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Research Article

Approximate Periodic Solution for the Nonlinear Helmholtz-Duffing Oscillator via Analytical Approaches

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The conservative Helmholtz-Duffing oscillator is analyzed by means of three analytical techniques. The max-min, second-order of the Hamiltonian, and the global error minimization approaches are applied to achieve natural frequencies. The obtained results are compared with the homotopy perturbation method and numerical solutions. The results show that second-order of the global error minimization method is very accurate, so it can be widely applicable in engineering problems.

1. Introduction

Mathematical modeling and frequency analysis of the nonlinear vibrational systems are an important and interesting field of mechanics. A lot of researchers have worked in this field and have proposed a lot of methods for demonstrating the dynamics responses of these systems [1–4]. They have developed this field of science and have analyzed the responses of the nonlinear vibration problems such as Duffing oscillators [5–10], nonlinear dynamics of a particle on a rotating parabola [11], nonlinear oscillators with discontinuity [12], oscillators with noninteger order nonlinear connection [13], the plasma physics equation [14], and van der Pol oscillator [15, 16]. The Helmholtz-Duffing equation is a nonlinear problem with the quadratic and cubic nonlinear terms. Surveying the literature shows that this equation has wide applications in the engineering problems. For example, due to different vibration behavior of functionally graded materials (FGMs) at positive and negative amplitudes, the governing equations of FGM beams, plates, and shells are conduced to a second-order nonlinear ordinary equation with quadratic and cubic nonlinear terms [17–20]. Moreover, Sharabiani and Yazdi [21] obtained a Helmholtz-Duffing type equation within studying of nonlinear free vibrations of functionally graded nanobeams with surface effects. On the other hand, they revealed application of this equation in FG nanostructures.

In this paper, the frequency-amplitude relationship of the conservative Helmholtz-Duffing oscillator is obtained by means of the max-min [22–26], Hamiltonian [27–31], and global error minimization methods [32–35]. The Hamiltonian approach is a kind of energy method and is proposed by He [27]. It is a simple method and can be used for the conservative nonlinear equations. Recently, it is applied for dynamic analysis of an electromechanical resonator [36]. Moreover, Akbarzade and Khan [37] employed the second-order Hamiltonian approach for nonlinear dynamic analysis of conservative coupled systems of mass-spring. The max-min approach is made on the base of Chengtian's inequality [38]. It is a valuable method for obtaining the frequency responses of the nonlinear problems, and many researchers are attracted to use this method for studying the nonlinear systems. The global error minimization method is a modified type of variational approach and converts the nonlinear equation to an equivalent minimization problem.

The Helmholtz-Duffing equation is analyzed by many researchers. For instance, Leung and Guo [39] have applied the homotopy perturbation method (HPM) to this equation

and have obtained accurate responses. Askari et al. [40] studied approximate periodic solutions of this equation using He's energy balance method (HEBM) and He's frequency-amplitude formulation (HFAF). Akbarzade et al. [41] used the first-order of the Hamiltonian approach and coupled homotopy-variational formulation to study the periodic solutions of the Helmholtz-Duffing oscillator; they also discussed the stability of the system for selected constant parameters. Recently, Li et al. [42] determined limit cycles and homoclinic orbits of this oscillator via a generalized harmonic function perturbation method. In the next sections, the max-min, the Hamiltonian, and the global error minimization methods are applied for evaluating the dynamics responses of the Helmholtz-Duffing equation. The results of these methods are compared with the exact ones and HPM solutions.

2. Solution Procedure

Let us consider the Helmholtz-Duffing equation

$$\ddot{u} + u + (1-\sigma)u^2 + \sigma u^3 = 0, \qquad u(0) = A, \qquad \dot{u}(0) = 0. \tag{1}$$

