \frac{r_{00}s_0}{\alpha} \left[2(v\Psi_{ss} + \Psi v_{ss} - \Psi_s v_s)(B - s^2) \right. \\
& - 2\Psi_{Bs} - v_{ss} + 2v\Psi \\
& \left. - 2s\Psi v_s - 5sv\Psi_s + 2Q\Psi_{ss} - Q_s \Psi_s \right] \\
& + \frac{2r_{00}r_0}{\alpha} (-\Psi_{Bs} + \Psi \Psi_s) \\
& + 2s_0r_0(-\Psi v_s + 2\Psi_s v - v_{Bs})
\end{aligned}$$

$$\begin{aligned}
& + s_0^2 \left[(2v v_{ss} - v_s^2) (B - s^2) - Q_s v_s \right. \\
& \quad \left. - 2v_{Bs} + 2v^2 + 2Q v_{ss} - 3s v v_s \right] \\
& - \frac{r_{00|0}}{\alpha} \Psi_s - s_{0|0} v_s + \alpha s_k s_0^k (2Q v_s - Q_s v) \\
& + 2r_{k0} s_0^k (2Q \Psi_s - Q_s \Psi), \\
C_{23} &= r_{00} \left[(-v \Psi_s + 2\Psi v_s) (B - s^2) \right. \\
& \quad \left. + 4\Psi_B - Q \Psi_s - v_s + 2s v \Psi \right] \\
& + \alpha s_0 \left[v v_s (B - s^2) + 2v_B + Q v_s + s v^2 \right] \\
& - 2\alpha r_0 (\Psi v + v_B), \\
C_{24} &= 4r_{00} (\Psi^2 + \Psi_B) + 4\alpha s_0 (v_B + \Psi v), \\
C_{230} &= \frac{3r_{00}}{\alpha} (1 + sQ) \Psi_s + 3s_0 [v_s - (v - s v_s) Q], \\
C_{240} &= \frac{r_{00}}{\alpha} [2\Psi \Psi_s (B - s^2) - \Psi_s] \\
& + s_0 \left[(4v \Psi_s - 2\Psi v_s) (B - s^2) + v_s \right. \\
& \quad \left. - 2s v \Psi + 4Q \Psi_s - 4\Psi_B \right] \\
& - 4r_0 (\Psi^2 + \Psi_B), \\
C_3 &= 2\alpha s_0 [Q_s v (B - s^2) + Q Q_s + s Q v + v] \\
& + r_{00} [2Q_s \Psi (B - s^2) + 2\Psi - Q_s + 2s Q \Psi], \\
C_{31} &= r_{00} (-2Q \Psi + 2s Q_s \Psi - s Q \Psi_s) \\
& - \alpha s_0 (v Q + s Q v_s - 2s Q_s v), \\
C_{32} &= r_{00} (-2Q_s \Psi + Q \Psi_s) - \alpha s_0 (2Q_s v - Q v_s), \\
C_{310} &= \frac{r_{00}}{\alpha} [(Q_s s \Psi_s - 2s \Psi Q_{ss}) (B - s^2) \\
& \quad + 2Q_s s^2 \Psi - 2s Q \Psi + Q_s^2 \Psi_s + s Q_{ss} + s \Psi_s] \\
& + s_0 [(s Q_s v_s - Q_s v - 2s Q_{ss} v) (B - s^2) \\
& \quad + Q_s^2 v_s - Q Q_s + s Q_s^2 - 3s Q v \\
& \quad - v + s v_s - 2s Q Q_{ss} + 2Q_s s^2 v], \\
C_{320} &= \frac{r_{00}}{\alpha} [(2Q_{ss} \Psi - Q_s \Psi_s) (B - s^2) \\
& \quad + 2Q \Psi - 2s \Psi Q_s - s Q \Psi_s - Q_{ss} - \Psi_s] \\
& + s_0 [(2Q_{ss} v - Q_s v_s) (B - s^2) - v_s - Q_s^2 \\
& \quad + 2v Q - 2s Q_s v + 2Q Q_{ss} - s Q v_s],
\end{aligned}$$

$$\begin{aligned}
C_{330} &= -\alpha [v Q_s (B - s^2) + Q Q_s + v + s Q v], \\
C_{331} &= -3Q^2 + 3s Q Q_s + 3Q_s, \\
C_{340} &= -2 [Q_s (B - s^2) + s Q + 1] \Psi + Q_s, \\
C_4 &= (2\Psi) r_{00} + (2v) \alpha s_0, \\
C_{410} &= \frac{r_{00}}{\alpha} s \Psi_s + s_0 (s v_s - v), \\
C_{420} &= -\frac{r_{00}}{\alpha} \Psi_s - s_0 v_s, \\
C_{332} &= \alpha (Q - s Q_s) Q, \quad C_{333} = \alpha Q Q_s, \\
C_{311} &= s Q_s - 1.
\end{aligned} \tag{14}$$

