Research Article

Asymptotic Behavior of the Solutions of System of Difference Equations of Exponential Form

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The goal of this paper is to study the boundedness, the persistence, and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form: $x_{n+1} = (a + be^{-y_n} + ce^{-x_n}) / (d + hy_n)$, $y_{n+1} = (a + be^{-x_n} + ce^{-y_n}) / (d + hx_n)$, where a, b, c, d , and h are positive constants and the initial values x_0, y_0 are positive real values. Also, we determine the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of this system.

1. Introduction

In [1], the authors studied the boundedness, the asymptotic behavior, the periodicity, and the stability of the positive solutions of the difference equation:

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \quad (1)$$

where α, β, γ are positive constants and the initial values y_{-1}, y_0 are positive numbers.

Motivated by the above paper we will extend the above difference equation to a system of difference equations; our goal will be to investigate the boundedness, the persistence, and the asymptotic behavior of the positive solutions of the following system of exponential form:

$$\begin{aligned} x_{n+1} &= \frac{a + be^{-y_n} + ce^{-x_n}}{d + hy_n}, \\ y_{n+1} &= \frac{a + be^{-x_n} + ce^{-y_n}}{d + hx_n}, \end{aligned} \quad (2)$$

where a, b, c, d, h are positive constants and the initial values x_0, y_0 are positive real values.

Difference equations and systems of difference equations of exponential form can be found in [2–6]. Moreover, as difference equations have many applications in applied sciences, there are many papers and books that can be found concerning the theory and applications of difference equations; see [7–9] and the references cited therein.

2. Global Behavior of Solutions of System (2)

In the first lemma we study the boundedness and persistence of the positive solutions of (2).

Lemma 1. Every positive solution of (2) is bounded and persists.

Proof. Let (x_n, y_n) be an arbitrary solution of (2). From (2) we can see that

$$x_n \leq \frac{a + b + c}{d}, \quad y_n \leq \frac{a + b + c}{d}, \quad n = 1, 2, \dots \quad (3)$$

In addition, from (2) and (3) we get

$$\begin{aligned} x_n &\geq \frac{a + be^{-(a+b+c)/d} + ce^{-(a+b+c)/d}}{d + h \cdot ((a+b+c)/d)}, \\ y_n &\geq \frac{a + be^{-(a+b+c)/d} + ce^{-(a+b+c)/d}}{d + h \cdot ((a+b+c)/d)}, \end{aligned} \quad (4)$$

$n = 2, 3, \dots$

Therefore, from (3) and (4) the proof of the lemma is complete. $\square$

In order to prove the main result of this section, we recall the next theorem without its proof. See [10, 11].

Theorem 2. Let $\mathcal{R} = [a_1, b_1] \times [c_1, d_1]$ and

$$f: \mathcal{R} \longrightarrow [a_1, b_1], \quad g: \mathcal{R} \longrightarrow [c_1, d_1] \quad (5)$$

be a continuous functions such that the following hold:

- (a) $f(x, y)$ is decreasing in both variables and $g(x, y)$ is decreasing in both variables for each $(x, y) \in \mathcal{R}$;
- (b) if $(m_1, M_1, m_2, M_2) \in \mathcal{R}^2$ is a solution of

$$\begin{aligned} M_1 &= f(m_1, m_2), \quad m_1 = f(M_1, M_2), \\ M_2 &= g(m_1, m_2), \quad m_2 = g(M_1, M_2) \end{aligned} \quad (6)$$

then $m_1 = M_1$ and $m_2 = M_2$. Then the following system of difference equations,

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad (7)$$

has a unique equilibrium $(\bar{x}, \bar{y})$ and every solution (x_n, y_n) of the system (7) with $(x_0, y_0) \in \mathcal{R}$ converges to the unique equilibrium $(\bar{x}, \bar{y})$. In addition, the equilibrium $(\bar{x}, \bar{y})$ is globally asymptotically stable.

Now we state the main theorem of this section.

Theorem 3. Consider system (2). Suppose that the following relation holds true:

$$b + c < d. \quad (8)$$

Then system (2) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of (2) tends to the unique positive equilibrium $(\bar{x}, \bar{y})$ as $n \rightarrow \infty$. In addition, the equilibrium $(\bar{x}, \bar{y})$ is globally asymptotically stable.