The response oscillates between an asymmetric limit zone $[-b, A]$, where both A and b are positive. One can determine b in the form of [39]

$$b = \frac{1}{9A}(3A\sigma + 4 - 4A) + \frac{1}{9A}\Delta^{1/3} - \frac{2}{9A}\left(9A^2\sigma^2 + 6A\sigma - 6A\sigma^2 + 43\sigma - 8 - 8\sigma^2\right)\Delta^{-1/3}, \tag{2}$$

where

$$\begin{aligned}\Delta = {} & 270A^2\sigma^2(1 + A\sigma - \sigma) - 72A\sigma\left(1+\sigma^2\right) \\ & - 516\sigma(1-\sigma) + 64\left(1-\sigma^3\right) + 630A\sigma^2 \\ & + 54\sigma\Big[-12\left(1+\sigma^2\right) + 16A\left(a + 1 + A\sigma^4 - \sigma^3\right) \\ & + 78A\left(1 + A^3\sigma - A^3\sigma^2\right) - 8A^3\sigma\left(1-\sigma^3\right) \\ & - 172A^2\sigma\left(1+\sigma^2\right) - 120A\sigma(1-\sigma) + 9A^4\sigma^2 \\ & \times\left(1 + 5A\sigma + 10\sigma + 3A^2\sigma^2 - 5A\sigma^2 + \sigma^2\right) \\ & + 447A^2\sigma^2\Big]^{1/2}.\end{aligned} \tag{3}$$

2.1. Max-Min Approach (MMA). We can rewrite (1) in the following form:

$$\ddot{u} + \left(1 + (1-\sigma)u + \sigma u^2\right)u = 0. \tag{4}$$

By choosing $u(t) = A\cos(\omega t)$ as a trial function that satisfied the initial conditions, the maximum and minimum values of $1 + (1-\sigma)u + \sigma u^2$ can be calculated with Maple software as $1 + A - \sigma A + \sigma A^2$ and $-(\sigma^2 - 6\sigma + 1)/4\sigma$, respectively, so we can write

$$-\frac{\sigma^2 - 6\sigma + 1}{4\sigma} < \omega^2 < 1 + A - \sigma A + \sigma A^2. \tag{5}$$

By using Chengtian interpolation [38], we have

$$\begin{aligned}\omega^2 &= \frac{-m\left(\left(\sigma^2 - 6\sigma + 1\right)/4\sigma\right) + n\left(1 + A - \sigma A + \sigma A^2\right)}{m+n} \\ &= 1 - (1-k)\left(\frac{\sigma^2 - 2\sigma + 1}{4\sigma}\right) + k\left(A - \sigma A + \sigma A^2\right),\end{aligned} \tag{6}$$

where m, n are weighting factor, $k = n/(m+n)$. Therefore the frequency can be approximated as

$$\omega = \sqrt{1 + (1-k)\left(-\frac{\sigma^2 - 2\sigma + 1}{4\sigma}\right) + k\left(A - \sigma A + \sigma A^2\right)}. \tag{7}$$

Then the solution of (1) can be written:

$$u(t) = A\cos\left(\left(1 - (1-k)\left(\frac{\sigma^2 - 2\sigma + 1}{4\sigma}\right) + k\left(A - \sigma A + \sigma A^2\right)t\right)^{1/2}\right). \tag{8}$$

By using the approximate solution, (4) can be written in the following form:

$$\begin{aligned}\ddot{u} + \left(1 - (1-k)\left(\frac{\sigma^2 - 2\sigma + 1}{4\sigma}\right) + k\left(A - \sigma A + \sigma A^2\right)\right)u \\ = -(1-k)\left(\frac{\sigma^2 - 2\sigma + 1}{4\sigma}\right)u + k\left(A - \sigma A + \sigma A^2\right)u \\ - (1-\sigma)u^2 - \sigma u^3.\end{aligned} \tag{9}$$

If by chance (8) is the exact solution of (1), then the right-hand side of (9) vanishes. According to [22], we set

$$\int_0^{T/4}\left((k-1)\left(\frac{\sigma^2 - 2\sigma + 1}{4\sigma}\right)u + k\left(A - \sigma A + \sigma A^2\right)u - (1-\sigma)u^2 - \sigma u^3\right)\cos(\omega t)\,dt = 0, \tag{10}$$

where $T = 2\pi/\omega$. By substituting (8) into (10), we obtain the following expression for k:

$$k = \frac{1}{3}\frac{3\pi - 6\pi\sigma + 3\pi\sigma^2 + 32A\sigma - 32A\sigma^2 + 9\pi A^2\sigma^2}{\pi\left(1 - 2\sigma + \sigma^2 + 4A\sigma - 4A\sigma^2 + 4A^2\sigma^2\right)}. \tag{11}$$