Proposition 4 (see [6]). *For an (α, β) -metric $F = \alpha\phi(s)$, $s = (\beta/\alpha)$, the Ricci curvature of F is related to the Ricci curvature ${}^\alpha \text{Ric}$ of α by*

$$\text{Ric} = {}^\alpha \text{Ric} + T_m^m, \tag{15}$$

where

$$\begin{aligned}
T_m^m &= \frac{r_{00}^2}{\alpha^2} [(n-1) c_1 + c_2] \\
& + \frac{1}{\alpha} \{ r_{00} s_0 [(n-1) c_3 + c_4] \\
& \quad + r_{00} r_0 [(n-1) c_5 + c_6] \\
& \quad + r_{00|0} [(n-1) c_7 + c_8] \} \\
& + \{ s_0^2 [(n-1) c_9 + c_{10}] + (r r_{00} - r_0^2) c_{11} \\
& \quad + r_0 s_0 [(n-1) c_{12} + c_{13}] \\
& \quad + (r_{00} r_m^m - r_{0m} r_0^m + r_{00|m} b^m - r_{0m|0} b^m) c_{14} \\
& \quad + r_{0m} s_0^m [(n-1) c_{15} + c_{16}] \\
& \quad + s_{0|0} [(n-1) c_{17} + c_{18}] + s_{0m} s_0^m c_{19} \} \\
& + \alpha \{ r s_0 c_{20} + s_m s_0^m [(n-1) c_{21} + c_{22}] \\
& \quad + (3s_m r_0^m - 2s_0 r_m^m + 2r_m s_0^m \\
& \quad - 2s_{0|m} b^m + s_{m|0} b^m) c_{23} + s_{0|m}^m c_{24} \} \\
& + \alpha^2 (s_m s_m^m c_{25} + s_m^i s_i^m c_{26}), \\
c_1 &= 2\Psi \Theta_s (B - s^2) - 2s \Psi \Theta + \Theta^2 - \Theta_s, \\
c_2 &= (2\Psi \Psi_{ss} - \Psi_s^2) (B - s^2)^2 \\
& - (6s \Psi \Psi_s + \Psi_{ss}) (B - s^2) + 2s \Psi_s,
\end{aligned}$$

$$\begin{aligned}
c_3 &= -4(2Q\Psi\Theta_s + Q_s\Psi\Theta)(B - s^2) \\
&\quad + 2(Q_s\Theta + 2Q\Theta_s) + 4Q\Theta(s\Psi - \Theta) - 2\Theta_B, \\
c_4 &= \left[-4\Psi(2Q\Psi_{ss} + Q_s\Psi_s + Q_{ss}\Psi) \right. \\
&\quad \left. + 4Q\Psi_s^2 \right] (B - s^2)^2 \\
&\quad + \left[-4\Psi^2(Q - sQ_s) + 2(2Q_{ss}\Psi + Q_s\Psi_s + 2Q\Psi_{ss}) \right. \\
&\quad \left. - 2\Psi_{sB} + 20sQ\Psi\Psi_s \right] (B - s^2) \\
&\quad + 2\Psi(Q - sQ_s) - 4\Psi_s - Q_{ss} - 10sQ\Psi_s, \\
c_5 &= 4\Psi\Theta - 2\Theta_B, \quad c_6 = 2(2\Psi\Psi_s - \Psi_{sB})(B - s^2) - 2\Psi_s, \\
c_7 &= -\Theta, \quad c_8 = -\Psi_s(B - s^2), \\
c_9 &= 8Q\Psi(Q\Theta_s + Q_s\Theta)(B - s^2) \\
&\quad + 4Q^2(\Theta^2 - \Theta_s) + 4Q(\Theta_B - \Theta Q_s), \\
c_{10} &= \left[4\Psi^2(2QQ_{ss} - Q_s^2) + 8Q\Psi(Q\Psi_{ss} + Q_s\Psi_s) \right. \\
&\quad \left. - 4Q^2\Psi_s^2 \right] (B - s^2)^2 \\
&\quad + \left[-16sQ\Psi(Q\Psi_s + Q_s\Psi) - 4\Psi(2QQ_{ss} - Q_s^2) \right. \\
&\quad \left. - 4Q(Q\Psi_{ss} + Q_s\Psi_s) \right. \\
&\quad \left. + 4(Q\Psi_{sB} + Q_s\Psi_B) \right] (B - s^2) \\
&\quad - 12s^2Q^2\Psi^2 + 4(2 + 3sQ)(Q\Psi_s + Q_s\Psi) \\
&\quad - 8Q^2\Psi + 2QQ_{ss} - Q_s^2 + 4sQ\Psi_B + 8B\Psi^2Q^2, \\
c_{11} &= 4\Psi^2 + 4\Psi_B, \quad c_{12} = 4Q(-2\Psi\Theta + \Theta_B), \\
c_{13} &= \left[8\Psi(Q_s\Psi - Q\Psi_s) + 4(Q\Psi_{sB} + Q_s\Psi_B) \right] (B - s^2) \\
&\quad + 8sQ\Psi^2 + 4Q\Psi_s - 4(1 - sQ)\Psi_B, \\
c_{14} &= 2\Psi, \quad c_{15} = 4Q\Theta, \\
c_{16} &= -4(Q_s\Psi - Q\Psi_s)(B - s^2) + 2Q_s - 2(1 + 2sQ)\Psi, \\
c_{17} &= 2Q\Theta, \quad c_{18} = 2(Q_s\Psi + Q\Psi_s)(B - s^2) - Q_s + 2sQ\Psi, \\
c_{19} &= -2Q^2 + 2(1 + sQ)Q_s, \quad c_{20} = -8Q(\Psi^2 + \Psi_B), \\
c_{21} &= -4Q^2\Theta, \quad c_{22} = -4Q^2\Psi_s(B - s^2) + 2Q\Psi, \\
c_{23} &= 2Q\Psi, \quad c_{24} = 2Q, \\
c_{25} &= -4Q^2\Psi, \quad c_{26} = -Q^2.
\end{aligned} \tag{16}$$