Proof. We consider the functions

$$\begin{aligned} f(u, v) &= \frac{a + be^{-v} + ce^{-u}}{d + hv}, \\ g(u, v) &= \frac{a + be^{-u} + ce^{-v}}{d + hu}, \end{aligned} \quad (9)$$

where

$$u, v \in I = \left[\frac{a + (b+c)e^{-(a+b+c)/d}}{c + h \cdot ((a+b+c)/d)}, \frac{a+b+c}{d} \right]. \quad (10)$$

It is easy to see that $f(u, v)$, $g(u, v)$ are decreasing in both variables for each $(u, v) \in I \times I$. In addition, from (9) and (10) we have $f(u, v) \in I$, $g(u, v) \in I$ as $(u, v) \in I \times I$ and so $f: I \times I \rightarrow I$, $g: I \times I \rightarrow I$.

Now let m_1, M_1, m_2, M_2 be positive real numbers such that

$$\begin{aligned} M_1 &= \frac{a + be^{-m_2} + ce^{-m_1}}{d + hm_2}, \\ M_2 &= \frac{a + be^{-m_1} + ce^{-m_2}}{d + hm_1}, \\ m_1 &= \frac{a + be^{-M_2} + ce^{-M_1}}{d + hM_2}, \\ m_2 &= \frac{a + be^{-M_1} + ce^{-M_2}}{d + hM_1}. \end{aligned} \quad (11)$$

Moreover arguing as in the proof of Theorem 2, it suffices to assume that

$$m_1 \leq M_1, \quad m_2 \leq M_2. \quad (12)$$

From (11) we get

$$\begin{aligned} be^{-m_2} + ce^{-m_1} &= (d + hm_2)M_1 - a, \\ be^{-M_2} + ce^{-M_1} &= (d + hM_2)m_1 - a, \\ be^{-m_1} + ce^{-m_2} &= (d + hm_1)M_2 - a, \\ be^{-M_1} + ce^{-M_2} &= (d + hM_1)m_2 - a, \end{aligned} \quad (13)$$

which imply that

$$\begin{aligned} d(M_1 - m_1) + h(M_1m_2 - M_2m_1) &= b(e^{-m_2} - e^{-M_2}) + c(e^{-m_1} - e^{-M_1}) \\ &= be^{-m_2-M_2}(e^{M_2} - e^{m_2}) + ce^{-m_1-M_1}(e^{M_1} - e^{m_1}), \\ d(M_2 - m_2) + h(M_2m_1 - M_2m_2) &= b(e^{-m_1} - e^{-M_1}) + c(e^{-m_2} - e^{-M_2}) \\ &= be^{-m_1-M_1}(e^{M_1} - e^{m_1}) + ce^{-m_2-M_2}(e^{M_2} - e^{m_2}). \end{aligned} \quad (14)$$

Moreover, we get

$$\begin{aligned} e^{M_1} - e^{m_1} &= e^\alpha (M_1 - m_1), \quad m_1 \leq \alpha \leq M_1, \\ e^{M_2} - e^{m_2} &= e^\beta (M_2 - m_2), \quad m_2 \leq \beta \leq M_2. \end{aligned} \quad (15)$$

Then by adding the two relations (14) we obtain

$$\begin{aligned} d(M_1 - m_1) + d(M_2 - m_2) &= (b+c) \\ &\cdot \left[e^{-m_1-M_1+\alpha} (M_1 - m_1) + e^{-m_2-M_2+\beta} (M_2 - m_2) \right]. \end{aligned} \quad (16)$$

Therefore from (16) we have

$$(M_1 - m_1) [d - (b + c) e^{-m_1 - M_1 + \alpha}] + (M_2 - m_2) [d - (b + c) e^{-m_2 - M_2 + \beta}] = 0. \quad (17)$$

Then using (8), (12), and (17) gives us $m_1 = M_1$ and $m_2 = M_2$. Hence from Theorem 2 system (2) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of (2) tends to the unique positive equilibrium $(\bar{x}, \bar{y})$ as $n \rightarrow \infty$. In addition, the equilibrium $(\bar{x}, \bar{y})$ is globally asymptotically stable. This completes the proof of the theorem. $\square$

3. Rate of Convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of the system (2) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [12, 13].