Substituting (11) into (7) and after some simplification, we have

$$\omega = \sqrt{1 + \frac{8}{3\pi}A - \frac{8}{3\pi}A\sigma + \frac{3}{4}A^2\sigma}. \tag{12}$$

2.2. Hamiltonian Approach

2.2.1. The First-Order Hamiltonian Approach (FHA). The Hamiltonian of (1) is constructed as

$$H = \frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{1}{3}(1-\sigma)u^3 + \frac{1}{4}\sigma u^4. \tag{13}$$

Assume the first approximate solution of (1) as

$$u(t) = A\cos(\omega t). \tag{14}$$

Then integrating (13) with respect to time from 0 to $T/4$, we have

$$\widetilde{H}(u) = \int_0^{T/4}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{1}{3}(1-\sigma)u^3 + \frac{1}{4}\sigma u^4\right)dt. \tag{15}$$

Substituting (14) in (15) yields

$$\begin{aligned}\widetilde{H}(u) &= \int_0^{T/4}\left(\frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{2}A^2\cos^2(\omega t)\right.\\ &\quad\left.+\frac{1}{3}(1-\sigma)A^3\cos^3(\omega t) + \frac{1}{4}\sigma A^4\cos^4(\omega t)\right)dt\\ &= \int_0^{\pi/2}\left(\frac{1}{2}A^2\omega\sin^2(t) + \frac{1}{2\omega}A^2\cos^2(t)\right.\\ &\quad\left.+\frac{1}{3\omega}(1-\sigma)A^3\cos^3(t) + \frac{1}{4\omega}\sigma A^4\cos^4(t)\right)dt\\ &= \frac{1}{576}\frac{A^2\left(72\pi + 128A - 128A\sigma + 27\pi A^2\sigma + 72\pi\omega^2\right)}{\omega}.\end{aligned} \tag{16}$$

Set

$$\begin{aligned}&\frac{\partial}{\partial A}\left(\frac{\partial\widetilde{H}}{\partial(1/\omega)}\right)\\ &\quad= \frac{1}{48}A\left(-12\pi\omega^2 + 12\pi + 32A - 32A\sigma + 9\pi A^2\sigma\right) = 0.\end{aligned} \tag{17}$$

Finally,

$$\omega_{\mathrm{FHA}} = \sqrt{1 + \frac{8}{3\pi}A - \frac{8}{3\pi}A\sigma + \frac{3}{4}A^2\sigma}. \tag{18}$$

The result of first-order Hamiltonian and max-min approaches is alike.

2.2.2. The Second-Order Hamiltonian Approach (SHA). For the second-order Hamiltonian approach, we consider the following equation as the response of the system:

$$u = a\cos(\omega t) + b\cos(3\omega t). \tag{19}$$

Whereas (19) must satisfy the initial condition we have

$$A = a + b. \tag{20}$$

Substituting (19) into (15) yields

$$\begin{aligned}\widetilde{H}(u) &= \int_0^{T/4}\left[\frac{1}{2}\omega^2(a\sin(\omega t) + 3b\sin(3\omega t))^2\right.\\ &\quad+\frac{1}{2}(a\cos(\omega t) + b\cos(3\omega t))^2\\ &\quad+\frac{1}{3}(1-\sigma)(a\cos(\omega t) + b\cos(3\omega t))^3\\ &\quad\left.+\frac{1}{4}\sigma(a\cos(\omega t) + b\cos(3\omega t))^4\right]dt\\ \widetilde{H}(u) &= \int_0^{\pi/2}\left[\frac{1}{2}\omega(a\sin(t) + 3b\sin(3t))^2\right.\\ &\quad+\frac{1}{2\omega}(a\cos(t) + b\cos(3t))^2\\ &\quad+\frac{1}{3\omega}(1-\sigma)(a\cos(t) + b\cos(3t))^3\\ &\quad\left.+\frac{1}{4\omega}\sigma(a\cos(t) + b\cos(3t))^4\right]dt\end{aligned} \tag{21}$$