Remark 5. In Proposition 4, we have corrected some terms in the formulas for coefficients c_9 and c_{10} , which are not printed.

Definition 6 (see [7]). Let

$$D_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right), \tag{17}$$

where G^i are the spray coefficients of F . The tensor $D := D_{jkl}^i \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called *Douglas tensor*. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

Note that an (α, β) -metric $F = \alpha + \varepsilon\beta + (2k\beta^2/\alpha) - (k^2\beta^4/3\alpha^3)$ is a Douglas metric if and only if $b_{lij} = \sigma[(1 + 4kB)a_{ij} - 5kb_jb_i]$ holds for some scalar function $\sigma = \sigma(x)$ [8].

Definition 7 (see [7]). Put

$$W_k^i := A_k^i - \frac{1}{n+1} \frac{\partial A_k^m}{\partial y^m} y^i, \tag{18}$$

where $R = (R_k^i/(n-1))$, $A_j^i = R_j^i - R\delta_j^i$. Then $W = W_k^i(\partial/\partial x^i) \otimes dx^k$ is a tensor on $TM \setminus \{0\}$. It is called the *Weyl curvature tensor*.

It is Hilbert's fourth problem in the regular case to study and characterize Finsler metrics on an open domain $u \subset \mathbf{R}^n$ whose geodesics are straight lines. Finsler metrics with this property are called projectively flat metrics. A famous theorem of Douglas is in the following.

Theorem 8 (see [9]). *A Finsler metric F on a manifold M ($\dim M > 2$) is locally projectively flat if and only if $D = 0$ and $W = 0$.*

3. Proof of Theorem 1

In this section, we will prove Theorem 1. First, we will study the following lemma, because Lemma 4.1 [3] is found to have some wrong. For example, when $k = -(1/4b^2)$, F does not satisfy the definition of (α, β) -metrics at $s = 0$.

Lemma 9. *$F = \alpha + \varepsilon\beta + (2k\beta^2/\alpha) - (k^2\beta^4/3\alpha^3)$ is a Finsler metric if and only if one of the following holds.*

- (a) If $k > 0$, then $b < 1/\sqrt{k}$ and $\varepsilon \in [-\sqrt{k}, \sqrt{k}]$.
- (b) If $k < 0$, then $b < (1/2\sqrt{-k})$ and $\varepsilon \in [-(23/24)\sqrt{-k}, (23/24)\sqrt{-k}]$,
where $b := \|\beta_x\|_\alpha$, for any $x \in M$.

Proof. Let

$$\phi = 1 + \varepsilon s + 2ks^2 - \frac{k^2s^4}{3}, \quad s = \frac{\beta}{\alpha}. \tag{19}$$

Direct computations yield

$$\begin{aligned}
\phi - s\phi' &= (ks^2 - 1)^2, \\
\phi - s\phi' + (b^2 - s^2)\phi'' &= (ks^2 - 1)(5ks^2 - 1 - 4kb^2).
\end{aligned} \tag{20}$$

From the definition of (α, β) , we know that, for all $|s| \leq b < b_0$, ϕ must satisfy

$$\phi > 0, \quad \phi - s\phi' + (b^2 - s^2)\phi'' > 0. \quad (21)$$

That is,

$$1 + \varepsilon s + 2ks^2 - \frac{k^2 s^4}{3} > 0, \quad (22)$$

$$(ks^2 - 1)(5ks^2 - 1 - 4kb^2) > 0.$$

Particularly, when $s = 0$ and $s = b$, it is easy to see that

$$1 + 4kb^2 > 0, \quad kb^2 - 1 \neq 0. \quad (23)$$

Now we first discuss the second inequality of (22). It is equivalent to the following two cases.