The following results give the rate of convergence of solutions of a system of difference equations:

$$\mathbf{x}_{n+1} = [A + B(n)] \mathbf{x}_n, \quad (18)$$

where $\mathbf{x}_n$ is a k -dimensional vector, $A \in \mathbb{C}^{k \times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \rightarrow \mathbb{C}^{k \times k}$ is a matrix function satisfying

$$\|B(n)\| \rightarrow 0 \quad \text{when } n \rightarrow \infty, \quad (19)$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm; $\|\cdot\|$ also denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|\mathbf{x}\| = \|(x, y)\| = \sqrt{x^2 + y^2}. \quad (20)$$

Theorem 4 (see [14]). Assume that condition (19) holds. If $\mathbf{x}_n$ is a solution of system (18), then either $\mathbf{x}_n = 0$ for all large n or

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\|\mathbf{x}_n\|} \quad (21)$$

exists and is equal to the modulus of one of the eigenvalues of matrix A .

Theorem 5 (see [14]). Assume that condition (19) holds. If $\mathbf{x}_n$ is a solution of system (18), then either $\mathbf{x}_n = 0$ for all large n or

$$\rho = \lim_{n \rightarrow \infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \quad (22)$$

exists and is equal to the modulus of one of the eigenvalues of matrix A .

The equilibrium point of the system (2) satisfies the following system of equations:

$$\begin{aligned} \bar{x} &= \frac{a + be^{-\bar{y}} + ce^{-\bar{x}}}{d + h \cdot \bar{y}}, \\ \bar{y} &= \frac{a + be^{-\bar{x}} + ce^{-\bar{y}}}{d + h \cdot \bar{x}}. \end{aligned} \quad (23)$$

If $b + c < d$, we can easily see that the system (23) has an unique equilibrium $E = (\bar{x}, \bar{x})$.

The map T associated with the system (2) is

$$T(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} \frac{a + be^{-y} + ce^{-x}}{d + hy} \\ \frac{a + be^{-x} + ce^{-y}}{d + hx} \end{pmatrix}. \quad (24)$$

The Jacobian matrix of T is

$$J_T = \begin{pmatrix} \frac{-ce^{-x}}{d + hy} & \frac{-be^{-y}(d + hy) - h \cdot (a + be^{-y} + ce^{-x})}{(d + hy)^2} \\ \frac{-be^{-x}(d + hx) - h \cdot (a + be^{-x} + ce^{-y})}{(d + hx)^2} & \frac{-ce^{-y}}{d + hx} \end{pmatrix}. \quad (25)$$

By using the system (23), value of the Jacobian matrix of T at the equilibrium point $E = (\bar{x}, \bar{y}) = (\bar{x}, \bar{x})$ is

$$J_T = \begin{pmatrix} \frac{-ce^{-\bar{x}}}{d + h\bar{x}} & \frac{-be^{-\bar{x}}(d + h\bar{x}) - h \cdot [a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2} \\ \frac{-be^{-\bar{x}}(d + h\bar{x}) - h \cdot [a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2} & \frac{-ce^{-\bar{x}}}{d + h\bar{x}} \end{pmatrix}. \quad (26)$$

Our goal in this section is to determine the rate of convergence of every solution of the system (2) in the regions where the parameters $a, b, c, d, h \in (0, \infty)$, $(b + c < d)$ and initial conditions x_0 and y_0 are arbitrary, nonnegative numbers.

Theorem 6. The error vector $\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$ of every solution $\mathbf{x}_n \neq \mathbf{0}$ of (2) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{\|\mathbf{e}_n\|} &= |\lambda_i(J_T(E))| \quad \text{for some } i = 1, 2, \\ \lim_{n \rightarrow \infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} &= |\lambda_i(J_T(E))| \quad \text{for some } i = 1, 2, \end{aligned} \quad (27)$$

where $|\lambda_i(J_T(E))|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $J_T(E)$.