$$\begin{aligned}\widetilde{H}(u) &= \frac{1}{60480\omega}\\ &\quad\times\left(13440a^3 - 4480b^3 - 13440a^3\sigma\right.\\ &\quad- 8046a^2b\sigma + 2835\pi a^4\sigma - 31104ab^2\sigma\\ &\quad+ 8064a^2b + 2835\pi b^4\sigma + 4480b^3\sigma\\ &\quad+ 31104ab^2 + 3780\pi a^3b\sigma + 11340\pi a^2b^2\sigma\\ &\quad\left.+7560\pi a^2 + 7560\pi b^2\right) + \frac{\pi\omega}{8}\left(a^2 + 9b^2\right).\end{aligned}$$

Set

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{\partial\widetilde{H}}{\partial(1/\omega)}\right) &= -\frac{1}{4}\pi a\omega^2 + \frac{2}{3}a^2 - \frac{2}{3}a^2\sigma\\ &\quad- \frac{4}{15}ab\sigma + \frac{3}{16}\pi a^3\sigma - \frac{18}{35}b^2\sigma + \frac{4}{15}ab\\ &\quad+ \frac{18}{35}b^2 + \frac{3}{16}\pi a^2b\sigma + \frac{3}{8}\pi ab^2\sigma + \frac{1}{4}\pi a = 0\\ \frac{\partial}{\partial b}\left(\frac{\partial\widetilde{H}}{\partial(1/\omega)}\right) &= -\frac{9}{4}\pi b^2\omega^2 - \frac{2}{9}b^2 - \frac{2}{15}a^2\sigma\\ &\quad- \frac{36}{35}ab\sigma + \frac{2}{15}a^2 + \frac{3}{16}\pi b^3\sigma + \frac{2}{9}b^2\sigma\\ &\quad+ \frac{36}{35}ab + \frac{1}{16}\pi a^3\sigma + \frac{3}{8}\pi a^2b\sigma + \frac{1}{4}\pi b = 0.\end{aligned} \tag{22}$$

Since finding a relation between different parameters from (22) is not easy, the value of other parameters is determined in Table 1 for several values of σ.

Table 1: Frequency-amplitude relationships for several values of σ.

σ	a	b	ω_{SHA}
0.1	$0.987791887A$	$0.012208112A$	$\sqrt{1+0.758436858A+0.074106747A^2}$
0.3	$0.986559761A$	$0.013440239A$	$\sqrt{1+0.589470841A+0.222057234A^2}$
0.5	$0.985302609A$	$0.014697391A$	$\sqrt{1+0.420742301A+0.369650489A^2}$
0.7	$0.984019498A$	$0.015980502A$	$\sqrt{1+0.252257249A+0.516878382A^2}$
0.9	$0.982709435A$	$0.017290564A$	$\sqrt{1+0.084021956A+0.663732471A^2}$
1	$0.982043978A$	$0.017956021A$	$\sqrt{1+0.737016661A^2}$

2.3. Global Error Minimization Method

2.3.1. The First-Order Global Error Minimization Method (FGEM). Based on standard procedure of modified variational approach, the minimization problem is

$$\text{Minimize } E(\dot{u},u,t) = \int_0^T \left(\ddot{u}+u+(1-\sigma)u^2+\sigma u^3\right)^2 dt, \qquad T=\frac{2\pi}{\omega}. \tag{23}$$

For first-order approximation, consider a trial function as follows:

$$u(t) = A\cos(\omega t). \tag{24}$$

Substituting (24) into (23) yields

$$\begin{aligned}\text{Minimize } E(\dot{u},u,t) &= \frac{\pi A^2}{8\omega}\left(5A^4\sigma^2-12A^2\sigma\omega^2+6A^2\sigma^2+8\omega^4\right.\\ &\quad\left.-16\omega^2+6A^2+8\right).\end{aligned} \tag{25}$$