Let $y_1 = ks^2 - 1$, $y_2 = 5ks^2 - 1 - 4kb^2$; it is easy to see that y_1 and y_2 are quadratic functions with respect to s .

Case 1. When

$$ks^2 - 1 > 0, \quad (24)$$

$$5ks^2 - 1 - 4kb^2 > 0,$$

(I) $k > 0$.

By the graph of quadratic function and (23), we find that there is no b such that $ks^2 - 1 > 0$ and $5ks^2 - 1 - 4kb^2 > 0$ always hold for all $|s| \leq b$.

Consider (II) $k < 0$.

By the graph of quadratic function and (23), we conclude the same result with (I).

In any case, Case 1 cannot exist for all $|s| \leq b$.

Case 2. When

$$ks^2 - 1 < 0, \quad (25)$$

$$5ks^2 - 1 - 4kb^2 < 0,$$

(III) $k > 0$.

By the graph of quadratic function and (23), the above system of inequality always holds if and only if $b < (1/\sqrt{k})$.

Consider (IV) $k < 0$.

Similarly, the above system of inequality always holds if and only if $b < (1/2\sqrt{-k})$.

In the following, we will discuss the condition satisfying the first inequality of (22). Since

$$1 + \varepsilon s + 2ks^2 - \frac{k^2 s^4}{3} > 0 \iff (ks^2 - 3)^2 - 3\varepsilon s - 12 < 0, \quad (26)$$

$$\text{let } y = (ks^2 - 3)^2 - 3\varepsilon s - 12.$$

Consider (1) $k > 0$.

In this case, by (III), we know that $|s| \leq b < (1/\sqrt{k})$. By the graph and monotonicity of quadratic functions, we find

$$-8 - 3\varepsilon s < y \leq -3 - 3\varepsilon s. \quad (27)$$

Hence, for all $|s| \leq b < (1/\sqrt{k})$, (26) always holds if and only if $1 + \varepsilon s > 0$. Moreover,

if $\varepsilon > 0$, we have $\varepsilon \leq \sqrt{k}$. If $\varepsilon = 0$, $1 + \varepsilon s > 0$ holds forever for all $|s| \leq b < (1/\sqrt{k})$. If $\varepsilon < 0$, we get $\varepsilon \geq -\sqrt{k}$.

In a word, $\varepsilon \in [-\sqrt{k}, \sqrt{k}]$.

Consider (2) $k < 0$.

In this case, by (IV), we know that $|s| \leq b < (1/2\sqrt{-k})$. By the same way as above, we have

$$-3 - 3\varepsilon s \leq y < -\frac{23}{16} - 3\varepsilon s. \quad (28)$$

Hence, for all $|s| \leq b < (1/2\sqrt{-k})$, (26) always holds if and only if $(23/16) + 3\varepsilon s \geq 0$. Furthermore, if $\varepsilon > 0$, we have $\varepsilon \leq (23/24)\sqrt{-k}$. If $\varepsilon = 0$, $(23/16) + 3\varepsilon s \geq 0$ holds forever for all $|s| \leq b < (1/2\sqrt{-k})$. If $\varepsilon < 0$, we get $\varepsilon \geq -(23/24)\sqrt{-k}$.

In any case, $\varepsilon \in [-(23/24)\sqrt{-k}, (23/24)\sqrt{-k}]$. We complete the proof of the lemma. $\square$

Lemma 10. Let $F = \alpha + \varepsilon\beta + (2k\beta^2/\alpha) - (k^2\beta^4/3\alpha^3)$ be a Finsler metric on a manifold M^n , where ε is a constant and k is a nonzero constant. If F is of constant flag curvature, then it must satisfy

$$(a) \quad r_{00} = \sigma(1 + 4kB - 5ks^2)\alpha^2,$$

$$(b) \quad s_0 = 0,$$

$$(c) \quad s_0^k s_{0k} = 0,$$

where $\sigma = \sigma(x)$ is a smooth function on M .