Proof. First, we will find a system satisfied by the error terms. The error terms are given as

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{a + be^{-y_n} + ce^{-x_n}}{d + hy_n} - \frac{a + be^{-\bar{y}} + ce^{-\bar{x}}}{d + h\bar{y}} \\ &= \left((a + be^{-y_n} + ce^{-x_n})(d + h\bar{y}) \right. \\ &\quad \left. - (a + be^{-\bar{y}} + ce^{-\bar{x}})(d + hy_n) \right) \\ &\quad \times ((d + hy_n)(d + h\bar{y}))^{-1} \\ &= (bd(e^{-y_n} - e^{-\bar{y}}) + cd(e^{-x_n} - e^{-\bar{x}}) \\ &\quad + ah(\bar{y} - y_n) + bh(e^{-y_n}\bar{y} - e^{-\bar{y}}y_n) \\ &\quad + ch(e^{-x_n}\bar{y} - e^{-\bar{x}}y_n)) \\ &\quad \times ((d + hy_n)(d + h\bar{y}))^{-1} \\ &= \frac{-bd(e^{y_n} - e^{\bar{y}})}{e^{y_n+\bar{y}}(d + hy_n)(d + h\bar{y})} \\ &\quad + \frac{-cd(e^{x_n} - e^{\bar{x}})}{e^{x_n+\bar{x}}(d + hy_n)(d + h\bar{y})} \\ &\quad + \frac{bh}{(d + hy_n)(d + h\bar{y})} \\ &\quad \times (e^{-y_n}\bar{y} - e^{-y_n}y_n + e^{-y_n}y_n - e^{-\bar{y}}y_n) \\ &\quad + \frac{ch}{(d + hy_n)(d + h\bar{y})} \\ &\quad \times (e^{-x_n}\bar{y} - e^{-x_n}y_n + e^{-x_n}y_n - e^{-\bar{x}}y_n) \\ &\quad - \frac{ah}{(d + hy_n)(d + h\bar{y})}(y_n - \bar{y}) \\ &= \frac{-b}{e^{y_n+\bar{y}}(d + h\bar{y})}(e^{y_n} - e^{\bar{y}}) \end{aligned}$$

$$\begin{aligned} &- \frac{c}{e^{x_n+\bar{x}}(d + h\bar{y})}(e^{x_n} - e^{\bar{x}}) \\ &- h \cdot \frac{a + be^{-y_n} + ce^{-x_n}}{(d + hy_n)(d + h\bar{y})}(y_n - \bar{y}) \\ &= \frac{-b}{e^{y_n}(d + h\bar{y})}(e^{y_n-\bar{y}} - 1) \\ &- \frac{c}{e^{x_n}(d + h\bar{y})}(e^{x_n-\bar{x}} - 1) \\ &- h \cdot \frac{a + be^{-y_n} + ce^{-x_n}}{(d + hy_n)(d + h\bar{y})}(y_n - \bar{y}) \\ &= \frac{-b}{e^{y_n}(d + h\bar{y})}[(y_n - \bar{y}) + \mathcal{O}_2((y_n - \bar{y})^2)] \\ &- \frac{c}{e^{x_n}(d + h\bar{y})}[(x_n - \bar{x}) + \mathcal{O}_1((x_n - \bar{x})^2)] \\ &- h \cdot \frac{a + be^{-y_n} + ce^{-x_n}}{(d + hy_n)(d + h\bar{y})}(y_n - \bar{y}) \\ &= \frac{-c}{e^{x_n}(d + h\bar{y})}(x_n - \bar{x}) \\ &\quad + \frac{-be^{-y_n}(d + hy_n) - h(a + be^{-y_n} + ce^{-x_n})}{(d + hy_n)(d + h\bar{y})} \\ &\quad \times (y_n - \bar{y}) \\ &\quad + \mathcal{O}_1((x_n - \bar{x})^2) + \mathcal{O}_2((y_n - \bar{y})^2). \end{aligned} \quad (28)$$

By calculating similarly, we get

$$\begin{aligned} y_{n+1} - \bar{y} &= \frac{-be^{-x_n}(d + hx_n) - h(a + be^{-x_n} + ce^{-y_n})}{(d + hx_n)(d + h\bar{x})}(x_n - \bar{x}) \\ &\quad + \frac{-c}{e^{y_n}(d + h\bar{x})}(y_n - \bar{y}) \\ &\quad + \mathcal{O}_3((x_n - \bar{x})^2) + \mathcal{O}_4((y_n - \bar{y})^2). \end{aligned} \quad (29)$$