The solution of (25) could be found through

$$\frac{\partial E(\dot{u},u,t)}{\partial\omega}=0. \tag{26}$$

By some simplifications, the following equation is obtained:

$$\begin{aligned}&-24\omega^4+12A^2\omega^2\sigma+16\omega^2+5A^4\sigma^2\\ &\quad+6A^2\sigma^2+6A^2+8=0.\end{aligned} \tag{27}$$

One may find the first-order approximation by solving (27) as

$$\begin{aligned}\omega_{\mathrm{FGEM}} &= \frac{1}{6}\left(9A^2\sigma+12\right.\\ &\quad\left.+3\left(39A^4\sigma^2+36A^2\sigma^2+24A^2\sigma\right.\right.\\ &\quad\left.\left.+36A^2+64\right)^{1/2}\right)^{1/2}.\end{aligned} \tag{28}$$

2.3.2. The Second-Order Global Error Minimization Method (SGEM). For the second-order approximation, consider a trial function as

$$u(t) = a\cos(\omega t)+b\cos(3\omega t). \tag{29}$$

Whereas (29) must satisfy the initial condition, we have

$$A = a+b. \tag{30}$$

Substituting (29) into (23) yields

$$\begin{aligned}\text{Minimize } E(\dot{u},u,t) &= \frac{\pi}{8\omega}\left(5\sigma^2a^6-144b^2\omega^2\right.\\ &\quad+45\sigma^2a^2b^4+45\sigma^2a^4b^2+24\sigma^2a^2b^2\\ &\quad+8\sigma^2a^3b+6\sigma^2b^4+5\sigma^2b^6+8a^2\omega^4\\ &\quad+30\sigma^2a^3b^3+24a^2b^2-16a^2\omega^2\\ &\quad+8a^2+8b^2+6a^4-240a^2\omega^2\sigma b^2\\ &\quad+15\sigma^2a^5b-12a^4\omega^2\sigma+6\sigma^2a^4\\ &\quad+648b^2\omega^4+6b^4-108b^4\omega^2\sigma\\ &\quad\left.+8a^3b-48a^3\omega^2\sigma b\right).\end{aligned} \tag{31}$$

The solution of (31) could be found through

$$\frac{\partial E(\dot{u},u,t)}{\partial\omega}=0, \qquad \frac{\partial E(\dot{u},u,t)}{\partial a}=0, \qquad \frac{\partial E(\dot{u},u,t)}{\partial b}=0, \tag{32}$$

which yield the following:

$$\begin{aligned}\frac{\partial E(\dot{u},u,t)}{\partial\omega} &= -\frac{\pi}{8\omega^2}\left(5\sigma^2a^6+8a^3b+45\sigma^2a^2b^4\right.\\ &\quad+8a^2+8b^2+5\sigma^2b^6+45\sigma^2a^4b^2\\ &\quad-24a^2\omega^4+16a^2\omega^2+6b^4\\ &\quad+6a^4+24\sigma^2a^2b^2+15\sigma^2a^5b\\ &\quad+30\sigma^2a^3b^3-1944b^2\omega^4\\ &\quad+144b^2\omega^3+6\sigma^2a^4+6\sigma^2b^4\\ &\quad+48a^3\omega^2\sigma b+240a^2\omega^2\sigma b^2\\ &\quad+8\sigma^2a^3b+12a^4\omega^2\sigma\\ &\quad\left.+108b^4\omega^2\sigma+24a^2b^2\right)=0\end{aligned}$$

Table 2: Comparison between MMA, FHA, SHA, FGEM, and SGEM results with exact ones and HPM responses ($\sigma = 0.5$).