Proof. By a direct computation, we have

$$Q = \frac{3\varepsilon + 12ks - 4k^2 s^3}{3(-1 + ks^2)^2},$$

$$\Psi = \frac{2k}{1 + 4kB - 5ks^2}, \quad (29)$$

$$\Theta = \frac{3\varepsilon - 40k^2 s^3 - 15\varepsilon ks^2 + 8k^3 s^5}{2(1 + 4kB - 5ks^2)(3 + 3\varepsilon s + 6ks^2 - k^2 s^4)}.$$

Since F is of constant flag curvature, F is also of constant Ricci curvature; that is, the following system of equations holds

$${}^\alpha R_i^j + T_i^j - K(n-1)F^2 = 0. \quad (30)$$

By Proposition 4, we can calculate T_i^i by Maple

$$\begin{aligned}
T_i^i = & \frac{r_{00}^2}{\alpha^2} \left[(n-1) \frac{\bar{c}_1}{A_1^3 A_3^2} + \frac{\bar{c}_2}{A_1^4} \right] \\
& + \frac{r_{00}s_0}{\alpha} \left[(n-1) \frac{\bar{c}_3}{A_1^3 A_2^2 A_3^2} + \frac{\bar{c}_4}{A_1^4 A_2^2} \right] \\
& + \frac{1}{\alpha} \left\{ r_{00}r_0 \left[(n-1) \frac{\bar{c}_5}{A_1^2 A_3} + \frac{\bar{c}_6}{A_1^3} \right] \right. \\
& \quad \left. + r_{00|0} \left[(n-1) \frac{\bar{c}_7}{A_1 A_3} + \frac{\bar{c}_8}{A_1^2} \right] \right\} \\
& + s_0^2 \left[(n-1) \frac{\bar{c}_9}{A_1^3 A_2^4 A_3^2} + \frac{\bar{c}_{10}}{A_1^4 A_2^5} \right] - \frac{16k^2}{A_1^2} (rr_{00} - r_0^2) \\
& + r_0 s_0 \left[(n-1) \frac{\bar{c}_{12}}{A_1^2 A_2^2 A_3} + \frac{\bar{c}_{13}}{A_1^3 A_2^2} \right] \\
& + \frac{4k}{A_1} (r_{00}r_i^i - r_{0i}r_0^i + r_{00|i}b^i - r_{0i|0}b^i) \\
& + r_{0i}s_0^i \left[(n-1) \frac{\bar{c}_{15}}{A_1 A_2^2 A_3} + \frac{\bar{c}_{16}}{A_1^2 A_2^2} \right] \\
& + s_{0|0} \left[(n-1) \frac{\bar{c}_{17}}{A_1 A_2^2 A_3} + \frac{\bar{c}_{18}}{A_1^2 A_2^2} \right] + \frac{\bar{c}_{19}}{A_2^5} s_{0i}s_0^i \\
& + \alpha \left\{ r s_0 \frac{\bar{c}_{20}}{A_1^2 A_2^2} + s_i s_0^i \left[(n-1) \frac{\bar{c}_{21}}{A_1 A_2^4 A_3} + \frac{\bar{c}_{22}}{A_1^2 A_2^4} \right] \right\} \\
& + \alpha \left[(3s_i r_0^i - 2s_0 r_i^i + 2r_i s_0^i - 2s_{0|i}b^i + s_{i|0}b^i) \right. \\
& \quad \left. \times \frac{\bar{c}_{23}}{A_1 A_2^2} + \frac{\bar{c}_{24}}{A_2^2} s_{0|i}^i \right] + \alpha^2 \left(\frac{\bar{c}_{25}}{A_1 A_2^4} s_i s^i + \frac{\bar{c}_{26}}{A_2^4} s_i^j s_j^i \right), \tag{31}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= 1 + 4kB - 5ks^2, & A_2 &= -1 + ks^2, \\
A_3 &= 3 + 3\varepsilon s + 6ks^2 - k^2 s^4 \tag{32}
\end{aligned}$$

and $\bar{c}_i$ ($i = 1, 2, \dots, 26$) are polynomials of s .

Plugging (31) into (30) and multiplying it by A_1^4 yield

$$\begin{aligned}
& \alpha R_i^i (1 + 4kB - 5ks^2)^4 + T_i^i (1 + 4kB - 5ks^2)^4 \\
& - K(n-1)F^2(1 + 4kB - 5ks^2)^4 = 0. \tag{33}
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
& \alpha R_i^i (1 + 4kB - 5ks^2)^4 - K(n-1)F^2(1 + 4kB - 5ks^2)^4 \\
& \equiv 0 \pmod{(1 + 4kB - 5ks^2)}. \tag{34}
\end{aligned}$$

Hence,

$$T_i^i (1 + 4kB - 5ks^2)^4 \equiv 0 \pmod{(1 + 4kB - 5ks^2)}. \tag{35}$$

From (31) we find that

$$\frac{r_{00}^2}{\alpha^2} \bar{c}_2 + \frac{r_{00}s_0}{\alpha} \frac{\bar{c}_4}{A_2^2} + \frac{\bar{c}_{10}}{A_2^5} s_0^2 \equiv 0 \pmod{(1 + 4kB - 5ks^2)}. \tag{36}$$

By Lemma 9, we need to divide two cases.

Consider (1) $k > 0$.

In this case, we know that $k \in (0, (1/B))$, where $B := b^2 = \|\beta\|_\alpha^2$. Hence, $((1 + 4kB)/5k) > 0$.