From (28) and (29) we have

$$\begin{aligned} x_{n+1} - \bar{x} &\approx \frac{-c}{e^{x_n}(d + h\bar{y})}(x_n - \bar{x}) \\ &\quad + \frac{-be^{-y_n}(d + hy_n) - h(a + be^{-y_n} + ce^{-x_n})}{(d + hy_n)(d + h\bar{y})} \\ &\quad \times (y_n - \bar{y}), \end{aligned}$$

$$y_{n+1} - \bar{y} \approx \frac{-be^{-x_n}(d + hx_n) - h(a + be^{-x_n} + ce^{-y_n})}{(d + hx_n)(d + h\bar{x})} \times (x_n - \bar{x}) + \frac{-c}{e^{y_n}(d + h\bar{x})} (y_n - \bar{y}). \quad (30)$$

Set

$$e_n^1 = x_n - \bar{x}, \quad e_n^2 = y_n - \bar{y}. \quad (31)$$

Then system (30) can be represented as

$$\begin{aligned} e_{n+1}^1 &\approx a_n e_n^1 + b_n e_n^2, \\ e_{n+1}^2 &\approx c_n e_n^1 + d_n e_n^2, \end{aligned} \quad (32)$$

where

$$\begin{aligned} a_n &= \frac{-c}{e^{x_n}(d + h\bar{y})}, \\ b_n &= \frac{-be^{-y_n}(d + hy_n) - h(a + be^{-y_n} + ce^{-x_n})}{(d + hy_n)(d + h\bar{y})}, \\ c_n &= \frac{-be^{-x_n}(d + hx_n) - h(a + be^{-x_n} + ce^{-y_n})}{(d + hx_n)(d + h\bar{x})}, \\ d_n &= \frac{-c}{e^{y_n}(d + h\bar{x})}. \end{aligned} \quad (33)$$

Taking the limits of a_n , b_n , c_n , and d_n as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \frac{-c}{e^{\bar{x}}(d + h\bar{x})}, \\ \lim_{n \rightarrow \infty} b_n &= \frac{-be^{-\bar{x}}(d + h\bar{x}) - h[a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2}, \\ \lim_{n \rightarrow \infty} c_n &= \frac{-be^{-\bar{x}}(d + h\bar{x}) - h[a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2}, \\ \lim_{n \rightarrow \infty} d_n &= \frac{-c}{e^{\bar{x}}(d + h\bar{x})}; \end{aligned} \quad (34)$$

that is

$$\begin{aligned} a_n &= \frac{-c}{e^{\bar{x}}(d + h\bar{x})} + \alpha_n, \\ b_n &= \frac{-be^{-\bar{x}}(d + h\bar{x}) - h[a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2} + \beta_n, \\ c_n &= \frac{-be^{-\bar{x}}(d + h\bar{x}) - h[a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2} + \gamma_n, \\ d_n &= \frac{-c}{e^{\bar{x}}(d + h\bar{x})} + \delta_n, \end{aligned} \quad (35)$$

where $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0$, $\gamma_n \rightarrow 0$, and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.

Now, we have system of the form (18):

$$\mathbf{e}_{n+1} = (A + B(n)) \mathbf{e}_n, \quad (36)$$

where

$$A = \begin{pmatrix} \frac{-ce^{-\bar{x}}}{d + h\bar{x}} & \frac{-be^{-\bar{x}}(d + h\bar{x}) - h[a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2} \\ \frac{-be^{-\bar{x}}(d + h\bar{x}) - h[a + (b + c)e^{-\bar{x}}]}{(d + h\bar{x})^2} & \frac{-ce^{-\bar{x}}}{d + h\bar{x}} \end{pmatrix}, \quad (37)$$

$$B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix}, \quad \|B(n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, the limiting system of error terms can be written as

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{pmatrix} = A \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix}. \quad (38)$$

The system is exactly linearized system of (2) evaluated at the equilibrium $E = (\bar{x}, \bar{y}) = (\bar{x}, \bar{x})$. Then Theorems 4 and 5 imply the result. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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