A	T_{exact}	$T_{\text{MMA}} = T_{\text{FHA}}$ (relative error %)	T_{FGEM} (relative error %)	T_{SHA} (relative error %)	T_{SGEM} (relative error %)	T_{HPM} [39] (relative error %)
0.01	6.283133	6.26977721 (0.2125657)	6.28305278 (0.0012767)	6.2698934 (0.2107154)	6.28305278 (0.0012767)	6.283132 (0.0000159)
0.05	6.281851	6.21470929 (1.0688203)	6.27987482 (0.0314585)	6.21530801 (1.0592895)	6.27987515 (0.0314630)	6.281831 (0.0003184)
0.1	6.277721	6.14291226 (2.1474152)	6.26997826 (0.1233367)	6.14414741 (2.1277402)	6.26998356 (0.1234043)	6.277642 (0.0012584)
10	1.004172	0.9610403 (4.2952461)	0.99830911 (0.5838523)	0.96753201 (3.6487768)	1.02917191 (2.4291285)	0.98321194 (2.087298)
20	0.5143428	0.4975256 (3.2696323)	0.50439032 (1.9349883)	0.50101412 (2.5914005)	0.52050519 (1.1839256)	0.50328477 (2.14993)
50	0.2082766	0.2028184 (2.6206285)	0.20235575 (2.8427777)	0.20426634 (1.9254484)	0.20888151 (0.2895968)	0.20375994 (2.168587)
100	0.1045213	0.1020148 (2.3980756)	0.10122095 (3.1575832)	0.10274684 (1.6976947)	0.10448959 (0.0303460)	0.1022518 (2.171328)

Table 3: Comparison between MMA, FHA, SHA, FGEM, and SGEM results with exact ones and HPM responses ($\sigma = 0.9$).

A	T_{exact}	$T_{\text{MMA}} = T_{\text{FHA}}$ (relative error %)	T_{FGEM} (relative error %)	T_{SHA} (relative error %)	T_{SGEM} (relative error %)	T_{HPM} [39] (relative error %)
0.01	6.2829757	6.28030856 (0.0424502)	6.28297267 (0.0000482)	6.28033909 (0.0419642)	6.28297267 (0.0000482)	6.2829757 (0.00000005)
0.1	6.2622091	6.23585315 (0.42087294)	6.26202447 (0.0029481)	6.23646361 (0.4111248)	6.26203439 (0.0027898)	6.26220029 (0.0001407)
10	0.7727359	0.75450185 (2.3596739)	0.74930385 (3.032348)	0.76075498 (1.5504538)	0.77284969 (0.0147255)	0.7564511 (2.1074)
50	0.1561833	0.15271588 (2.2200941)	0.15087022 (3.4018226)	0.15400462 (1.3949487)	0.15573888 (0.2845491)	0.1527948 (2.1696)
100	0.0781394	0.07642280 (2.1968399)	0.07545104 (3.4404631)	0.07706833 (1.3707164)	0.07788792 (0.3218291)	0.07644255 (2.1716)

$$\begin{aligned}
&\frac{\partial E(\dot{u},u,t)}{\partial a} \\
&= \frac{\pi}{8\omega}\Big(30\sigma^2a^5 + 90\sigma^2ab^4 + 180\sigma^2a^3b^2 \\
&\quad + 24\sigma^2a^2b + 16a\omega^4 + 90\sigma^2a^2b^3 + 48ab^2 \\
&\quad - 32a\omega^2 + 16a + 24a^3 - 480a\omega^2\sigma b^2 \\
&\quad + 75\sigma^2a^4b - 48a^3\omega^2\sigma + 24a^2b \\
&\quad -144a^2\omega^2\sigma b + 24\sigma^2a^3 + 48\sigma^2ab^2\Big) = 0 \\
&\frac{\partial E(\dot{u},u,t)}{\partial b} \\
&= \frac{\pi}{8\omega}\Big(-288b\omega^2 + 180\sigma^2a^2b^3 + 90\sigma^2a^4b \\
&\quad + 48\sigma^2a^2b + 8\sigma^2a^3 + 24\sigma^2b^3 + 30\sigma^2b^5 \\
&\quad + 90\sigma^2a^3b^2 + 48a^2b + 16b - 480a^2\omega^2\sigma b \\
&\quad + 15\sigma^2a^5 + 1296b\omega^4 + 24b^3 \\
&\quad -432b^3\omega^2\sigma + 8a^3 - 48a^3\omega^2\sigma\Big) = 0.
\end{aligned} \tag{33}$$

Finally, by simultaneously solving (30) and (33) unknown parameters are determined for different values of A and σ as the second-order global error minimization method (SGEM) solution.