For

$$\begin{aligned}
1 + 4kB - 5ks^2 &= -5k \left(s^2 - \frac{1 + 4kB}{5k} \right) \\
&= -5k \left(s - \sqrt{\frac{1 + 4kB}{5k}} \right) \left(s + \sqrt{\frac{1 + 4kB}{5k}} \right), \tag{37}
\end{aligned}$$

we get

$$\begin{aligned}
\frac{r_{00}^2}{\alpha^2} \bar{c}_2 + \frac{r_{00}s_0}{\alpha} \frac{\bar{c}_4}{A_2^2} + \frac{\bar{c}_{10}}{A_2^5} s_0^2 &\equiv 0 \pmod{\left(s + \sqrt{\frac{1 + 4kB}{5k}} \right)}, \\
\frac{r_{00}^2}{\alpha^2} \bar{c}_2 + \frac{r_{00}s_0}{\alpha} \frac{\bar{c}_4}{A_2^2} + \frac{\bar{c}_{10}}{A_2^5} s_0^2 &\equiv 0 \pmod{\left(s - \sqrt{\frac{1 + 4kB}{5k}} \right)}. \tag{38}
\end{aligned}$$

Simplifying (38) by Maple yields

$$\begin{aligned}
& \left(\frac{r_{00}}{\alpha} - \frac{8k\sqrt{(5(1 + 4kB)/k)(2kB - 7)} + 75\varepsilon}{24(kB - 1)^2} s_0 \right)^2 \\
& \equiv 0 \pmod{\left(s + \sqrt{\frac{1 + 4kB}{5k}} \right)}, \\
& \left(\frac{r_{00}}{\alpha} + \frac{8k\sqrt{(5(1 + 4kB)/k)(2kB - 7)} - 75\varepsilon}{24(kB - 1)^2} s_0 \right)^2 \\
& \equiv 0 \pmod{\left(s - \sqrt{\frac{1 + 4kB}{5k}} \right)}. \tag{39}
\end{aligned}$$

Since $k \in (0, (1/B))$, $2kB - 7 \neq 0$. Moreover, we note that $\varepsilon \neq \pm(8k/75)\sqrt{(5(1 + 4kB)/k)(2kB - 7)}$. Otherwise, by Lemma 9, F is not a Finsler metric.

From (39), and Zhou's Lemma 4.1 [2], we obtain that

$$\begin{aligned} r_{00} + \frac{\sqrt{5k} \left[8k \sqrt{(5(1+4kB)/k)} (2kB-7) + 75\varepsilon \right]}{24(kB-1)^2 \sqrt{1+4kB}} \alpha s s_0 \\ = \sigma_1 (1+4kB-5ks^2) \alpha^2, \\ r_{00} + \frac{\sqrt{5k} \left[8k \sqrt{(5(1+4kB)/k)} (2kB-7) - 75\varepsilon \right]}{24(kB-1)^2 \sqrt{1+4kB}} \alpha s s_0 \\ = \sigma_2 (1+4kB-5ks^2) \alpha^2, \end{aligned} \quad (40)$$

where $\sigma_1 = \sigma_1(x)$ and $\sigma_2 = \sigma_2(x)$ are smooth functions on a manifold M^n .

It is easy to see that

$$ss_0 \equiv 0 \pmod{(1+4kB-5ks^2)}. \quad (41)$$

The above equation holds if and only if

$$s_0 = 0. \quad (42)$$

Therefore, we have

$$r_{00} = \sigma (1+4kB-5ks^2) \alpha^2, \quad (43)$$

where $\sigma = \sigma(x)$ is a smooth function on a manifold M^n .

Consider (2) $k < 0$.

In this case, we know that $k \in (-1/4B, 0)$, where $B := b^2 = \|\beta\|_\alpha^2$. Hence, $((1+4kB)/5k) < 0$. Moreover, $1+4kB-5ks^2$ is irreducible with respect to s .

Simplifying (36) directly by Maple yields

$$\begin{aligned} & \left[\frac{r_{00}}{\alpha} + \frac{8(2kB-7) \left(5ks + \sqrt{5k(5ks^2-4kB-1)} \right) - 75\varepsilon}{24(1-kB)^2} s_0 \right] \\ & \times \left[\frac{r_{00}}{\alpha} + \frac{8(2kB-7) \left(5ks - \sqrt{5k(5ks^2-4kB-1)} \right) - 75\varepsilon}{24(1-kB)^2} s_0 \right] \\ & \equiv 0 \pmod{(1+4kB-5ks^2)}. \end{aligned} \quad (44)$$

By Lemma 9, we note that $5ks^2 - 4kB - 1 < 0$. The above equation holds if and only if

$$\begin{aligned} r_{00} + \frac{8(2kB-7) \left(5ks + \sqrt{5k(5ks^2-4kB-1)} \right) - 75\varepsilon}{24(1-kB)^2} \alpha s_0 \\ = \sigma_3 (1+4kB-5ks^2) \alpha^2 \end{aligned} \quad (45)$$

or

$$\begin{aligned} r_{00} + \frac{8(2kB-7) \left(5ks - \sqrt{5k(5ks^2-4kB-1)} \right) - 75\varepsilon}{24(1-kB)^2} \alpha s_0 \\ = \sigma_4 (1+4kB-5ks^2) \alpha^2, \end{aligned} \quad (46)$$

where $\sigma_3 = \sigma_3(x)$ and $\sigma_4 = \sigma_4(x)$ are smooth functions on a manifold M^n .