3. Results

The relative errors of the max-min, first- and second-orders of the Hamiltonian, and the global error minimization methods are shown in Tables 2, 3, and 4 for $\sigma = 0.5$, $\sigma = 0.9$, and $\sigma = 1$, respectively. It is seen that the second-order global error minimization method results are closer to the exact ones than the other mentioned methods for large initial amplitudes.

The differences of obtained responses and velocities with exact ones are plotted in Figures 1 and 2, respectively. It is found that the second-order global error minimization method illustrates a better accuracy than the aforementioned techniques.

4. Conclusion

The Helmholtz-Duffing equation is investigated via the analytical approaches; the accuracy and validity of the obtained

Table 4: Comparison between MMA, FHA, SHA, FGEM, and SGEM results with exact ones ($\sigma = 1$).

A	T_{exact}	$T_{\text{MMA}} = T_{\text{FHA}}$ (relative error %)	T_{FGEM} (relative error %)	T_{SHA} (relative error %)	T_{SGEM} (relative error %)
0.01	6.282949702	6.282949704 (0.000000032)	6.282949698 (0.000000063)	6.282953782 (0.000064937)	6.282949704 (0.000000032)
0.1	6.259762302	6.259755078 (0.000115404)	6.259750255 (0.0001924513)	6.260158455 (0.006328562)	6.259762297 (0.00000008)
10	0.736288961	0.720730784 (2.113053139)	0.7113585054 (3.385960801)	0.7269670459 (1.266067481)	0.7337722018 (0.3418167776)
50	0.148282774	0.1450652702 (2.169843275)	0.1431321874 (3.473489510)	0.1463367452 (1.312376851)	0.1477516413 (0.3581890773)
100	0.074157585	0.0725471382 (2.171654781)	0.07157966046 (3.476278981)	0.0731832605 (1.313856823)	0.07389156930 (0.3587167786)

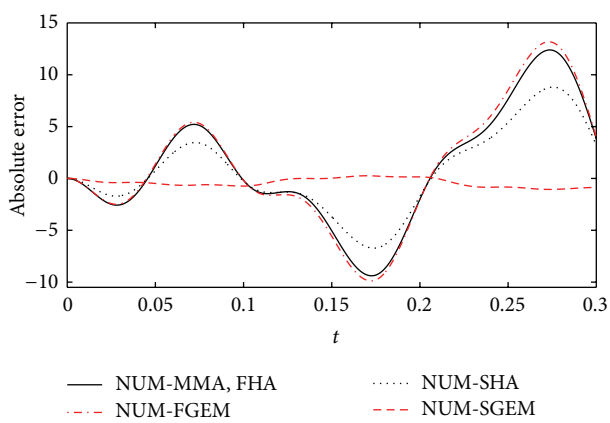

Figure 1: Difference between obtained responses and the numerical solution ($A = 50$, $\sigma = 0.5$).

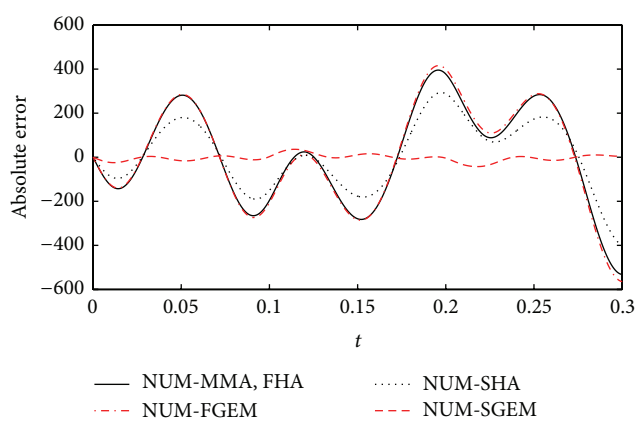

Figure 2: Difference between obtained velocities and the numerical solution ($A = 50$, $\sigma = 0.5$).

results have been examined by comparing to the exact ones and HPM solutions. The second-order of the global error minimization method achieved better approximate solutions for this equation. In present study, it is demonstrated that higher order of the modified variational approach is accurate and simple for solving asymmetric nonlinear conservative oscillatory systems.

Conflict of Interests

The authors declare that they have no competing interests.

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