In any case, it is easy to find that $s_0 = 0$. Hence, $r_{00} = \sigma(1+4kB-5ks^2)\alpha^2$, where $\sigma = \sigma(x)$ is a smooth function on a manifold M^n .

Plugging (31) and $s_0 = 0$ into (30) and multiplying it by A_2^5 , we have

$$s_{0i} s_0^i \bar{c}_{19} \equiv 0 \pmod{(-1+ks^2)}, \quad (47)$$

where

$$\begin{aligned} \bar{c}_{19} = -\frac{2}{9} (20k^5 s^8 - 160k^4 s^6 - 48\varepsilon k^3 s^5 + 360k^3 s^4 \\ + 240\varepsilon k^2 s^3 + 45\varepsilon^2 k s^2 + 36k - 9\varepsilon^2). \end{aligned} \quad (48)$$

Simplifying (47) by Maple, we have

$$s_{0i} s_0^i (192\varepsilon k s + 256k + 36\varepsilon^2) \equiv 0 \pmod{(-1+ks^2)}. \quad (49)$$

So

$$s_{0i} s_0^i = 0. \quad (50)$$

□

Lemma 11. Let $F = \alpha + \varepsilon\beta + (2k\beta^2/\alpha) - (k^2\beta^4/3\alpha^3)$ be a Finsler metric on an n -dimensional manifold M , where ε is a constant and k is a nonzero constant. Suppose that F is of constant flag curvature; then β is closed and $K = 0$.

Proof. By Lemma 10, we have

$$\begin{aligned} r_{ij} &= \sigma \left[(1+4kB) a_{ij} - 5k b_i b_j \right], \\ s_i &= 0, \quad s_k^i s_j^k = 0, \end{aligned} \quad (51)$$

where $\sigma = \sigma(x)$ is a smooth function on an n -dimensional manifold M .

Thus, by some computations, we have

$$r_0 = \sigma\beta(1-kB), \quad r = \sigma(1-kB)B,$$

$$r_{00|0} = \sigma_0 \alpha^2 (1+4kB-5ks^2) + 2k\sigma\alpha(4\alpha r_0 - 5sr_{00}),$$

$$r_{00|j} b^j = \sigma_b \alpha^2 (1+4kB-5ks^2) + 2k\sigma\alpha(4\alpha r - 5sr_0),$$

$$\begin{aligned}
r_{j0|0} b^j &= \sigma_0 (1 - kB) \beta + k\sigma (3\beta r_0 - 5Br_{00}), \\
r_i^i &= \sigma [(1 + 4kB)n - 5kB], \\
r_{i0} r_0^i &= \sigma^2 \alpha^2 [(1 + 4kB)^2 - 5ks^2 (2 + 3kB)], \\
r_{j0} &= \sigma \alpha [(1 + 4kB)l_j - 5ksb_j], \quad r_j = \sigma (1 - kB) b_j, \\
r_{j0|0} &= \sigma_0 \alpha [(1 + 4kB)l_j - 5ksb_j] \\
&\quad + \sigma k \alpha \left(8r_0 l_j - 5 \frac{r_{00}}{\alpha} b_j - 5sr_{j0} + 5ss_{0j} \right), \\
r_{00|j} &= \sigma_j \alpha^2 (1 + 4kB - 5ks^2) \\
&\quad + 2\sigma k \alpha (4r_j \alpha - 5sr_{j0} - 5ss_{0j}), \\
r_0^i &= \sigma \alpha [(1 + 4kB)l^i - 5ksb^i], \\
r_j^i &= \sigma [(1 + 4kB)\delta_j^i - 5kb^i b_j],
\end{aligned} \tag{52}$$

where $\sigma_0 := \sigma_{x^i} y^i$, $\sigma_b := \sigma_{x^i} b^i$, $r_0 := r_i y^i$, $l_j := (y_j/\alpha)$, $l^i := a^{ij} l_j = (y^i/\alpha)$, $(a^{ij}) := (a_{ij})^{-1}$.

Case 1 ($\varepsilon \neq 0$). Plugging the above equations into the expression of T_j^i in Proposition 3 and simplifying it by Maple, we have

$$\begin{aligned}
T_j^i &= \left(\frac{T_1}{A_3} \sigma^2 + T_2 \sigma_0 \right) \frac{\alpha^2}{A_3} \delta_j^i \\
&\quad + \left[\frac{T_3}{A_3^2} \alpha \sigma^2 + \left(\frac{T_4}{A_3} \alpha + T_5 \right) \frac{\sigma_0}{A_1} \right] \frac{\alpha}{A_3} l^i l_j \\
&\quad + \left[\frac{T_6}{A_3^2} \alpha \sigma^2 + \left(\frac{T_7}{A_3} \alpha + T_8 \right) \frac{\sigma_0}{A_1} \right] \frac{\alpha}{A_3} l^i b_j \\
&\quad + \frac{\alpha}{A_2^2 A_3} (T_9 \sigma + T_{10}) l^i s_{0j} + \frac{T_{11}}{A_3} \alpha^2 l^i \sigma_j \\
&\quad + 4\alpha k \left(\frac{5\alpha k s^2 - 4kB - 1}{A_1} \sigma_0 + \alpha k s \sigma^2 \right) b^i l_j \\
&\quad + 4\alpha k^2 \left[\frac{5s(1 - \alpha)}{A_1} \sigma_0 - \alpha \sigma^2 \right] b^i b_j \\
&\quad + \frac{\alpha \sigma T_{12}}{A_2^2} b^i s_{0j} + 4k\alpha^2 b^i \sigma_j - \frac{20k^2 s A_3}{A_2^3} (s l_j - b_j) s_0^i \\
&\quad + \frac{T_{13}}{A_2^5} s_0^i s_{0j} + \frac{T_{14} \alpha}{A_2^2} (2s_{0|j}^i - s_{j|0}^i) \\
&\quad + \frac{T_{15}}{A_2^3} s_{0|0}^i l_j + \frac{T_{16}}{A_2^3} s_{0|0}^i b_j,
\end{aligned} \tag{53}$$

where T_i ($i = 1, \dots, 16$) are polynomials of s , $A_1 = 1 + 4kB - 5ks^2$, $A_2 = -1 + ks^2$, and $A_3 = 3 + 3\varepsilon s + 6ks^2 - k^2 s^4$.

Because F has constant flag curvature, it is equivalent to

$${}^\alpha R_j^i + T_j^i = KF^2 \left(\delta_j^i - \frac{y^i}{F} F_{y^j} \right). \tag{54}$$

Substitute (53) into (54) and multiplying it by A_2^5 , we find

$$s_0^i s_{0j} T_{13} \equiv 0 \pmod{(ks^2 - 1)}, \tag{55}$$

where $T_{14} = -(1/3)(20k^5 s^8 - 160k^4 s^6 - 48\varepsilon k^3 s^5 + 360k^3 s^4 + 240\varepsilon k^2 s^3 + 45\varepsilon^2 k s^2 + 36k - 9\varepsilon^2)$.

By the same way as Lemma 10, we conclude $s_0^i = 0$ or $s_{0j} = 0$; that is, β is closed.

Case 2 ($\varepsilon = 0$). By the same approach as the above, we still conclude that β is closed.

Now we will prove $K = 0$.

Let $\phi = 1 + \varepsilon s + 2ks^2 - (k^2 s^4/3)$. If $F = \alpha\phi(s)$, $s = \beta/\alpha$, has constant flag curvature K , then F has constant Ricci curvature. That means that $Ric = (n - 1)KF^2$. Hence, F is an Einstein metric with Ricci constant K .

Note that $\phi = 1 + \varepsilon s + 2ks^2 - (k^2 s^4/3)$ is a polynomial in s of degree 4. According to Theorem 1.1 (see [6]), we know that it is Ricci flat. So $K = 0$. $\square$

Proof of Theorem 1. By Lemma 10, we have

$$r_{ij} = \sigma [(1 + 4kB) a_{ij} - 5kb_i b_j], \tag{56}$$

where $\sigma = \sigma(x)$ is a smooth function on an n -dimensional manifold.

Moreover, by Lemma 11, we know that β is closed; that is, $s_{ij} = 0$. Hence,

$$b_{i|j} = \sigma [(1 + 4kB) a_{ij} - 5kb_i b_j]. \tag{57}$$

(1) If $\sigma = 0$, then $b_{i|j} = 0$; that is, β is parallel with respect to α . By the proof of Lemma 11, we find that the sectional curvature of α equals 0. That means that α is flat. Therefore, F is locally Minkowskian.

(2) If $\sigma \neq 0$.

By (57), we obtain that F is a Douglas metric. That means that the Douglas curvature tensor $D = 0$.

Since F has constant flag curvature, the Weyl curvature tensor $W = 0$. By Theorem 8, we know that F must be locally projectively flat. Hence, by Theorem 1.3 (see [3]), F is locally Minkowskian. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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